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Coherent Instabilities in Particle Accelerators: Conventional and Novel Approaches

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Abstract

Recently, the Thermal Wave Model (TWM) has been proposed for describing the coherent instabilities in particle accelerators. It has been shown that the coasting beam stability criterion can be obtained as "modulational stability" occurring in the propagation of an electromagnetic pulse in a nonlinear medium. Moreover, the TWM indicates the possible existence of soliton-like solutions. In this paper a review of the classical and TWM approaches is presented.

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1. – INTRODUCTION

To describe the dynamics of a beam of charged particles in a circular accelerator one considers the beam as a collection of non–interacting single particles moving in the external guide magnetic fields of the machine. With these fields the effects of linear and non–linear dynamics of a single particle can be studied in detail.

Accelerators quite often require beams of high intensities. As the beam intensity is increased, the electromagnetic fields generated by the interaction of the beam with its environment must be taken into account. Such self–fields act back on the beam perturbing its motion. To describe this effect the single–particle model does not suffice and a multiparticle representation is necessary. The beam, described by a distribution function, can, under unfavourable conditions, produce an electromagnetic field that leads to instabilities known as "coherent instabilities".

In this paper we review the mechanism giving rise to these instabilities with the classical approach, and compare the results with those of the "Thermal Wave Model".

Classically, the study of the instabilities can be divided into three steps. The first one is to find the stationary distribution of the beam from the properties of the external guide fields. Assuming then a small perturbation in the equilibrium distribution, one finds the forces acting back on the beam. Finally one studies the effects of these self–forces on the perturbation itself.

This procedure implies that the behaviour of the beam is mainly determined by the external guide fields, and the self–field represents a small perturbation. Under this linear approximation one finds the instability limits due to the coherent effects.

2. – COHERENT INSTABILITIES OF COASTING BEAM WITH THE CONVENTIONAL APPROACH

A particle with nominal energy E_o moves with velocity βc on a closed orbit, called reference orbit, of length $L_o = 2\,\pi\,R_o$ where R_o is the average machine radius. A small energy deviation $\Delta E>0$, because of the dispertion in the guide fields, makes the charge to travel a longer distance with a higher speed. The change $\Delta\omega$ in its revolution angular frequency is a combination of the two effects[1]:

$$\frac{\Delta\omega}{\omega_o} = -\frac{\eta}{\beta^2}\rho \qquad (1)$$

with

$$\eta = \alpha_c - \frac{1}{\gamma^2}\ ,\ \gamma = \frac{E_o}{m_o c^2}\ ,\ \rho = \frac{\Delta E}{E_o} \qquad (2)$$

where α_c is the momentum compaction and is a property of the guide fields.

We describe the azimuthal position of the particle along its orbit with the longitudinal coordinate x with respect to a moving frame rotating at angular frequency ω_o. The distribution function of the beam is $G(x,\rho;s)$, where s is the independent variable and is equal to βct. The integration of $G(x,\rho;s)$ over the whole ring and all the energies gives the total number of particles N

$$\iint G(x,\rho;s)dxd\rho = N \tag{3}$$

Given the distribution function, we define the charge line density

$$\lambda(x;s) = \int G(x,\rho;s)d\rho \tag{4}$$

and the instantaneous current

$$I(x;s) = e\beta c\lambda(x;s) \tag{5}$$

Let us consider the case of coasting beam. The stationary distribution does not depend on time and on the longitudinal coordinate x, therefore we can write

$$G(x,\rho;s) = Ng_o(\rho) \tag{6}$$

with

$$\int g_o(\rho)d\rho = \frac{1}{2\pi R_o} \tag{7}$$

We now assume a disturbance in the stationary distribution of the form

$$G(x,\rho;s) = N\left[g_o(\rho) + g_1(\rho)e^{i\left(n\frac{x}{R_o} - \Delta\Omega\frac{s}{\beta c}\right)}\right] \tag{8}$$

where n is the number of waves per turn of the disturbance and

$$\Delta\Omega = \Omega - n\omega_o \tag{9}$$

The line density and the current are

$$\lambda(x;s) = \lambda_o + \lambda_1 e^{i\left(n\frac{x}{R_o} - \Delta\Omega\frac{s}{\beta c}\right)} \tag{10}$$

$$I(x;s) = I_o + I_1 e^{i\left(n\frac{x}{R_o} - \Delta\Omega\frac{s}{\beta c}\right)} \tag{11}$$

