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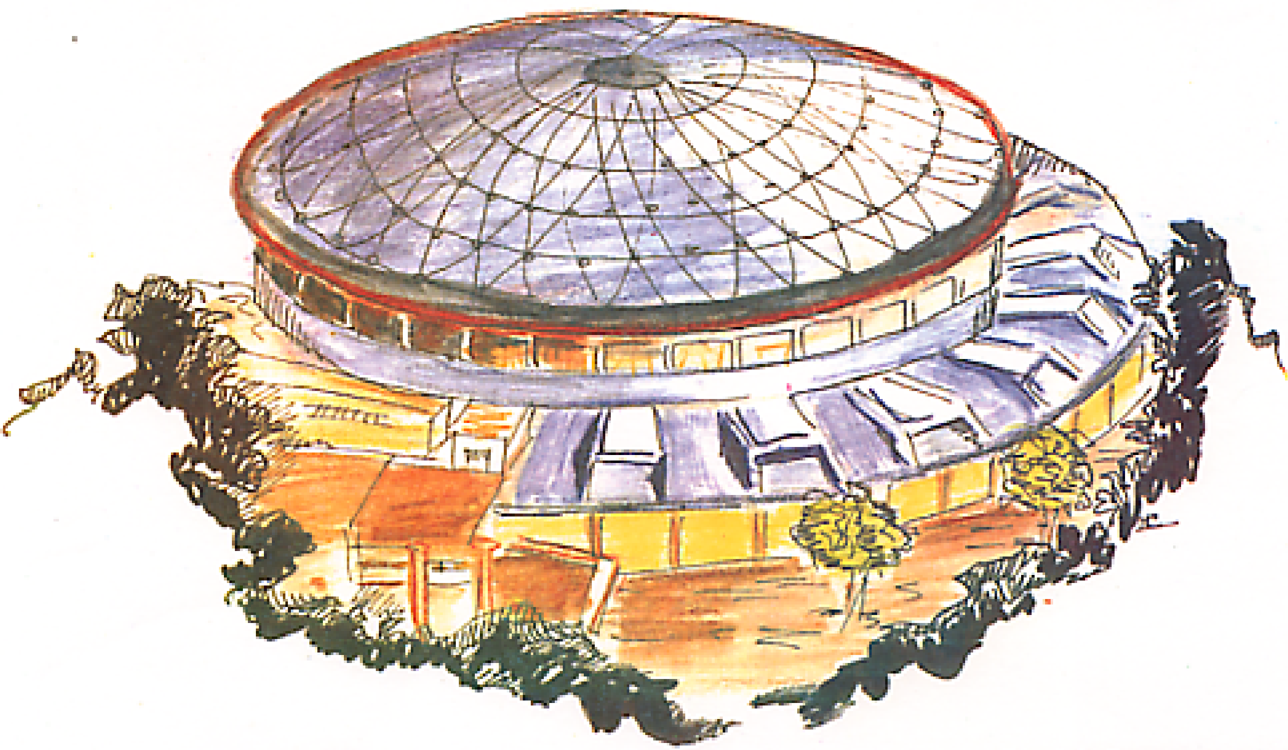

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ON THE CONVERGENCE OF THE INVERSE DIFFRACTION TRANSFORM KERNEL USING CESÀRO SUMMABILITY

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ABSTRACT

In diffraction tomography, optical information processing and, more generally, Fourier optics, the diffraction transform solves both the direct and the inverse boundary value propagation problem for the Helmholtz equation. Its kernel is itself an integral. It is the representation of the evolution operator associated with translations of a constrained Cartesian coordinate. This fact threatens the inverse scattering problem with a divergence if the transform kernel is understood as a Cauchy integral. The kernel is, however, everywhere convergent if its integral representation is interpreted as a Cesàro summable integral.

