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## Linearizing $W_{2,4}$ and $WB_2$ Algebras

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### Abstract

It has recently been shown that the  $W_3$  and  $W_3^{(2)}$  algebras can be considered as subalgebras in some linear conformal algebras. In this paper we show that the nonlinear algebras  $W_{2,4}$  and  $WB_2$  as well as Zamolodchikov's spin 5/2 superalgebra also can be embedded as subalgebras into some linear conformal algebras with a finite set of currents. These linear algebras give rise to new realizations of the nonlinear algebras which could be suitable in the construction of  $W$ -string theories.

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## 1 Introduction

In the last few years a lot of attention has been devoted to nonlinear algebras. A great success was achieved in the construction of  $W$ -algebras from the linear Kac-Moody (affine) algebras by gauging some set of constraints imposed on the currents of the latter (see Refs.[1] and references therein for reviews). In this way Miura-like free-fields realizations [2] of  $W$ -algebras were also obtained. This deep connection between  $W$ - and affine algebras by no means implies that the currents of former algebra can be constructed directly from the currents of the latter one [3,4]. This remarkable relation between linear and nonlinear algebras allows one to analyze the properties of  $W$ -algebras and the corresponding  $W$ -strings from the affine algebras point of view.

A different way to connect  $W$ -algebras to linear ones was proposed by two of us [5]. We addressed the question, whether linear conformal Lie algebras with a finite number of currents exist, such that they contain nonlinear  $W$ -algebras as subalgebras in some nonlinear basis? It has been proved, for the simplest case of  $W_3$  and  $W_3^{(2)}$  algebras, that such linear algebras exist, have the structure of an extension of the  $W$ -currents by some set of currents forming a realization of the given  $W$ -algebra.

In an effort to show that the existence of the linear structure is a general feature of the  $W$ -type algebras, rather than an exception valid only for the simple cases mentioned above, and motivated by the expectation that the properties of the theories based on nonlinear algebras can be understood in the simplest way from the corresponding linear structures, we extend in the present paper the list of nonlinear algebras which admit linearization. We analyze from this point of view three  $W$ -type algebras:  $W_{2,4}$  algebra [6], the simplest fermionic nonlinear  $WB_2$  algebra [7], and Zamolodchikov spin 5/2 algebra [8] which exists only for some particular value of the central charge. We find that in all cases the linear conformal algebras indeed exist and are connected with the aforementioned types of  $W$ -realizations, and contain the considered  $W$ -algebras as subalgebras. Using the linear conformal algebras obtained we find, as a first application, a new field realization of  $W_{2,4}$  algebra. At the level of the linear algebras, the intrinsic relation between the classical  $W_{2,4}$  and  $WB_2$  algebras is more transparent, in the sense that the linear algebra for classical  $W_{2,4}$  is simply a subalgebra of the one for  $WB_2$ .

## 2 $WB_2$ and $W_{2,4}$ algebras

In this Section we fix our notation by displaying the explicit structure of  $WB_2$  [7] (or, using a different notation,  $W(2, \frac{5}{2}, 4)$ ) and  $W_{2,4}$  [6] algebras and their relations in both the quantum and the classical cases.

The  $WB_2$  algebra is the simplest example of a superalgebra with a quadratic nonlinearity. Besides the standard Virasoro stress-tensor  $T(z)$  it contains two additional currents: the bosonic current  $U(z)$  with spin 4 and the fermionic current  $Q(z)$  with spin 5/2. This algebra exists for the generic central charge  $c$  and their currents obey the following operator product

expansions (OPE's)<sup>1</sup>:

$$T(z_1)T(z_2) = \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad (2.1)$$

$$T(z_1)Q(z_2) = \frac{5/2 Q}{z_{12}^2} + \frac{Q'}{z_{12}} \quad , \quad (2.2)$$

$$T(z_1)U(z_2) = \frac{4U}{z_{12}^2} + \frac{U'}{z_{12}} \quad , \quad (2.3)$$

$$Q(z_1)Q(z_2) = \frac{2c/5}{z_{12}^5} + \frac{2T}{z_{12}^3} + \frac{T'}{z_{12}^2} + [b_1 T'' + b_2 U + b_3 L_4] \frac{1}{z_{12}} \quad , \quad (2.4)$$

$$\begin{aligned} U(z_1)Q(z_2) &= \frac{d_1 Q}{z_{12}^4} + \frac{d_2 Q'}{z_{12}^3} + [d_3 Q'' + d_4 (T Q)] \frac{1}{z_{12}^2} + \\ &\quad [d_5 Q''' + d_6 (T' Q) + d_7 (T Q')] \frac{1}{z_{12}} \quad , \\ U(z_1)U(z_2) &= \frac{c/4}{z_{12}^8} + \frac{2T}{z_{12}^6} + \frac{T'}{z_{12}^5} + [a_1 U + a_2 L_4 + a_3 T''] \frac{1}{z_{12}^4} + \\ &\quad [a_1/2 U' + a_2/2 L'_4 + a_4 T'''] \frac{1}{z_{12}^3} + [a_5 U'' + a_6 L''_4 + a_7 T'''' + \\ &\quad a_8 L_{61} + a_9 L_{62} + a_{10} L_{63} + a_{11} L_{64}] \frac{1}{z_{12}^2} + [a_{12} U''' + a_{13} L'''_4 + a_{14} T'''' + \\ &\quad a_8/2 L'_{61} + a_9/2 L'_{62} + a_{10}/2 L'_{63} + a_{11}/2 L'_{64}] \frac{1}{z_{12}} \quad , \end{aligned} \quad (2.5)$$

