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Composite Operators as Integration Variables in Berezin Integrals

G. De Franceschi
Dipartimento di Fisica dell'Università "La Sapienza", I–00185 Rome, Italy
INFN – Laboratori Nazionali di Frascati, P.O. Box 13, I–00044 Frascati (Rome) Italy

F. Palumbo[1]
INFN – Laboratori Nazionali di Frascati, P.O. Box 13, I–00044 Frascati (Rome) Italy

Abstract We define nonlinear changes of variables in Berezin integrals, assuming as new integration variables multilinear functions of the defining elements of the Grassmann algebra. We apply such a change of variables to QCD by introducing as integration variables trilinear and bilinear functions of the quark field, with the quantum numbers of the nucleon and the meson respectively, and we suggest a perturbative scheme using these functions as free states.

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1-Recently nonlinear changes of variables in Berezin integrals have been considered, in order to introduce multilinear functions of the defining elements of the Grassmann algebra as integration variables [1]. Restricting these functions to homogeneous polynomials we can distinguish them according to the parity of their degree. When this is odd we again have odd elements of the Grassmann algebra, and in this case the Berezin integral under the change of variables transforms into the Berezin integral over the new variables. When the order is even, the new variables are even elements of the Grassmann algebra, which are nilpotent commuting variables. In this case the Berezin integral transforms in a different type of integral [1]. One of the purposes of this paper is to define it in full generality.

In the preliminary investigation of such changes of variables the following program was outlined

i) to define an integral over even elements of a Grassmann algebra, in such a way that it is consistent with the interpretation of these variables as composites of fermionic constituents. This allows us to construct for phenomenological purposes a field theory in terms of even elements of a Grassmann algebra, namely nilpotent bosonic fields, disregarding the way they are related to the constituents and exploiting the specific features related to the nilpotency. Due to this property, for instance, a ϕ^4 theory exists also with attractive coupling and it is asymptotically free [1].

ii) to treat bosonization or bound states by performing the change of variables in the Berezin integral which defines the partition function of a fermionic system. This application is relevant to field theory as well as to the theory of many-body systems. In this paper we will work out it in the context of QCD [2]. The hope is to solve the conceptual difficulty of perturbation theory whith free quark states, introducing instead as free states the physical ones.

To actually do some perturbative expansion, the action should contain quadratic terms in the composites, and we should know the corresponding propagators. Quadratic terms can be introduced either as irrelevant terms, as we will suggest later, or by appropriate manipulations. One example of the latter option is reported, although in an exceedingly schematic model, in the second reference [1]. As far as the propagators are concerned,

their evaluation does not present any difficulty for trilinear composites, but for bilinear composites the problem is not yet completely solved. The propagator of a nilpotent commuting scalar field has in fact been evaluated only in the case in which the index of nilpotency (see below) is 1. We regard as a promising feature of the formalism the fact that free propagators of composites can be exactely evaluated.

The present paper is devoted to some of the issues of the above program. We will discuss the change of integration variables from the defining elements of the Grassmann algebra to trilinear and bilinear functions of them. Then we will apply such change of variables to QCD introducing composites with the quantum numbers of the nucleon or the meson. Finally we will sketch how to set up a perturbative expansion for hadrons.

We will assume the action defined in Euclidean space and regularized on a lattice.

2-Let us consider a typical hadronic correlation function

$$< \psi^*_{I_1}(x_1)...\phi^*_{J_1}(y_1)... >= \frac{1}{Z_0}\int [d\lambda^* d\lambda][dU]\psi^*_{I_1}(x_1)...\phi^*_{J_1}(y_1)...e^{-S_\lambda}. \tag{1}$$

In the above equation

$$Z_0 = \int [d\lambda^* d\lambda][dU]e^{-S_\lambda}, \tag{2}$$

S_λ is the euclidean action of the quark field λ coupled to the gauge field, $[dU]$ is the Haar measure over the gauge group and ψ and ϕ are trilinear and bilinear composites with hadronic quantum numbers

$$\psi_I(x) = \sum_{i_1,i_2,i_3} F^I_{i_1,i_2,i_3}\lambda_{i_1}(x)\lambda_{i_2}(x)\lambda_{i_3}(x), \tag{3}$$

$$\phi_I(x) = \sum_{i_1,i_2} B^I_{i_1,i_2}\lambda_{i_1}(x)\lambda^*_{i_2}(x). \tag{4}$$