The interaction of the beam with the surroundings is described[2] by the "coupling impedance" Z. The energy lost by a particle due to the coherent disturbance is given by[1]

$$U(x;s) = ZI_1 e^{i\left(n\frac{x}{R_o} - \Delta\Omega\frac{s}{\beta c}\right)} = eNZ\beta c e^{i\left(n\frac{x}{R_o} - \Delta\Omega\frac{s}{\beta c}\right)} \int g_1(\rho)d\rho \tag{12}$$

To find the stability criterion we use the continuity equation

$$\frac{d}{ds}G(x,\rho;s) = 0 \tag{13}$$

that can be written as

$$\frac{\partial G}{\partial s}+\frac{\partial G}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial s}=0 \qquad (14)$$

where

$$\begin{cases}\dfrac{\partial x}{\partial s}=\dfrac{R_o}{\beta c}\Delta\omega \\ \dfrac{\partial \rho}{\partial s}=-\dfrac{eU}{E_o \beta c T_o}\end{cases} \qquad (15)$$

From the equation (14), neglecting the second order terms, one derives the well known dispertion integral

$$1=i\frac{\beta c Z I_o}{(E/e)}\int \frac{\partial g_o}{\partial \rho}\frac{d\rho}{\left(\Delta\Omega+n\omega_o\dfrac{\eta}{\beta^2}\rho\right)} \qquad (16)$$

2.1 – Monochromatic beam

Let us use the dispertion integral to find the stability of a beam without energy spread. In this case the stationary distribution is

$$g_o(\rho)=\frac{\delta(\rho)}{2\pi R_o} \qquad (17)$$

Substituting equation (17) into equation (16) we obtain

$$\frac{\Delta\Omega}{n\omega_o}=\pm\sqrt{i\frac{\eta\left(\dfrac{Z}{n}\right)I_o}{(E_o/e)2\pi\beta^2}} \qquad (18)$$

If $\Delta\Omega$ is imaginary we get a perturbation with exponentially growing and decaying amplitudes (equation (18) has two solutions) that leads to instability. The real part of $\Delta\Omega$ gives an angular frequency shift.

When the coupling impedance Z has a real part, that is a resistive component, $\Delta\Omega$ will always have an imaginary part and therefore the beam is unstable. For a pure imaginary impedance, instability or stability depend on the sign of η. Below transition energy (η<0) if Z_i is positive (capacitive due to space charge) we have stability. Above transition energy η changes its sign leading to the negative mass instability. The contrary happens if Z_i is negative. This behaviour is summarized in Table I.

Table I

Z_r=0	Z_i>0	Z_i<0
η>0	instability	stability
η<0	stability	instability

In Figure 1 we show the contour plot $\Delta\Omega_i$ = constant of equation (18) on the plane Z_r, Z_i. This allows us to read the grow rate for any impedance.