where the composites  $L_4$  (spin 4), and  $L_{61} - L_{64}$  (all with spin 6) are defined by

$$\begin{aligned} L_4 &= (T T), \\ L_{61} &= (T L_4), \quad L_{62} = (T' T'), \quad L_{63} = (T U), \quad L_{64} = (Q' Q) \end{aligned} \quad (2.6)$$

with normal ordering understood for the products of currents. The values of all coefficients in the quantum and classical cases are given in Appendix.

The  $W_{2,4}$  algebra contains only the bosonic currents  $\{T(z), U(z)\}$  with spins  $\{2, 4\}$ . The current  $U(z)$  is primary with respect to the Virasoro stress-tensor  $T$ . Therefore the only OPE we need, in order to specify this algebra, is  $U(z_1)U(z_2)$ . This OPE has the same form as (2.5) with  $a_{11} = 0$  and all other coefficients as given in Appendix.

We would like to stress that for both  $WB_2$  and  $W_{2,4}$  algebras we use the non standard definition of composite currents (2.6), which are *non* primary with respect to the corresponding stress-tensor  $T(z)$ , in contrast with the papers [6,7] where primary composites have been used. Despite the less evident structure of the coefficients in the OPE's (2.4)-(2.5), the use of non primary composites gives a significant speed up of all calculations (at least twice as fast) which have been done with the help of the Mathematica [9] package OPEdefs [10].

Let us summarize some peculiarities of these algebras.

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<sup>1</sup>As usual, we will write OPE's in the quantum as well as in the classical case keeping in mind that for the classical case only one pair contractions took place. The currents in the r.h.s. of the OPE's are evaluated at the point  $z_2$ , and  $z_{12} = z_1 - z_2$ .

First of all, let us notice that in the quantum  $WB_2$  algebra the coefficient  $b_2 = \sqrt{\frac{6(13c+14)}{5c+22}}$  before the spin 4 current  $U(z)$  in the r.h.s (2.4) vanishes, for the particular value  $c = -\frac{13}{14}$  of the central charge. This means that for this value of the central charge the currents  $T(z)$  and  $Q(z)$  form (modulo null-fields) the closed quantum algebra which is the spin  $\frac{5}{2}$  algebra of Zamolodchikov [8].

Secondly, we would like to stress that for the classical algebras all coefficients in the OPE  $U(z_1)U(z_2)$  for  $WB_2$  and  $W_{2,4}$  coincide (with the obvious exception of  $a_{11}$  which vanishes for  $W_{2,4}$ ). This means that the classical  $W_{2,4}$  algebra can be obtained from  $WB_2$  by truncation, namely by dropping from OPE's the current  $Q(z)$  and all composites constructed from  $Q(z)$ . In other words, in the classical case the composite current  $(Q' Q)$  transforms homogeneously through itself and  $Q(z)$ , and so it can be consistently dropped, together with  $Q(z)$ . In the quantum case the situation changes drastically. As shown in the Appendix, the values of the coefficients  $a_1 - a_{14}$  for  $WB_2$  and  $W_{2,4}$  are quite different and there exists no truncation procedure connecting these algebras. The reason that singles out the quantum case is the following: despite the existence of a primary spin 6 current which looks like  $(Q' Q) +$  corrections, its quantum OPE with  $U(z)$  necessarily contains a term proportional to the  $U(z)$  current in the right-hand side. So, in the quantum  $WB_2$  algebra we cannot exclude from the OPE's the current  $Q(z)$  and all composites constructed from  $Q(z)$ , without contradicting the Jacobi identities. That is why we need to consider the quantum  $W_{2,4}$  algebra independently from the  $WB_2$  one.

In the next Section we construct the linear algebra which contains the  $W_{2,4}$  one as a subalgebra.

### 3 Linearization of $W_{2,4}$ algebra

In this Section we show that there exists a linear algebra  $W_{2,4}^{lin}$  with a finite number of current which contains  $W_{2,4}$  as a subalgebra.

The main idea of our approach [5] is to extend the set of currents of the given nonlinear algebra by adding some new currents in such a way that

- the resulting set of currents still form a closed algebra,
- the extended algebra contains the Virasoro subalgebra and all other currents could be chosen to be primary with respect to Virasoro stress-tensor,
- there exists an *invertible* nonlinear transformation to another basis for the currents where all OPE's become linear.