1. – INTRODUCTION

In classical physics (Helmholtz equation), it is usual to approximate diffraction scattering as a boundary value problem associated with a linear wave equation [1]. According to this picture, a diffractive object (or target) is regarded as a passive geometrical obstacle in the path of the incident wave (or projectile). The incident wave field is specified as a boundary condition over the target profile. Because of the linearity of the wave equation, the scattered field is obtained as an integral transform of the incident field. This is the so–called direct scattering problem. The inverse problem consists in the recovery of the incident field, given the scattered field configuration. This solution provides information about the target profile geometry [2–5]. There are important applications (e.g. diffraction tomography [6], optical information processing and, more generally, Fourier optics [7]) which make extensive use of classical diffraction theory. This theory is, however, saddled with a long–standing, apparently unyielding, divergence problem [8, 9]. The divergence occurs only in a context where all integrals in the theory, independently of the formal manipulations which lead to them are naively assumed to be Cauchy summable. This, however, is not always the case. In particular, it is not true of the inverse integral transform which solves for the incident field in terms of the scattered field in classical diffraction theory [8, 9]. It is the Cauchy divergence of the kernel of this integral transform which constitutes the unsolved problem of this theory. We shall show that if this and other integrals in the theory are interpreted as Cesàro integrals [10, 11, 12, 13], then the divergence does not exist.

2. – DIFFRACTION TRANSFORMS

Consider the Helmholtz equation for a scalar field $\varphi(\vec{x})$

$$\left(\nabla^2 + k^2\right) \varphi(\vec{x}) = 0 \tag{1}$$

where

$$\nabla^2 \equiv \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^{\,2}} \tag{2}$$

is the Laplace operator in 3-space with coordinate vector $\vec{x}$ and $k^2 \geq 0$ is a positive parameter. In what follows it will be convenient to view the space spanned by the vector $\vec{x}$ in terms of two-dimensional plane slices orthogonal to the third or z-axis. A point on each such slice is described by the position z of the plane along the z-axis and a 2-vector coordinate $\vec{b}$ (the impact parameter) on the plane . Thus, the 3-vector $\vec{x}$ is decomposed into $\vec{x} \equiv (\vec{b}, z)$ and will be so understood throughout the rest of this paper. Eq(1) is to be solved with the boundary condition

$$\varphi(b, z{=}z_0) = \varphi_0(b) \tag{3}$$

The search for this solution in the half–space $z \geq z_0$ constitutes the direct exterior problem while in the half–space $z \leq z_0$ it becomes the direct interior problem. The exterior and interior solutions are related by a reflection principle (i.e. parity) as will emerge later. It is therefore sufficient to concentrate on the direct exterior problem.

In each half–space $z \geq z_0$ or $z \leq z_0$, there are two linearly independent solutions $\varphi_{\pm}(\vec{x})$ of Eq. (1). The one, e.g. $\varphi_{+}(\vec{x})$, is the regular solution in the half–space $z \geq z_0$ in the sense that it tends to zero for $z \to \infty$. We will refer to it as the outward propagating solution. The other $\varphi_{-}(\vec{x})$ is the singular solution. It does not vanish for $z \to \infty$, but rather for $z \to -\infty$. It is the inward propagating solution. $\varphi_{+}(\vec{x})$ and $\varphi_{-}(\vec{x})$ interchange roles in the half–space $z \leq z_0$, i.e. $\varphi_{-}(\vec{x})$ is there regular and outward propagating while $\varphi_{+}(\vec{x})$ is singular and inward propagating. To obtain these solutions, substitute the two–dimensional Fourier transform

$$\varphi(b, z) = \int d^2q\, e^{i\vec{q}\cdot\vec{b}}\, f(q, z) \tag{4}$$

into Eq. (1) to get the equation

$$\left[\frac{d^2}{dz^2} + p^2(q)\right] f(q, z) = 0 \tag{5}$$

where

$$p^2(q) := k^2 - q^2 \tag{6}$$

is not necessarily positive. Define the two real momentum variables

$$p_1(q) := \sqrt{k^2 - q^2} \quad p^2(q) \geq 0 \tag{7.i}$$

$$p_2(q) := \sqrt{q^2 - k^2} \quad p^2(q) < 0 \tag{7.ii}$$

The inverse

$$f(q, z) = \frac{1}{(2\pi)^2} \int d^2b\, e^{-i\vec{q}\cdot\vec{b}}\, \varphi(b, z) \tag{8}$$

of Eq. (4) will also be needed.