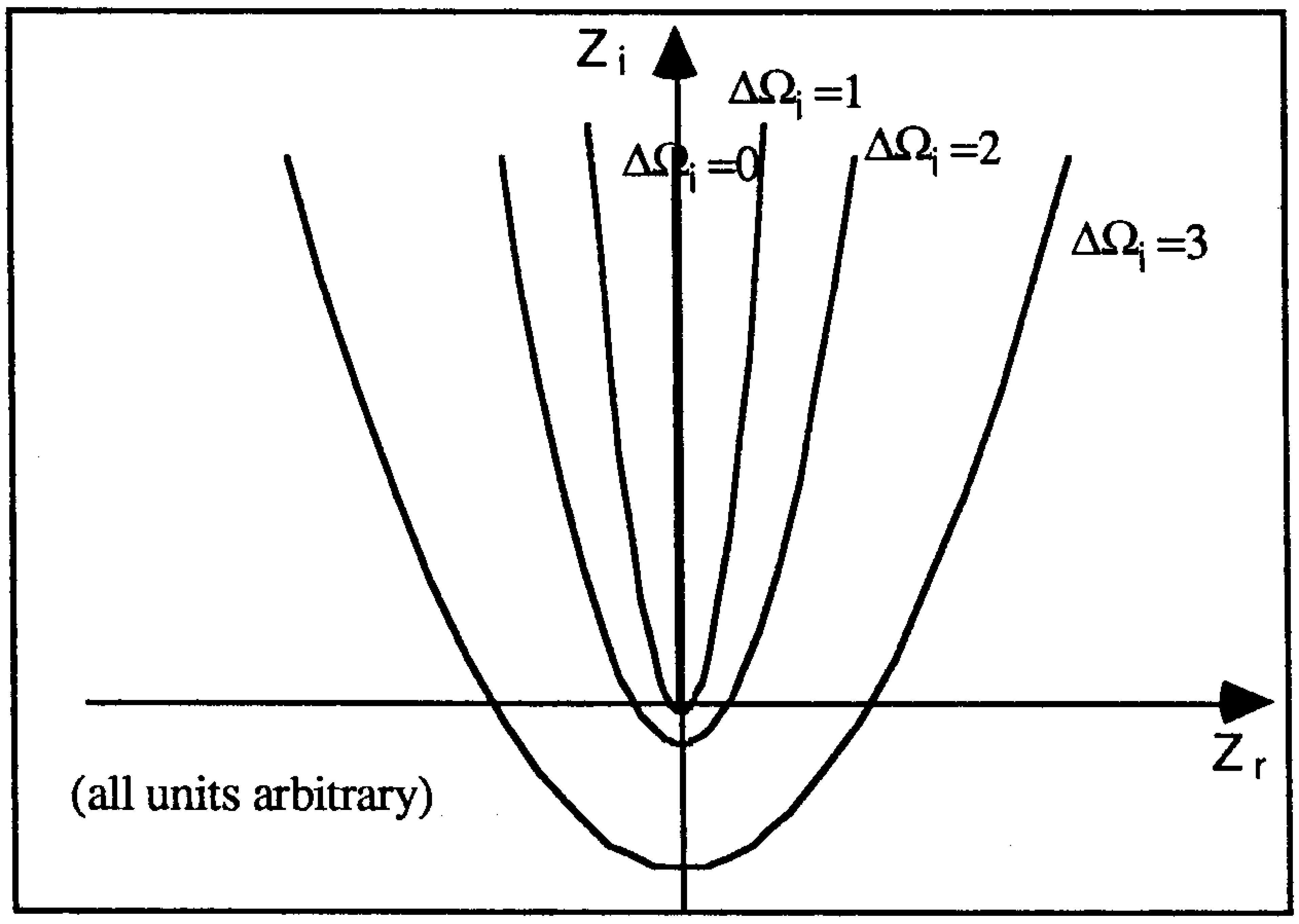

FIG. 1 – Stability diagram relating growth rate and impedance.

Qualitatively the instability can be explained as follows. Let us consider for instance the case of capacitive impedance. The electromagnetic force acting on the beam is proportional to $-\frac{\partial \lambda}{\partial x}$. A small perturbation in the form of a wave as shown in Figure 2, will produce a positive force acting on the front slope of the wave crest and will increase the energy of the particles. On the other side of the wave the force decreases the energy. If we assume to be above transition energy, an increase of the energy implies a decrease of the revolution frequency according to equation (1). Therefore the particles in the front crest will slow down and the particles in the back crest will speed up. The net result is an increase of the height of the crest. The original perturbation is thus increased leading to instability, known as negative mass instability. The analogous reasoning can be used to explain the case below transition energy with an inductive impedance for which the force is proportional to $\frac{\partial \lambda}{\partial x}$.