A list of the appropriate matrices F and B can be found in [3], and we will give explicit examples later on. The above equations define nonlinear changes of variables. Of course they cannot be inverted, and therefore S_λ cannot in general be expressed in terms of

the $\psi, \psi^*, \phi, \phi^*$. We will specify below in which sense such equations can be considered a change of variables. Notice that they involve only fields at the same site, whose indication will therefore be omitted in the sequel. Analogous transformations can be performed when the integration variables in the Berezin integral are the Fourier transforms of the λ field.

We want to define the integral of a function of the $\psi, \psi^*, \phi, \phi^*$ in such a way that its value be equal to that obtained by expressing these variables in terms of the λ, λ^*, and performing the Berezin integral over the latters. Now there is only one nonvanishing Berezin integral

$$\int d\lambda_1^* d\lambda_1 ... d\lambda_N^* d\lambda_N \lambda_1 \lambda_1^* ... \lambda_N \lambda_N^* = 1, \tag{5}$$

N being the number of degrees of freedom of the λ-field. It is economic to introduce the notation

$$\Lambda = \lambda_1 ... \lambda_N, \quad \Lambda^* = \lambda_N^* ... \lambda_1^*. \tag{6}$$

We must first determine all the functions of the $\psi, \psi^*, \phi, \phi^*$ which, when expressed in terms of the λ, λ^*, are proportional to $\Lambda\Lambda^*$ (with nonzero coefficient). We call them relevant. Only when relevant functions can actually be constructed by means of the given composites, can the latter be introduced as new variables of integration.

The most general function of nilpotent variables is a polynomial. It is therefore sufficient to determine all the relevant monomials, which are the monomials of maximum degree. It is easy to see that we can construct only one relevant monomial in terms of trilinear variables

$$\Psi\Psi^* = F^* F \Lambda\Lambda^*, \quad \Psi = \prod_{I=1}^{3} \psi_I = F\Lambda, \tag{7}$$

where F is a numerical factor which we call the weight of Ψ.

In the case of bilinear composites the situation is different. In order to properly explain this difference we must introduce the notion of order of nilpotency. This is the smallest

integer n^* such that

$$\phi^n = o, \quad for \ \ n > n^*. \tag{8}$$

Now trilinear composites necessarily have index of nilpotency 1 ($(\lambda_1\lambda_2\lambda_3 + \lambda_4\lambda_5\lambda_6)^2 = 0$), while bilinear composites can have different index of nilpotency ($(\lambda_1\lambda_2 + \lambda_3\lambda_4)^2 \neq 0$). This is why we can construct only one relevant monomial in terms of trilinear composites, while we construct many in terms of bilinear

$$\Phi_m = \prod_I \phi_I^{m_I} = b_m \Lambda\Lambda^*. \tag{9}$$

In the above equation the index m is a vector with components m_I ($\sum_I m_I = N$) and $b_m \neq 0$ is the weight of Φ_m. We can now define the integral over the new variables.

Let us start by the bilinear composites. Their most general function can be written

$$f(\phi, \phi^*) = \sum_m f_m \Phi_m + irrelevant\ terms. \tag{10}$$

If we think of f as expressed, via the definition of the ϕ, ϕ^*, in terms of the λ, λ^*, its Berezin integral is

$$\int [d\lambda^* d\lambda] f(\phi(\lambda, \lambda^*), \phi^*(\lambda, \lambda^*)) = \sum_m f_m b_m. \tag{11}$$

This result can be directly obtained if we integrate f over the new variables ϕ, ϕ^* provided we give as rule of integration

$$\int [d\phi d\phi^*] \Phi_m = b_m, \tag{12}$$

for all the relevant monomials, all other integrals being zero. Note that, although in general different expansions

$$f(\phi, \phi^*) = \sum_m f_m \Phi_m + irr.\ terms = \sum_m f'_m \Phi_m + irr.\ terms \tag{13}$$

may exist, the above equality implies

$$\sum_m f_m b_m = \sum_m f'_m b_m, \tag{14}$$

so that the value of the integral does not depend on the particular expression for f.