Two linearly independent solutions of Eq. (5) are

$$f_\pm(q, z) = \theta\,(k^2 - q^2)\, a_\pm(q)\, e^{\pm i p_1 z} + \theta\,(q^2 - k^2)\, b_\pm(q)\, e^{\mp p_2 z} \tag{9}$$

where the coefficients $a_\pm(q)$ and $b_\pm(q)$ are arbitrary functions of $\vec{q}$ (*) . Rather than fix these coefficients by means of boundary conditions one can instead invert Eq. (9) and solves for them in terms of $f_\pm(q, z)$. Multiplying in turn Eq.(9) by $\theta\,(k^2 - q^2)$ and $\theta\,(q^2 - k^2)$ end taking in account Eq. (8), one obtains

$$\theta(k^2 - q^2)\, a_\pm(q) = \frac{\theta(k^2 - q^2)}{e^{\pm i p_1 z}}\, \frac{1}{(2\pi)^2} \int d^2b\, e^{\mp i\vec{q}\cdot\vec{b}}\, \varphi_\pm(b, z) \tag{10.i}$$

(*) $\theta\,(k^2 - q^2)\, a_\pm(q)\, e^{\pm i p_1 z}$ is referred to as homogeneous wave, whereas $\theta\,(q^2 - k^2)\, b_\pm(q)\, e^{\mp p_2 z}$ is called Inomogeneous (Evanescent) wave.

$$\theta(q^2 - k^2)\, b_\pm(q) = \frac{\theta(q^2-k^2)}{e^{\mp p_2 z}} \frac{1}{(2\pi)^2} \int d^2 b\, e^{\mp i \vec{q} \cdot \vec{b}}\, \varphi_\pm(b, z) \qquad (10.ii)$$

Finally making use of Eqs. (10) in (9) and the latter in (4), one arrives at the homogeneous integral equation

$$\varphi_\pm(b, z) = \int d^2 b'\, G_\pm(b, z \mid b', z')\, \varphi_\pm(b', z') \qquad (11)$$

where

$$G_\pm(b, z \mid b', z') = \frac{1}{(2\pi)^2} \int d^2 q\, e^{\pm i \vec{q}(\vec{b} - \vec{b}')} \bullet$$

$$\bullet \left[\theta(k^2 - q^2)\, e^{\pm i p_1 (z-z')} + \theta(q^2 - k^2)\, e^{\mp p_2 (z-z')} \right] \qquad (12)$$

To arrive at Eq. (11) we have interchanged the order of the $\vec{b}$–integration coming from Eqs.(10) with that of the $\vec{q}$–integration coming from Eq. (4). This interchange is usually (Sherman [8] ; Shewell and Wolf [9]) held responsible for the divergence in $G_\pm(b, z \mid b', z')$.

Now it is immediate to verify that $G_\pm^{-1}(b, z \mid b', z') \equiv G_\pm(b', z' \mid b, z)$ is the inverses of $G_\pm(b, z \mid b', z')$. Since one has

$$\int d^2 b'\, G_\pm(b, z \mid b', z')\, G_\pm(b', z' \mid b'', z) = \delta^{(2)}(\vec{b} - \vec{b}'') \qquad (13)$$

On account of Eq. (13), for any fixed pair of parameters (z, z'), Eq(11) has the inverse,

$$\varphi_\pm(b', z') = \int d^2 b\, G_\pm(b', z' \mid b, z)\, \varphi_\pm(b, z) \qquad (14)$$

The structure of Eqs. (11) and (14) coincide on account of the equality $G_\pm^{-1}(b, z \mid b', z') \equiv G_\pm(b', z' \mid b, z)$. If the integral representation in Eq.(12) is regarded as a Cauchy integral, then the forward (backward) propagator $G_+(b', z' \mid b, z)$ ($G_-(b', z' \mid b, z)$) is convergent (divergent) for $z<z'$. If, on the other hand, this integral representation is interpreted as a Cesàro summable integral, then $G_\pm(b', z' \mid b, z)$ exists and is finite for all finite values of $z-z'$.