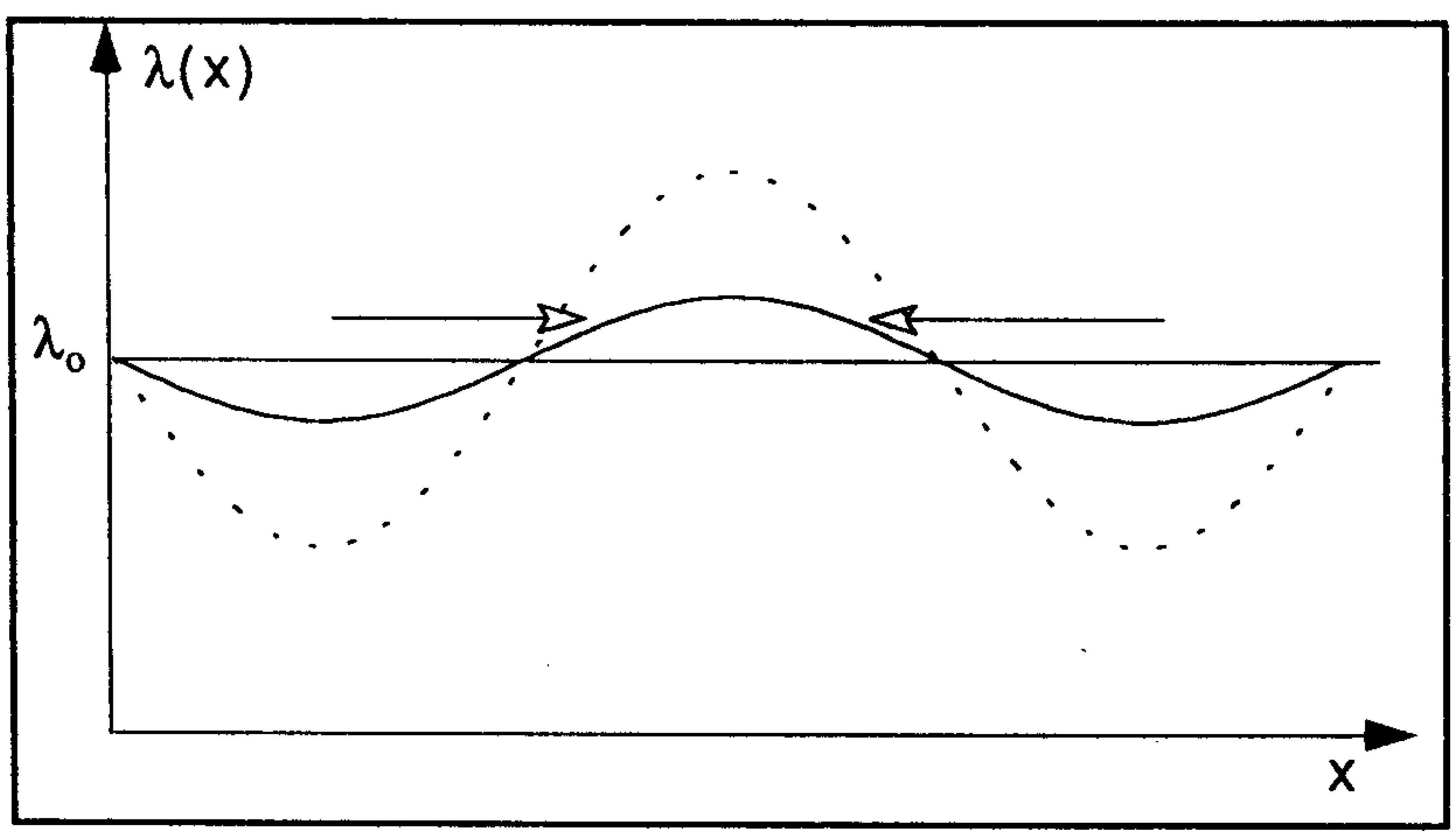

FIG. 2 – Instability in a beam without energy spread.