In other words, here we would like to demonstrate that there is some *linear, conformal* algebra which contains the  $W_{2,4}$  algebra as a subalgebra in a certain nonlinear basis.

It is well known [3,4], that the currents of the nonlinear algebra obtained through Drinfeld-Sokolov reduction, can be expressed in terms of the corresponding WZNW sigma-model currents. The novelties of our approach are as follows:

- the number of currents in the linear algebra is the same as in the extended nonlinear one and the transformation from one set of currents to another is invertible,

- the linear algebra is conformal, i.e. it contains the Virasoro subalgebra and all other currents are primary with respect to it.

Let us demonstrate that for the  $W_{2,4}$  algebra the corresponding linear algebra with the aforementioned properties does really exist.

The starting point of our construction is the following linear algebra  $W_{2,4}^{lin}$ :

$$\begin{aligned}
 T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad T(z_1)J(z_2) = \frac{c_1}{z_{12}^3} + \frac{J}{z_{12}^2} + \frac{J'}{z_{12}} \quad , \\
 T(z_1)U_1(z_2) &= \frac{4U_1}{z_{12}^2} + \frac{U_1'}{z_{12}} \quad , \quad J(z_1)U_1(z_2) = \frac{2q_1 U_1}{z_{12}} \quad , \\
 J(z_1)J(z_2) &= \frac{1}{z_{12}^2} \quad , \quad U_1(z_1)U_1(z_2) = \text{regular} \quad ,
 \end{aligned} \tag{3.1}$$

where the central charges  $c$  and  $c_1$  are connected with the  $U(1)$  charge  $q_1$  as follows:

$$c = -\frac{120q_1^4 - 86q_1^2 + 15}{q_1^2} \quad , \quad c_1 = \frac{2 - 6q_1^2}{q_1} . \tag{3.2}$$

Let us remark that we demand that the stress-tensor of the nonlinear  $W_{2,4}$  algebra coincides with the stress-tensor of the linear algebra (3.1). That is why we need to have the central charge  $c_1$  in the OPE  $T(z_1)J(z_2)$ . Of course, we are free to pass by the transformation  $T \rightarrow T + \alpha J'$  to another basis where all currents are primary. However, in the basis under consideration all expressions look more tractable.

Now it is a matter of straightforward calculations to show that after the following *invertible* nonlinear transformation to the new basis  $\{J, T, U\}$ , where

$$\begin{aligned}
 U &= U_1 + z_1(T T) + z_2(T' J) + z_3(T J') + z_4(T J J) + z_5 T'' + \\
 &\quad z_6(J J J J) + z_7(J' J J) + z_8(J'' J) + z_9(J' J') + z_{10} J''' \quad ,
 \end{aligned} \tag{3.3}$$

the currents  $T$  and  $U$  form the  $W_{2,4}$  algebra (2.1),(2.3),(2.5) with the central charge  $c$  parametrized by  $q_1$  (3.2), provided the coefficients  $\{z_1 - z_{10}\}$  are chosen to be

$$\begin{aligned}
 z_1 &= \frac{(8 - 27q_1^2)(3 - 4q_1^2)}{84(1 - 8q_1^4)} a_{10} \\
 z_2 &= \frac{(1 - 4q_1^2)(600q_1^4 - 452q_1^2 + 75)}{168q_1(1 - 8q_1^4)} a_{10} \\
 z_3 &= \frac{(1 - 2q_1^2)(600q_1^4 - 452q_1^2 + 75)}{84q_1(1 - 8q_1^4)} a_{10} \\
 z_4 &= \frac{600q_1^4 - 452q_1^2 + 75}{84(8q_1^4 - 1)} a_{10} \\
 z_5 &= \frac{1440q_1^8 - 1860q_1^6 + 865q_1^4 - 181q_1^2 + 15}{168q_1^2(8q_1^4 - 1)} a_{10} \\
 z_6 &= -\frac{z_4}{2} \\
 z_7 &= \frac{(3q_1^2 - 1)(600q_1^4 - 452q_1^2 + 75)}{42q_1(1 - 8q_1^4)} a_{10}
 \end{aligned}$$

$$\begin{aligned}
z_8 &= \frac{(32q_1^4 - 16q_1^2 + 3)(600q_1^4 - 452q_1^2 + 75)}{336q_1^2(1 - 8q_1^4)} a_{10} \\
z_9 &= \frac{(48q_1^4 - 36q_1^2 + 7)(600q_1^4 - 452q_1^2 + 75)}{672q_1^2(1 - 8q_1^4)} a_{10} \\
z_{10} &= \frac{(96q_1^6 - 68q_1^4 + 19q_1^2 - 3)(600q_1^4 - 452q_1^2 + 75)}{2016q_1^3(1 - 8q_1^4)} a_{10}
\end{aligned} \tag{3.4}$$

and  $a_{10}$  is defined as in the Appendix:

$$a_{10} = \frac{8(7c - 115)}{2c + 25} \sqrt{\frac{6}{(5c + 22)(14c + 13)}} .$$

To obtain the classical limit, we need to make the following renormalizations of the  $U(1)$  current  $J$ , central charge  $c_1$ , and  $U(1)$  charge  $q_1$ :

$$J \rightarrow \frac{1}{\sqrt{c}} J \quad , \quad c_1 \rightarrow \sqrt{c} c_1 \quad , \quad q_1 \rightarrow \sqrt{c} q_1 \quad , \tag{3.5}$$

and consider, as usual, the values of all coefficients in the limit

$$c \rightarrow \infty \quad \text{or} \quad q_1 \rightarrow 0 \quad . \tag{3.6}$$