In the case of trilinear composites the same criterion leads to the definition

$$\int [d\psi^* d\psi] \Psi \Psi^* = F^* F, \tag{15}$$

all other integrals vanishing.

It should now be clear in which sense we can talk about a change of variables. Even though the λ, λ^* cannot be expressed in terms of the $\psi, \psi^*, \phi, \phi^*$, any nonvanishing Berezin integral must contain the product $\Lambda\Lambda^*$, and we can always replace this product by relevant functions of the composite variables.

3-We now apply the formalism developed so far to introduce trilinear or bilinear functions of the quark field whith the quantum numbers of the nucleon and the meson resp. as integration variables in the partition function of QCD.

Let us start by trilinear functions whith the quantum numbers of the nucleon. The quark field $\lambda^a_{\alpha,f}$ has 3 quantum numbers corresponding to colour (a=1,...3), spinor index (α=1,...4) and flavour (f). We will confine ourselves to two flavours. The four components of λ for given colour and flavour will be denoted according to

$$\lambda^a_{s,1} = u^a_s, \quad \lambda^a_{s+2,1} = (\bar{u}^a_s)^*, \quad \lambda^a_{s,2} = d^a_s, \quad \lambda^a_{s+2,2} = (\bar{d}^a_s)^*, \quad s = 1, 2. \tag{16}$$

A trilinear function whith the quantum numbers of the proton [3] is

$$\psi_{p,s} = \epsilon_{abc}(\sigma_2\sigma_h)_{ij}(\sigma_h)_{sk} u^a_i u^b_j d^c_k, \quad s = 1, 2, \tag{17}$$

with the convention of summation over repeated indices. The neutron function $\psi_{n,s}$ is obtained by the replacement $u \leftrightarrow d$. We want to show that these trilinear functions can

be assumed as integration variables. Analogous proof can be given for the change from the $\bar{u}, \bar{d}$ to the antiproton, antineutron functions.

We must show that Eq.(7) holds with

$$\Psi = \psi_{p,1}\psi_{p,2}\psi_{n,1}\psi_{n,2}, \quad \Lambda = P(u_1)P(u_2)P(d_1)P(d_2), \quad P(u) = u^1u^2u^3, \quad F \neq 0. \tag{18}$$

Inserting in Eq.(17) the values of the matrix elements of the Pauli matrices one obtains

$$\psi_{p,s} = -2i[(u_s u_1 d_2) - (u_s u_2 d_1)]. \tag{19}$$

In the above equation

$$(xyz) = \epsilon_{abc}x^a y^b z^c, \tag{20}$$

x, y, z being three-component vectors. Note that for vectors x, y, z with anticommuting components (xyz) is completely symmetric.

We first evaluate

$$\psi_{p,1}\psi_{p,2} = -4[(u_1u_1d_2) - (u_1u_2d_1)][(u_1u_2d_2) - (u_2u_2d_1)], \tag{21}$$

noting that, if u, x, z are 3-vectors with anticommuting components, we have

$$\begin{aligned} (uux)(uyz) &= -\epsilon_{abc}\epsilon_{def}u^a u^b u^d x^c y^e z^f = -\epsilon_{abc}\epsilon_{abd}\epsilon_{def}P(u)x^c y^e z^f = -2P(u)(xyz) \\ (xxx) &= 6P(x). \end{aligned} \tag{22}$$

Here we use $u^a u^b u^c = \epsilon_{abc}P(u)$. Hence

$$\psi_{p,1}\psi_{p,2} = -4[-2P(u_1)(u_2d_2d_2) - (u_1u_1d_2)(u_2u_2d_1) - (u_1u_2d_1)(u_1u_2d_2) + 2P(u_2)(u_1d_1d_1)]. \tag{23}$$