3. – CESÀRO SUMMABILITY OF THE DIFFRACTION TRANSFORM KERNEL

Formal manipulations of Cauchy integrals (e.g. interchange of order of integrations) are notoriously suspect because they do not always yield resultant integrals which are summable in the same way as the composite integrals. We make use of translation invariance and rewrite Eq. (12) as

$$G_{\pm}(b\,,z \mid b',z') \equiv G_{\pm}(r\,,\zeta) = \int_0^{\infty} dq\, q\, J_0(qr)\, u_{\pm}(q,\zeta) \tag{15}$$

where $\vec{r} := \vec{b} - \vec{b}'$, $\zeta := z - z'$ and

$$J_0(qr) := \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, e^{iqr\cos(\varphi)} \tag{16}$$

is the Bessel function of order zero. The functions $u_{\pm}(q,\zeta)$ are defined by

$$u_{\pm}(q,\zeta) = u_{\pm}^{(1)}(q,\zeta) + u_{\pm}^{(2)}(q,\zeta) \tag{17}$$

where

$$u_{\pm}^{(1)}(q,\zeta) := \frac{1}{2\pi}\, \theta\,(k^2-q^2)\, e^{\pm ip_1\zeta} \tag{18.i}$$

$$u_{\pm}^{(2)}(q,\zeta) := \frac{1}{2\pi}\, \theta\,(q^2-k^2)\, e^{\mp p_2\zeta} \tag{18.ii}$$

The integrals in Eq. (17) involving $u_{\pm}^{(1)}(q,\zeta)$ are convergent and will, therefore, not be discussed further. We concentrate on the contributions of $u_{\pm}^{(2)}(q,\zeta)$ to these integrals and restrict attention only to $G_{+}(r\,,\zeta)$ since $G_{-}(r\,,\zeta) = G_{+}(r\,,-\zeta)$. To lighten the notation, we represent this contribution as

$$G_2(r,\zeta) := \frac{1}{2\pi} \int_k^{\infty} dq\, q\, J_0(qr)\, e^{-p_2\zeta} = \frac{k^2}{2\pi} \int_0^{\infty} dt\, t\, J_0\big(R\sqrt{1+t^2}\,\big)\, e^{-t\,Y} \tag{19}$$

where $R := kr$ and $Y := k\,\zeta$.

The integral in Eq. (19) is convergent for $\zeta > 0$; for $\zeta = 0$ it becomes

$$G_2(r\,,\zeta) := \frac{1}{(2\pi)}\,\delta\,(r^2) - \frac{1}{(2\pi)} \int_0^{k} dq\, q\, J_0(qr) \tag{20}$$

where

$$\int_0^{\infty} dq\, q\, J_0(qr) = \delta(r^2) \tag{21}$$

is the Dirac delta function. For $\zeta < 0$, the integral in Eq. (19) is Cauchy divergent. We propose to regularize it by means of Cesàro summability. To this end, we associate with Eq. (21) the sequence of partial sums [10–13]

$$C_0(r, \zeta; Q) := \int_0^Q dq \left[qJ_0(qr)\, u_+^{(2)}(q, \zeta) \right] \tag{22.i}$$

$$C_n(r, \zeta; Q) := \int_0^Q dq\, C_{n-1}(r, \zeta; q)\,; \quad n \geq 1. \tag{22.ii}$$

Carrying out the implied iteration in Eq. (22.ii) and making use of (22.i) one finds that

$$C_n\,(r, \zeta; Q) := \frac{1}{n!} \int_0^Q dq\, (Q-q)^n \left[q\, J_0(qr)\, u_+^{(2)}(q, \zeta) \right] \tag{23}$$

Next, we define the n–th Cesàro mean by

$$C_2^{(n)}(r,\zeta;Q) := \frac{n!}{Q^n}\, C_n(r,\zeta;Q) = \int_0^Q dq\, (1-\frac{q}{Q})^n \left[q\, J_0(qr)\, u_+^{(2)}(q, \zeta) \right] \tag{24}$$