2.2 – Beam with energy spread

The beam has in reality an energy spread of finite width. Let us consider for example the case of a parabolic stationary distribution given by[3]

$$g_o(\rho) = \frac{3}{8\pi R_o \rho_m}\left[1-\left(\frac{\rho}{\rho_m}\right)^2\right] \tag{19}$$

Substituting equation (19) into equation (16), we obtain

$$1 = -i\frac{3ZI_o\beta c}{4\pi R_o(E_o/e)\rho_m^3}\int_{-\rho_m}^{\rho_m}\frac{\rho d\rho}{\Delta\Omega + n\omega_o\frac{\eta}{\beta^2}\rho} \tag{20}$$

If the frequency Ω of the perturbation lies within the frequency spread, namely $\frac{\beta^2\Delta\Omega}{n\omega_o\eta}$ is within the energy distribution, the dispertion integral has a singularity. In this case the integral is split in two parts, the principal value and the residual term, and the solution is[4]

$$1 = -i\frac{3ZI_o\beta^3 c}{4\pi R_o(E_o/e)\rho_m^2 n\omega_o\eta}\left[2 - y\ln\left(\frac{1+y}{1-y}\right) - i\pi y\right] \tag{21}$$

with

$$y = \frac{\beta^2\Delta\Omega}{n\omega_o\eta\rho_m} \tag{22}$$

The equation (21) relates the frequency Ω of the perturbation to the impedance of the surroundings. The stability condition requires that $\Delta\Omega_i$=0. In Figure 3 we show the real and imaginary impedance versus Ω at different values of $\Delta\Omega_i$. The shaded area is the stable one.

If the impedance Z is inside the stable area, then the beam coherent oscillation energy is transferred to the incoherent kinetic energy of the particles inside the beam, thus stabilising the perturbation. The damping of the oscillations is known as Landau damping effect[4].

The actual shape of the stability limit depends on the distribution edges. Sharp edge distributions, as the parabolic one, are less stable than the ones with long tail, such as the Gaussian distribution. When one neglects the effects of the edges, the stable area can be approximated by a circle obtaining the well–known stability criterion[5,6]

$$\left|\frac{Z}{n}\right| \le F\frac{(E_o/e)|\eta|\rho^2}{I_o\beta^2} \tag{23}$$

where the form factor F, of the order of unity, is determined by the radius of the approximating circle.