The classical linear algebra  $\mathcal{W}_{2,4}^{lin}$  has the following form<sup>2</sup>:

$$\begin{aligned}
\mathcal{T}(z_1)\mathcal{T}(z_2) &= \frac{\tilde{c}/2}{z_{12}^4} + \frac{2\mathcal{T}}{z_{12}^2} + \frac{\mathcal{T}'}{z_{12}} \quad , \quad \mathcal{T}(z_1)\mathcal{J}(z_2) = \frac{-2i\tilde{c}/\sqrt{15}}{z_{12}^3} + \frac{\mathcal{J}}{z_{12}^2} + \frac{\mathcal{J}'}{z_{12}} \quad , \\
\mathcal{T}(z_1)\mathcal{U}_1(z_2) &= \frac{4\mathcal{U}_1}{z_{12}^2} + \frac{\mathcal{U}_1'}{z_{12}} \quad , \quad \mathcal{J}(z_1)\mathcal{U}_1(z_2) = \frac{2i\sqrt{15}\mathcal{U}_1}{z_{12}} \quad , \\
\mathcal{J}(z_1)\mathcal{J}(z_2) &= \frac{\tilde{c}}{z_{12}^2} \quad , \quad \mathcal{U}_1(z_1)\mathcal{U}_1(z_2) = \text{regular} \quad .
\end{aligned} \tag{3.7}$$

The classical expression for the current  $\mathcal{U}$ , forming together with  $\mathcal{T}$  the classical  $\mathcal{W}_{2,4}$  algebra, reads

$$\begin{aligned}
\mathcal{U} &= \mathcal{U}_1 + \frac{24}{\tilde{c}\sqrt{105}} \mathcal{T}^2 - \frac{5i}{2\tilde{c}\sqrt{7}} \mathcal{T}'\mathcal{J} - \frac{5i}{\tilde{c}\sqrt{7}} \mathcal{T}\mathcal{J}' - \frac{75}{\tilde{c}^2\sqrt{105}} \mathcal{T}\mathcal{J}^2 + \frac{1}{2\sqrt{105}} \mathcal{T}'' + \\
&\quad \frac{75}{2\tilde{c}^3\sqrt{105}} \mathcal{J}^4 + \frac{10i}{\tilde{c}^2\sqrt{7}} \mathcal{J}'\mathcal{J}^2 - \frac{15}{4\tilde{c}\sqrt{105}} \mathcal{J}''\mathcal{J} - \frac{5}{8\tilde{c}} \sqrt{\frac{7}{15}} \mathcal{J}'\mathcal{J}' - \frac{i}{24\sqrt{7}} \mathcal{J}''' \quad .
\end{aligned} \tag{3.8}$$

So, we have explicitly shown that  $\mathcal{W}_{2,4}^{lin}$  algebra (3.1) ( $\mathcal{W}_{2,4}^{lin}$  (3.7)) contains the quantum (classical)  $\mathcal{W}_{2,4}$  algebra as its subalgebra in the nonlinear basis.

We postpone the discussion of the consequences of the linearization property to Section 5. In the next Section we show how to construct the linear algebra for the  $WB_2$  superalgebra.

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<sup>2</sup>We will use the calligraphic and tilded letters, to distinguish the classical currents and charges from the corresponding quantum ones.

## 4 Linearization of $WB_2$ algebra

In this Section we explicitly demonstrate that  $WB_2$  is a subalgebra of a very simple *linear* conformal superalgebra.

The main novelties of  $WB_2$  algebra are, in comparison with  $W_{2,4}$ ,

- the fermionic nature of the spin 5/2 current  $Q(z)$
- the appearance of the spin 4 current  $U(z)$  in the OPE of two spin 5/2 fermionic currents.

The last property means that the whole structure of  $WB_2$  algebra is encoded in the structure of the “supersymmetry” current  $Q(z)$ . Moreover, the linear algebra  $WB_2^{lin}$  we are going to construct must reproduce at  $c = -\frac{13}{14}$  the linear algebra for the Zamolodchikov’s spin 5/2 nonlinear algebra. In addition, we know that the classical  $\mathcal{W}_{2,4}$  algebra is a truncation of the classical  $\mathcal{WB}_2$ . In the case of the corresponding linear algebras, due to the absence of nonlinear composites, the  $\mathcal{W}_{2,4}^{lin}$  (3.7) must be a bosonic subalgebra of  $\mathcal{WB}_2^{lin}$ .