By the exchange $u \leftrightarrow d$ we obtain

$$\psi_{n,1}\psi_{n,2} = -4[-2P(d_1)(u_2u_2d_2) - (u_2d_1d_1)(u_1d_2d_2) - (u_1d_1d_2)(u_2d_1d_2) + 2P(d_2)(u_1u_1d_1)]. \tag{24}$$

We now evaluate Ψ disregarding all the terms which are more than cubic in anyone of the vectors u_s, d_s, since they are necessarily zero. The resulting expression for Ψ is a sum of 6 terms, 5 of which can be immediately shown to be proportional to Λ using only the identities (22). Ψ has therefore the expression

$$\Psi = 16\{7 \cdot 48\ \Lambda + (u_1u_2d_1)(u_1u_2d_2)(u_1d_1d_2)(u_2d_1d_2)\}. \tag{25}$$

Now

$$\begin{aligned}(ux_1y_1)(ux_2y_2)(ux_3y_3) &= \epsilon_{abc}\epsilon_{a_1b_1c_1}\epsilon_{a_2b_2c}u^a u^{a_1}u^{a_2}x_1^b y_1^c x_2^{b_1}y_2^{c_1}x_3^{b_2}y_3^{c_2} \\ &= \epsilon_{abc}\epsilon_{a_1b_1c_1}\epsilon_{a_2b_2c_2}\epsilon_{aa_1a_2}P(u)x_1^b y_1^c x_2^{b_1}y_2^{c_1}x_3^{b_2}y_3^{c_2} \\ &= P(u)[(x_1x_2y_2)(y_1x_3y_3) + (y_1x_2y_2)(x_1x_3y_3)]. \end{aligned} \tag{26}$$

(We have used the well known formula $\sum_a \epsilon_{abc}\epsilon_{aa_1a_2} = \delta_{ba_1}\delta_{ca_2} - \delta_{ba_2}\delta_{ca_1}$). In the same way we reduce also the last term in Ψ to finally get

$$\Psi = 2^7 \cdot 3^2 \cdot 5\ \Lambda. \tag{27}$$

Finally we report without derivation the set of all the relevant monomials constructed in terms of bilinear variables with the quantum numbers of the pion. The pionic bilinear variables are [3]

$$\pi_+ = d^*\bar{u}^* - \bar{d}u, \quad \pi_- = u^*\bar{d}^* - \bar{u}d, \quad \pi_0 = u^*\bar{u}^* - d^*\bar{d}^* - (\bar{u}u - \bar{d}d). \tag{28}$$

The exponents of their relevant monomials

$$\Phi_m = (\pi_+)^{m_+}(\pi_-)^{m_-}(\pi_0)^{m_0}, \tag{29}$$

must be restricted according to Eq.(9). In the present case the quark-antiquark variables are involved at the same time, so that

$$\Lambda = P(u_1)P(u_2)P(\bar{u}_1)P(\bar{u}_2)P(d_1)P(d_2)P(\bar{d}_1)P(\bar{d}_2). \tag{30}$$

Inserting Eqs.(28)-(30) in Eq.(9) we find, after a lengthy but straightforward calculation, that Φ_m depends on only one of the components of the vector index m. Choosing this component to be m_+ we have

$$m_- = m_+, \quad m_0 = 24 - 2m_+, \quad 0 \leq m_+ \leq 12, \tag{31}$$

so that there are altogether 13 relevant monomials. The values of the corresponding weights are

$$b_{m_+} = (-1)^{m_+^2 - m_+}(m_+!)^2(24-2m_+)!\sum_{r=0}^{m_+}\theta(6-r)\theta(r-m_++6)\frac{6!}{(6-r)!r!}\frac{6!}{(6-m_++r)!(m_+-r)!} \tag{32}$$

where

$$\theta(r) = 1 \ \ for \ \ r \geq 0, \quad \theta(r) = 0 \ \ otherwise. \tag{33}$$

From Eqs. (28), (29) and (31) it follows that the index of nilpotency of the pion fields is

$$n^*_{\pi_+} = n^*_{\pi_-} = 12, \quad n^*_{\pi_0} = 24. \tag{34}$$

4-We end the paper by suggesting a way to use our change of variables to set up a perturbative scheme in QCD. Let us confine ourselves to nucleons and mesons.