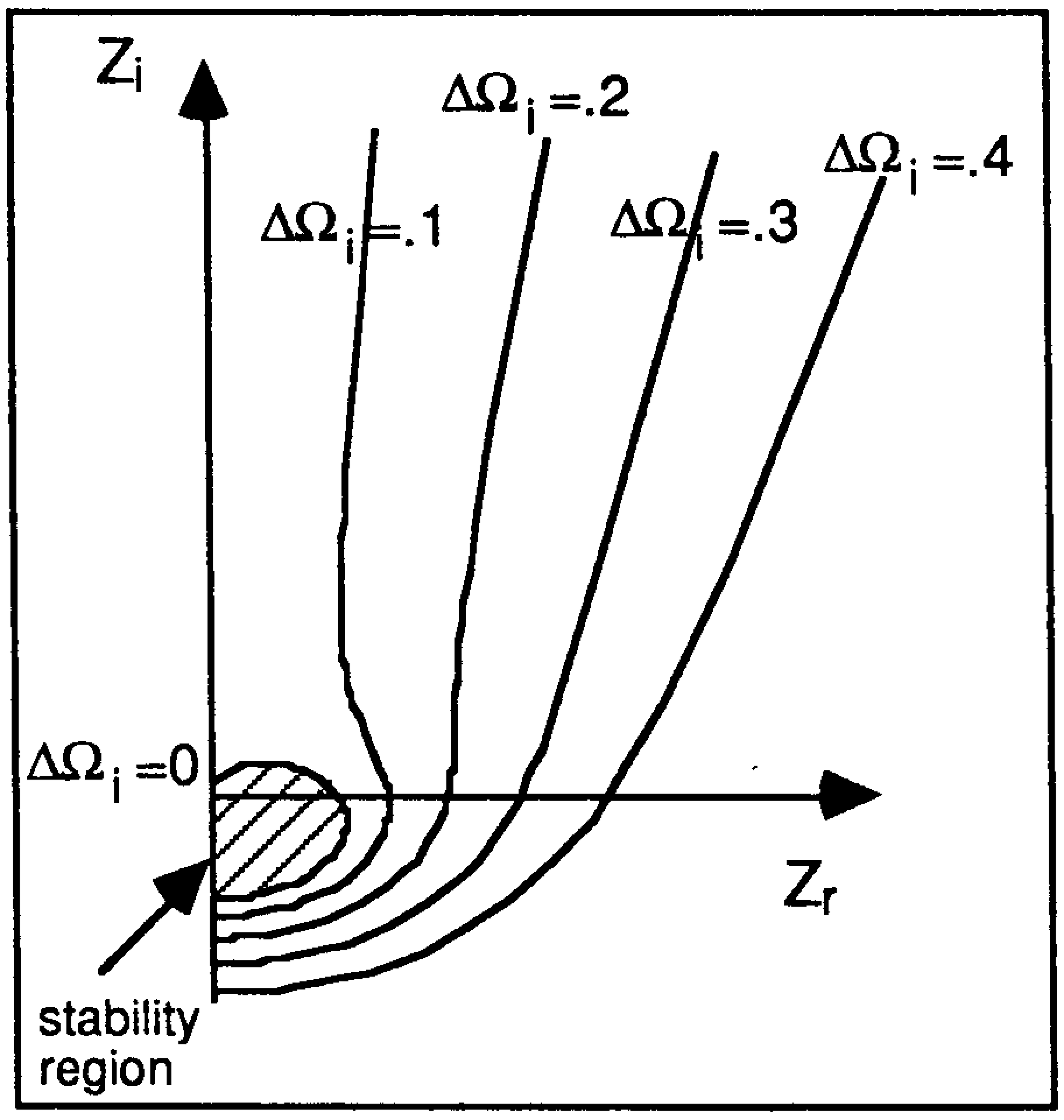

FIG. 3 – Stability diagram for a parabolic distribution.

3. – THE THERMAL WAVE MODEL APPROACH

The Thermal Wave Model[7] describes the beam dynamics by means of a complex wave function Ψ(x;s), the so–called beam wave function, satisfying a sort of Schrödinger equation. The square modulus of Ψ(x;s) gives the longitudinal density profile of the beam. We relate this beam wave function to the charge line density by

$$\lambda(x;s) = \frac{N}{2\pi R_o}|\Psi(x;s)|^2 \tag{24}$$

We start from the single particle equation of motion (15) with Δω given by (1) from which we get the dimensionless Hamiltonian

$$H(x,\rho;s) = -\frac{1}{2}\frac{\eta}{\beta^2}\rho^2 + \frac{\int_o^x U(x';s)dx'}{(E_o/e)\beta c T_o} \tag{25}$$

In order to find the Schrödinger equation, in complete analogy with quantum mechanics, we use the following correspondence rules

$$\rho \to \hat{\rho} \equiv -i\varepsilon\frac{\partial}{\partial x} \qquad H \to \hat{H} \equiv i\varepsilon\frac{\partial}{\partial s} \tag{26}$$

where ε is the longitudinal beam emittance.

Substituting equations (26) into equation (25) we obtain the Schrödinger–like equation for the beam wave function

$$i\varepsilon\frac{\partial\Psi(x;s)}{\partial s} = \frac{1}{2}\frac{\eta}{\beta^2}\varepsilon^2\frac{\partial^2\Psi(x;s)}{\partial x^2} + \frac{\Psi(x;s)}{(E_o/e)\beta c T_o}\int_o^x U(x';s)dx' \tag{27}$$

For a purely reactive impedance ($Z = i\,Z_i$), the energy lost by a particle in one turn is

$$U(x;s) = e\beta c R_o \frac{Z_i}{n}\frac{\partial \lambda(x;s)}{\partial x} \tag{28}$$