Without going into details, let us show that  $WB_2^{lin}$  contains the bosonic  $\{T, J, U_1\}$  and fermionic  $\{S, Q_1\}$  currents<sup>3</sup>, obeying the following OPE’s<sup>4</sup>:

$$\begin{aligned}
 T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \\
 T(z_1)J(z_2) &= \frac{c_2}{z_{12}^2} + \frac{J}{z_{12}} + \frac{J'}{z_{12}} \quad , \quad T(z_1)S(z_2) = \frac{1/2S}{z_{12}^2} + \frac{S'}{z_{12}} \quad , \\
 T(z_1)Q_1(z_2) &= \frac{5/2Q_1}{z_{12}^2} + \frac{Q_1'}{z_{12}} \quad , \quad T(z_1)U_1(z_2) = \frac{4U_1}{z_{12}^2} + \frac{U_1'}{z_{12}} \quad , \\
 J(z_1)Q_1(z_2) &= \frac{q_2Q_1}{z_{12}} \quad , \quad J(z_1)U_1(z_2) = \frac{2q_2U_1}{z_{12}} \quad , \\
 S(z_1)S(z_2) &= \frac{1}{z_{12}^2} \quad , \quad J(z_1)J(z_2) = \frac{1}{z_{12}^2} \quad , \quad Q_1(z_1)Q_1(z_2) = \frac{b_2U_1}{z_{12}} \quad , \quad (4.1)
 \end{aligned}$$

where the central charges  $c$  and  $c_2$  are parametrized by the  $U(1)$  charge  $q_2$

$$c = -\frac{120q_2^4 - 125q_2^2 + 30}{2q_2^2} \quad , \quad c_2 = \frac{2 - 4q_2^2}{q_2} \quad , \quad (4.2)$$

and the coefficient  $b_2$  defined in the Appendix reads

$$b_2 = \sqrt{\frac{6(14c + 13)}{5c + 22}} \quad .$$

In order to prove this, we do the following *invertible* transformation to the new basis  $\{T, J, S, Q, U\}$ , where

$$Q = Q_1 + z_1 S'' + z_2 (J S') + z_3 (J' S) + z_4 (J J S) + z_5 (T S) \quad (4.3)$$

<sup>3</sup>We hope that the use of the same notation for the bosonic currents as in the previous Section does not give rise to confusion as to which algebra we are dealing with.

<sup>4</sup>For the same reason of simplicity as in the previous Section we work in the non-primary basis.

and  $U$  is defined from the OPE in eqs. (2.4)

$$Q(z_1)Q(z_2) = \frac{2c/5}{z_{12}^5} + \frac{2T}{z_{12}^3} + \frac{T'}{z_{12}^2} + [b_1 T'' + b_2 U + b_3 (T T)] \frac{1}{z_{12}} .$$

If the parameters  $\{z_1 - z_5\}$  are chosen to be

$$\begin{aligned} z_1 &= \frac{8q_2^4 - 5q_2^2 + 2}{4q_2^2} \sqrt{\frac{30}{25 + 2c}} \\ z_2 &= \frac{2q_2^2 - 1}{q_2} \sqrt{\frac{30}{25 + 2c}} \\ z_3 &= \frac{6q_2^2 - 3}{2q_2} \sqrt{\frac{30}{25 + 2c}} \\ z_4 &= \sqrt{\frac{30}{25 + 2c}} \\ z_5 &= -\sqrt{\frac{30}{25 + 2c}} \end{aligned} \quad (4.4)$$

and the central charge  $c$  is connected with  $q$  as in (4.2), then the currents  $T, Q$  and  $U$  just form the  $WB_2$  algebra (2.5).

Thus, one concludes that  $WB_2^{lin}$  indeed contains  $WB_2$  as a subalgebra.

Let us close this Section with some comments.

First of all, we would like to remark that the linear  $WB_2^{lin}$  algebra (4.1) and the coefficients  $\{z_1 - z_5\}$  (4.4) do not contain singularities when  $c \rightarrow -13/14$  ( $q \rightarrow \sqrt{5/14}$ ). Therefore, we can immediately read off the linear algebra for the Zamolodchikov's spin 5/2 algebra with  $c = -13/14$

$$\begin{aligned} T(z_1)T(z_2) &= \frac{-13/28}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} , \\ T(z_1)J(z_2) &= \frac{4\sqrt{2/35}}{z_{12}^2} + \frac{J}{z_{12}^2} + \frac{J'}{z_{12}} , \quad T(z_1)S(z_2) = \frac{1/2S}{z_{12}^2} + \frac{S'}{z_{12}} , \\ T(z_1)Q_1(z_2) &= \frac{5/2Q_1}{z_{12}^2} + \frac{Q_1'}{z_{12}} , \quad J(z_1)Q_1(z_2) = \frac{\sqrt{5/14}Q_1}{z_{12}} , \\ S(z_1)S(z_2) &= \frac{1}{z_{12}^2} , \quad J(z_1)J(z_2) = \frac{1}{z_{12}^2} , \quad Q_1(z_1)Q_1(z_2) = \text{regular}. \end{aligned} \quad (4.5)$$