First of all we see that we cannot introduce all the corresponding composites at the same site because they are redundant. One way to avoid this redundancy is to introduce at some sites nucleons and at other sites mesons. This can be done in several ways. For the present illustrative purposes we choose to put them at staggered lattice sites, locating the nucleons at the sites x and the mesons at the sites y.

To do perturbation theory the action must contain quadratic terms in the hadronic variables, and we must know the corresponding propagators. In simple cases the quadratic terms can be introduced by appropriate manipulations [1], but due to the high index of nilpotency of the pion fields of Eq. (28) this way is certainly impractical in QCD.

In general the quadratic terms can be introduced by hands as irrelevant terms (from the point of view of the renormalization group, not of the integration!). We then consider the action

$$S = S_0 + S_\lambda, \tag{35}$$

where

$$\begin{aligned} S_0 \;=\; & \alpha_N a^4 \sum_x \sum_\mu a^5 \bar{N}(x) \frac{1}{4}[(\gamma_\mu - 1)N(x+2\mu) - (\gamma_\mu + 1)N(x-2\mu) + 4(\tilde{M}_N + 4)N(x)] \\ & + \alpha_M a^4 \sum_y \sum_\mu \sum_{\iota=+,-,0} a^2 \pi_\iota^*(y) \frac{1}{4}[\pi_\iota(y+2\mu) + \pi_\iota(y-2\mu) - 2\pi_\iota(y) - 4\tilde{M}_\pi^2]. \end{aligned} \tag{36}$$

In the above equation N is the nucleon field, a four spinor, isospin doublet (the sum over the corresponding quantum number being understood), π is the meson field, α_N and α_π are dimensionless couplings, M_N the nucleon mass, $\tilde{M}_\pi$ a parameter related to the pion mass (see ref.1), μ a vector with components $\mu_\nu = \delta_{\mu\nu}$ and a the lattice spacing. For the nucleon field we have assumed the Wilson action. In the formal continuum limit S_0 disappears, but at finite lattice spacing it provides the gaussian term necessary for a perturbative expansion. Of course one should add to the action all the terms of dimension

not greater than the dimension of S_0, and among them there will also be couplings of nucleons to mesons.

We therefore replace the partition function

$$Z_\lambda = \int [d\lambda^* d\lambda][dU] e^{-S_\lambda} \tag{37}$$

by

$$Z'_\lambda = \int [d\lambda^* d\lambda][dU] e^{-(S_0+S_\lambda)} = \int [d\psi^* d\psi][d\phi^* d\phi][dU] e^{-(S_0+S_\lambda)}, \tag{38}$$

where in the last step we have changed variables of integration.

The whole S_λ must now be treated as a perturbation. The evaluation of each term in the perturbative expansion requires the rearrangement of the product of the λ, λ^* fields appearing in S_λ into relevant functions of the ψ, ψ^* at the sites x and of the ϕ, ϕ^* at the sites y.

In the above example the quark field has been altogether eliminated as integration variable. But in some calculations a different change of variables might turn out to be more convenient. Let us for instance consider the coupling of the nucleon to the Maxwell field only, without mesons. In this case it might be advantageous to leave the quark variables at the sites y. The free part of the action now contains also the free action of the quark field, while the interaction part will contain new terms connecting the sites x and y, describing the fragmentation of the nucleon into quarks. A similar situation occurs if we are studying the positronium, in which case we must obviously keep the electron field.

Let us now come to the free propagators. In the case of trilinear variables the propagator can obviously be evaluated as for the defining elements. In the case of bilinear variables instead the propagator has been evaluated only for index of nilpotency 1. We must distinguish whether $\phi(x)^2 = 0$ (nilpotency in configuration space) or the square of the Fourier transform of ϕ vanishes (nilpotency in momentum space). In the first case the propagator can be related to that of the selfavoiding random walk, which is not known

analitically, while in the second one it is equal to that of a canonical scalar [1]. For this reason in actual calculations one should use as integration variables the Fourier transforms of the composites.

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