Writing U(x;s) as a function of Ψ(x;s) with equation (24), we finally get the cubic nonlinear Schrödinger equation

$$i\frac{\partial \Psi(x;s)}{\partial s} - \frac{1}{2}\varepsilon\frac{\eta}{\beta^2}\frac{\partial^2 \Psi(x;s)}{\partial x^2} - \frac{I_o}{\varepsilon(E_o/e)2\pi}\frac{Z_i}{n}|\Psi(x;s)|^2\Psi(x;s) = 0 \tag{29}$$

From the general theory of cubic nonlinear Schrödinger equation, it is possible to demonstrate that a small perturbation of Ψ(x;s) is stable when (Lighthill criterion)[8]

$$\left(-\frac{1}{2}\varepsilon\frac{\eta}{\beta^2}\right)\left(-\frac{I_o}{\varepsilon(E_o/e)2\pi}\frac{Z_i}{n}\right) < 0 \tag{30}$$

or equivalently

$$\eta Z_i < 0 \tag{31}$$

that is the same stability criterion of unbunched monochromatic beam summarized in Table I.

3.1 – Solitary waves

Another important property of the cubic nonlinear Schrödinger equation is that even in the unstable condition $\eta Z_i > 0$, solitary solutions are possible. They are found by looking for solutions depending on the moving coordinate $(x-\beta_o s)$ of the relativistic ($\beta_o \approx 1$) envelope form

$$\Psi(x;s) = G(x-\beta_o s)e^{i(k_o x - w_o s)} \tag{32}$$

with k_o and w_o real numbers.

Solitary waves preserve they shape during propagation through a dispersive medium, when nonlinear self–modulation occurs. The beam line density, under this condition, is[7]

$$\lambda(x;s) = \frac{N I_o R_o}{4\varepsilon^2(E_o/e)\eta}\frac{Z_i}{n}\mathrm{sech}^2\left[\frac{I_o R_o}{2\varepsilon^2(E_o/e)\eta}\frac{Z_i}{n}(x-\beta_o s)\right] \tag{33}$$

where

$$k_o = \frac{\beta_o}{\varepsilon\eta} \qquad w_o = \frac{1}{2}\varepsilon\eta k_o^2 - \frac{1}{2\varepsilon\eta}\left(\frac{I_o R_o}{2\varepsilon(E_o/e)}\frac{Z_i}{n}\right)^2 \tag{34}$$

So far there is no experimental proof of the existence of solitary waves. However, some "anomalous" behaviours of the beam dynamics have not been clearly explained. We mention, for instance, the early observation of the beam dynamics on the Cosmotron[9], where stable triangular distributions of the beam density develop in the unstable RF phase.

4. - CONCLUSIONS

The Thermal Wave Model applied to unbunched particle beams leads to a cubic nonlinear Schrödinger equation similar to the one governing the propagation of an electromagnetic pulse through a nonlinear medium (paraxial approximation). From the general properties of the nonlinear Schrödinger equation it is possible to derive the stability condition for monochromatic beams that reproduces the classical criterion. Furthermore, it indicates the possible existence of solitons in the unstable regions.

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References

1. A. Hofmann, "Single–Beam Collective Phenomena – Longitudinal", CERN 77–13, pp. 139–174, November 1976.
2. L. Palumbo, V. G. Vaccaro, "Wake Fields, Impedances and Green's Function", CERN 87–03, pp. 341–369, April 1987.
3. J. L. Laclare, "Coasting Beam Longitudinal Coherent Instabilities", CERN 94–01, pp. 349–384, January 1994.
4. H. G. Hereward, CERN Report 65–20, May 1965.
5. A. G. Ruggiero, V. G. Vaccaro, CERN Report ISR–TH/70–44, 1970.
6. E. Keil, W. Schnell, CERN Report ISR–TH–RF/69–48, 1969.
7. R. Fedele, et al., Physics Letters A 179 (1993), pp. 407–413.
8. G. B. Whitham, "Linear and nonlinear waves", New York, J. Wiley, 1974, pp. 601–603.
9. M. Q. Barton, C. E. Nielsen, "Longitudinal Instability and Cluster Formation in the Cosmotron", International Conference on High Energy Accelerators, 1961, p. 170.