The corresponding expression for the spin 5/2 current  $Q_Z$ , which together with the stress-tensor  $T$  forms the Zamolodchikov's spin 5/2 algebra, reads as follows:

$$Q_Z = Q_1 + \frac{121}{12\sqrt{105}} S'' - \frac{2}{3} \sqrt{\frac{2}{3}} (J S') - \sqrt{\frac{2}{3}} (J' S) + \frac{1}{3} \sqrt{\frac{35}{3}} (J J S) - \frac{1}{3} \sqrt{\frac{35}{3}} (T S) \quad (4.6)$$

Thus we showed that the linear superalgebra (4.5) contains the Zamolodchikov's spin 5/2 as a subalgebra.



Secondly, as pointed out in the Section 2, the classical  $\mathcal{WB}_2$  algebra contains the classical  $\mathcal{W}_{2,4}$  algebra, in the truncation limit  $Q \rightarrow 0$ . It can be easily shown that the analogous relation exists also for the linear algebra  $\mathcal{WB}_2^{lin}$ . In the classical limit  $c \rightarrow \infty$ , after the redefinitions

$$J \rightarrow \frac{1}{\sqrt{c}}J \quad , \quad S \rightarrow \frac{1}{\sqrt{c}}S \quad , \quad c_2 \rightarrow \sqrt{c}c_2 \quad , \quad q_2 \rightarrow \sqrt{c}q_2 \quad , \quad (4.7)$$

the classical linear algebra  $\mathcal{WB}_2^{lin}$  has the following form:

$$\begin{aligned} T(z_1)T(z_2) &= \frac{\tilde{c}/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \\ T(z_1)J(z_2) &= \frac{-2i\tilde{c}/\sqrt{15}}{z_{12}^2} + \frac{J}{z_{12}^2} + \frac{J'}{z_{12}} \quad , \quad T(z_1)S(z_2) = \frac{1/2S}{z_{12}^2} + \frac{S'}{z_{12}} \quad , \\ T(z_1)Q_1(z_2) &= \frac{5/2Q_1}{z_{12}^2} + \frac{Q_1'}{z_{12}} \quad , \quad T(z_1)U_1(z_2) = \frac{4U_1}{z_{12}^2} + \frac{U_1'}{z_{12}} \quad , \\ J(z_1)Q_1(z_2) &= \frac{i\sqrt{15}Q_1}{z_{12}} \quad , \quad J(z_1)U_1(z_2) = \frac{2i\sqrt{15}U_1}{z_{12}} \quad , \\ S(z_1)S(z_2) &= \frac{\tilde{c}}{z_{12}^2} \quad , \quad J(z_1)J(z_2) = \frac{\tilde{c}}{z_{12}^2} \quad , \quad Q_1(z_1)Q_1(z_2) = \frac{2\sqrt{21/5}U_1}{z_{12}}. \end{aligned} \quad (4.8)$$

One can immediately see that the bosonic subalgebra of  $\mathcal{WB}_2^{lin}$  coincides with  $\mathcal{W}_{2,4}^{lin}$  (3.7). While the classical  $W_{2,4}$  algebra is a truncation of  $WB_2$ , the corresponding linear algebra  $\mathcal{W}_{2,4}^{lin}$  is a simple subalgebra of  $\mathcal{WB}_2^{lin}$ .

In the next Section we consider some examples that can illustrate the use of the linearized algebra for both  $W_{2,4}$  and  $WB_2$  algebras.

## 5 Conclusions and discussion

In this paper we extend the list of nonlinear algebras which admit linearization, by explicitly constructing the linear conformal algebras, with a finite set of currents, which contain  $W_{2,4}$  and  $WB_2$  algebras as subalgebras in a nonlinear basis. Thus, now we know that the first representatives of both the  $WA_n$  ( $W_3$  [5]) and the  $WB_n$  ( $WB_2$ ) series of nonlinear algebras can be linearized. Despite the lack, up to now, of a general algorithmic procedure to construct the linear conformal algebra for any given nonlinear algebra, it makes sense to conjecture that the possibility of linearization is a general property of nonlinear algebras rather than a peculiarity of some exceptional cases.

Let us finish by discussing some common properties of the linearization and the linear algebras.

First of all, we would like to note that, in order to linearize a nonlinear algebra, we need to extend it by adding some currents ( $J(z)$  in the case of  $W_{2,4}$  and  $\{J(z), S(z)\}$  in the case of  $WB_2$  algebra). Thus, the first open question is as to which additional properties of the system are associated with these symmetries.

Secondly, it is interesting that the constructed linear algebras are *homogeneous* with respect to some currents ( $U_1(z)$  and  $\{U_1(z), Q_1(z)\}$  in the case of  $W_{2,4}$  and  $WB_2$ , respectively).