The limit

$$\text{Reg } G_2(r,\zeta) := \lim_{Q\to\infty} C_2^{(n)}(r,\zeta;Q) \quad (C, n) \tag{25}$$

when it exists, is said to define Reg $G_2(r, \zeta)$ as a Cesàro summable integral of order n (*). This is the significance of the symbol (C, n) in Eq. (25). By this definition, n=0 corresponds to Cauchy summability, whence the latter is a special case of Cesàro summability. If the limit in Eq.(25) exists for $n=N\geq0$ but not for $n<N$, then it exists for all $n=N+m$, $m \geq 0$. For $\zeta < 0$, this limit does not exist for n=0; in other words, the original integral in Eq. (19) is, as already noticed, not Cauchy summable. We have checked that for all finite $\zeta < 0$, there exists a finite n ≥ 1 for which the limit exists. In other words, the integral in Eq. (19) is Cesàro summable for all finite $\zeta < 0$. We illustrate this convergence graphically in Fig. 1 where Reg $G_2(r\,, \zeta)$ is plotted against ζ for various values of r.

(*) In the mathematical literature, it is usual to write $\mathfrak{G}_2$ (r, z) for Reg G_2 (r, z) in Eq. (25). We deviate from this convenient notation in favour of the usage in quantum field theory in order to emphasize that Reg G_2 (r, z) is different from the Cauchy integral representation for G_2 (r, z).