This property means that we could consistently put these currents equal to zero and be left with the realization of the algebras in terms of an arbitrary stress-tensor and a spin 1 current  $J(z)$  for  $W_{2,4}$ , and a spin 1 current  $J(z)$  and a spin 1/2 current  $S(z)$  for the  $WB_2$ .

Finally, let us remark that, owing to the invertible relation between the currents of the nonlinear and the linear algebras, every realization of  $W_{2,4}^{lin}$  or  $WB_2^{lin}$  is a realization of  $W_{2,4}$  or  $WB_2$  respectively. So, the problem of constructing the realizations of these algebras is reduced to the problem of constructing realizations of the linear algebras  $W_{2,4}^{lin}$  and  $WB_2^{lin}$ . In the rest of this Section we will present an example of such realization for the case of  $W_{2,4}$  algebra.

From the simple structure of the  $W_{2,4}^{lin}$  algebra (3.1) it is clear that its most general realization includes at least two scalar fields  $\phi_i$  ( $i = 1, 2$ ) with OPE's

$$\phi_i(z_1)\phi_j(z_2) = -\delta_{ij} \ln(z_{12}) \quad (5.1)$$

and, commuting with them, a Virasoro stress tensor  $\tilde{T}$  having a nonzero central charge which we will denote as  $c_T$ . Representing the bosonic primary current  $U_1(z)$  in the standard way by an exponential of  $\phi_i$ , we find the following expressions:

$$\begin{aligned} T &= \tilde{T} - \frac{1}{2}(\phi_1')^2 - \frac{1}{2}(\phi_2')^2 - \frac{i(1-3q_1^2)}{q_1}\phi_1'' - \\ &\quad \frac{i(4+12q_1^2-N)}{2\sqrt{N-4q_1^2}}\phi_2'' \quad , \\ J &= i\phi_1' \quad , \\ U_1 &= s \exp\left(i\sqrt{N-4q_1^2}\phi_2 + 2iq_1\phi_1\right) \quad , \\ c_T &= 3\left(\frac{(4-N+12q_1^2)^2}{N-4q_1^2} - \frac{(2q_1^2-1)^2}{q_1^2}\right) \quad , \end{aligned} \quad (5.2)$$

where  $N$  runs over non-negative natural numbers and  $s$  is an arbitrary parameter (due to a  $U_1$  rescaling invariance of the OPE's (3.1)  $s$  can be chosen to be 0 or 1).

In the case of  $s = 0$  the field  $\phi_2$  can be absorbed by the stress-tensor and we end with the following  $\phi_2$  independent realization:

$$\begin{aligned} T &= \tilde{T}_0 - \frac{1}{2}(\phi_1')^2 - \frac{i(1-3q_1^2)}{q_1}\phi_1'' \quad , \\ J &= i\phi_1' \quad , \\ U_1 &= 0 \quad , \\ c_{T_0} &= 1 - \frac{3(2q_1^2-1)^2}{q_1^2} \quad . \end{aligned} \quad (5.3)$$

After substituting eqs. (5.3) into (3.3), we get the known realization of  $W_{2,4}$  algebra [11], whereas the use of eqs. (5.2) in (3.3) yields a new realization of  $W_{2,4}$  algebra, which may play a role for  $W_{2,4}$  string theory [11].

A last comment concerns the realization (5.3), for which the  $W_{2,4}^{lin}$  algebra is reduced to a direct product of a  $U(1)$  algebra and the Virasoro one with the central charge  $c_{T_0}$  (5.3).

Then, the minimal Virasoro models [12] which correspond to the central charge

$$c_m = 1 - 6 \frac{(p-q)^2}{pq} \Rightarrow q_1^2 = \frac{p}{2q} \quad (5.4)$$

give rise to the following induced central charges (3.2):

$$c_{W_{2,4}}^{min} = -2 \frac{(5(2p) - 6q)(3(2p) - 5q)}{(2p)q} \quad (5.5)$$

They coincide with the central charges of  $W_{2,4}$  minimal models [13]

$$c^{min} = -2 \frac{(5\tilde{p} - 6\tilde{q})(3\tilde{p} - 5\tilde{q})}{\tilde{p}\tilde{q}} \quad (5.6)$$

for even values of  $\tilde{p}$ .

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## Appendix

Here we write down the expressions for the coefficients in the OPE's for  $WB_2$  and  $W_{2,4}$  algebras.