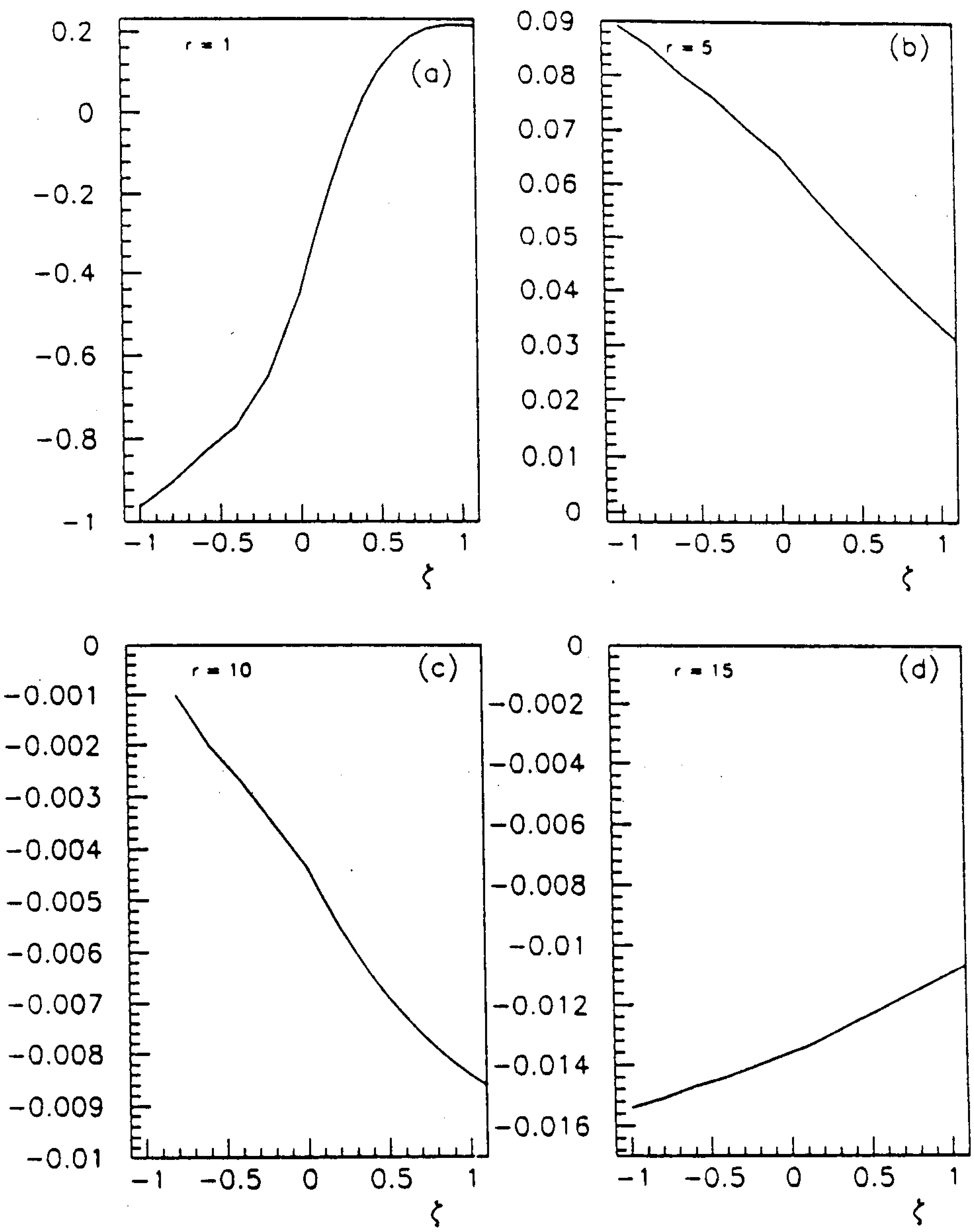

FIG. 1 – Plots of the Cesàro integral Reg $G_2(r, \zeta)$ as a function of ζ ($-\zeta \geq \zeta \geq +1$) for various values of r (r = 1, 5, 10, 15) and the parameter k = 1).

4. - CONCLUSIONS

The problem solved in this paper was first clearly formulated by Sherman [8] and by Shewell and Wolf [9] in optical phisics. The divergence, according to these authors, arises as a result of the interchange of the order of the $\vec{q}$–and $\vec{b}$–integrations, in the passage from Eqs. (4) and (10) to (11), which defines the kernels $G_{\pm}(b\ ,\ z \mid b'\ ,\ z')$. Shewell and Wolf propose to regularize the singularity most simply by means of a cut–off, a so called band–width limitation. The cut–off eliminates the higher frequencies $(q^2 > k^2)$. The arguments in support of this cut–off procedure are not theoretical but rather they make recourse to the behaviour of frequency detectors. The arguments claim that no detector can resolve frequencies that are arbitrarily high. Sherman is much more sophisticated in his regularization programme. He suggests to interpret the divergent integral as a distribution. In any case, the suggestion implies no more than using test functions to operate cut–offs. The work of Sherman and of Shewell and Wolf is widely used in applications. We quote, in this regard, the otherwise interesting paper of Devaney on diffraction tomography [6]. One encounters here too the divergent kernel $G_{+}(b\ ,\ z \mid b'\ ,\ z')$ $(z<z')$. To invert the diffraction transform and recover the object field from a given scattered field configuration, Devaney resorts to the construction of a set of filters upon which are imposed various band–width limitations. The operators corresponding to these filters are then expected to combine and yield a "good approximation" to the unit operator in function space. With this approximation one inverts the diffraction transform and recovers the required object field. The latter is then compared with the experimentally deduced target geometrical profile. Our aim in this paper has been to show that Cesàro summability offers a clean and uncluttered procedure for the inversion of the diffraction transform. The integral representation of the diffraction transform kernel is not everywhere Cauchy summable. It becomes therefore necessary to interpret it within a more general context which includes the Cauchy integral as a special case. The context we propose is that of Cesàro summability. This interpretation eliminates the divergence in the theory.

The Helmholz equation should be viewed as a particular case of more general situation. Starting from the Klein–Gordon equation the latter situation may arise. The solution of the scattering problem would then consist of only evanescent waves. Band–width limitations cannot be invoked to eliminate these waves. The interpretation of the integral representation of the diffraction transform kernel as a Cesàro integral then becomes a necessity. One may apply the same procedure to the solution of the classical scattering problem for the Klein–Gordon equation with $m^2 < 0$.

The scattering problem for the Helmholtz equation involves essentially the solution of the one dimensional Schrödinger equation with a potential barrier at the boundary plane $z = z_0$. The text book solution of this quantum mechanical problem is well known. One may approach the problem differently by taking issue with the non–Hermiticity of the translation operator conjugate to the constrained variable z. We propose to re–consider the problem from this point of view elsewhere.

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