Coeff.	Quantum $WB_2$	Classical $WB_2$	Quantum $W_{2,4}$	Classical $W_{2,4}$
$a_1$	$\frac{3(2c^2+83c-490)}{R_1} \sqrt{\frac{6}{R_2 R_3}}$	$3\sqrt{\frac{3}{35}}$	$3\sqrt{6}R_4$	$3\sqrt{\frac{3}{35}}$
$a_2$	$\frac{42}{R_2}$	$\frac{42}{5c}$	$\frac{42}{R_2}$	$\frac{42}{5c}$
$a_3$	$\frac{3(c-4)}{2R_2}$	$\frac{3}{10}$	$\frac{3(c-4)}{2R_2}$	$\frac{3}{10}$
$a_4$	$\frac{2c-29}{6R_2}$	$\frac{1}{15}$	$\frac{2c-29}{6R_2}$	$\frac{1}{15}$
$a_5$	$\frac{10c^2-197c-2810}{2R_1} \sqrt{\frac{1}{6R_2 R_3}}$	$\frac{1}{4} \sqrt{\frac{5}{21}}$	$\frac{5c+64}{2(c+24)\sqrt{6}} R_4$	$\frac{1}{4} \sqrt{\frac{5}{21}}$
$a_6$	$\frac{352c^2-704c-9125}{2R_1 R_2 R_3}$	$\frac{44}{35c}$	$\frac{176c^2+117c-2528}{2(2c-1)(7c+68)R_2}$	$\frac{44}{35c}$
$a_7$	$\frac{20c^3-462c^2-2862c+9865}{12R_1 R_2 R_3}$	$\frac{1}{84}$	$\frac{10c^3-166c^2-1645c+2296}{12(2c-1)(7c+68)R_2}$	$\frac{1}{84}$
$a_8$	$\frac{108(32c-5)}{R_1 R_2 R_3}$	$\frac{864}{35c^2}$	$\frac{24(72c+13)}{(2c-1)(7c+68)R_2}$	$\frac{864}{35c^2}$
$a_9$	$\frac{-3(38c^2-1669c-4930)}{2R_1 R_2 R_3}$	$\frac{-57}{140c}$	$\frac{-3(19c^2-786c-2368)}{2(2c-1)(7c+68)R_2}$	$\frac{-57}{140c}$
$a_{10}$	$\frac{8(7c-115)}{R_1} \sqrt{\frac{6}{R_2 R_3}}$	$\frac{4}{c} \sqrt{\frac{21}{5}}$	$\frac{28\sqrt{6}}{c+24} R_4$	$\frac{4}{c} \sqrt{\frac{21}{5}}$
$a_{11}$	$\frac{60R_2}{R_1 R_3}$	$\frac{75}{7c}$	0	0
$a_{12}$	$\frac{2c^2-223c-670}{2R_1} \sqrt{\frac{1}{6R_2 R_3}}$	$\frac{1}{4\sqrt{105}}$	$\frac{c-4}{2(c+24)\sqrt{6}} R_4$	$\frac{1}{4\sqrt{105}}$
$a_{13}$	$\frac{3(13c^2-278c-950)}{R_1 R_2 R_3}$	$\frac{39}{140c}$	$\frac{3(13c^2-131c-342)}{2(2c-1)(7c+68)R_2}$	$\frac{39}{140c}$
$a_{14}$	$\frac{20c^3-1196c^2+2953c+44150}{80R_1 R_2 R_3}$	$\frac{1}{560}$	$\frac{10c^3-283c^2-898c+5296}{80(2c-1)(7c+68)R_2}$	$\frac{1}{560}$
$b_1$	$\frac{3(c-1)}{2R_2}$	$\frac{3}{10}$	—	—
$b_2$	$\sqrt{\frac{6R_3}{R_2}}$	$2\sqrt{\frac{21}{5}}$	—	—
$b_3$	$\frac{27}{R_2}$	$\frac{27}{5c}$	—	—
$d_1$	$\frac{5}{4} \sqrt{\frac{3R_3}{2R_2}}$	$\frac{1}{4} \sqrt{105}$	—	—
$d_2$	$\sqrt{\frac{3R_3}{2R_2}}$	$\sqrt{\frac{21}{5}}$	—	—
$d_3$	$\frac{5(c+8)}{2R_1} \sqrt{\frac{R_3}{6R_2}}$	$\frac{1}{4} \sqrt{\frac{35}{3}}$	—	—
$d_4$	$\frac{15}{R_1} \sqrt{\frac{3R_3}{2R_2}}$	$\frac{3}{2c} \sqrt{105}$	—	—
$d_5$	$\frac{2c-5}{R_1} \sqrt{\frac{R_2}{6R_3}}$	$\frac{1}{2} \sqrt{\frac{5}{21}}$	—	—
$d_6$	$\frac{5(22c-49)}{R_1} \sqrt{\frac{3}{2R_2 R_3}}$	$\frac{11}{2c} \sqrt{\frac{15}{7}}$	—	—
$d_7$	$\frac{82c+215}{R_1} \sqrt{\frac{6}{R_2 R_3}}$	$\frac{41}{c} \sqrt{\frac{3}{35}}$	—	—

Here we set

$$R_1 = 2c + 25, R_2 = 5c + 22, R_3 = 14c + 13, R_4 = \sqrt{\frac{(c+24)(c^2-172c+196)}{(5c+22)(7c+68)(2c-1)}}$$

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