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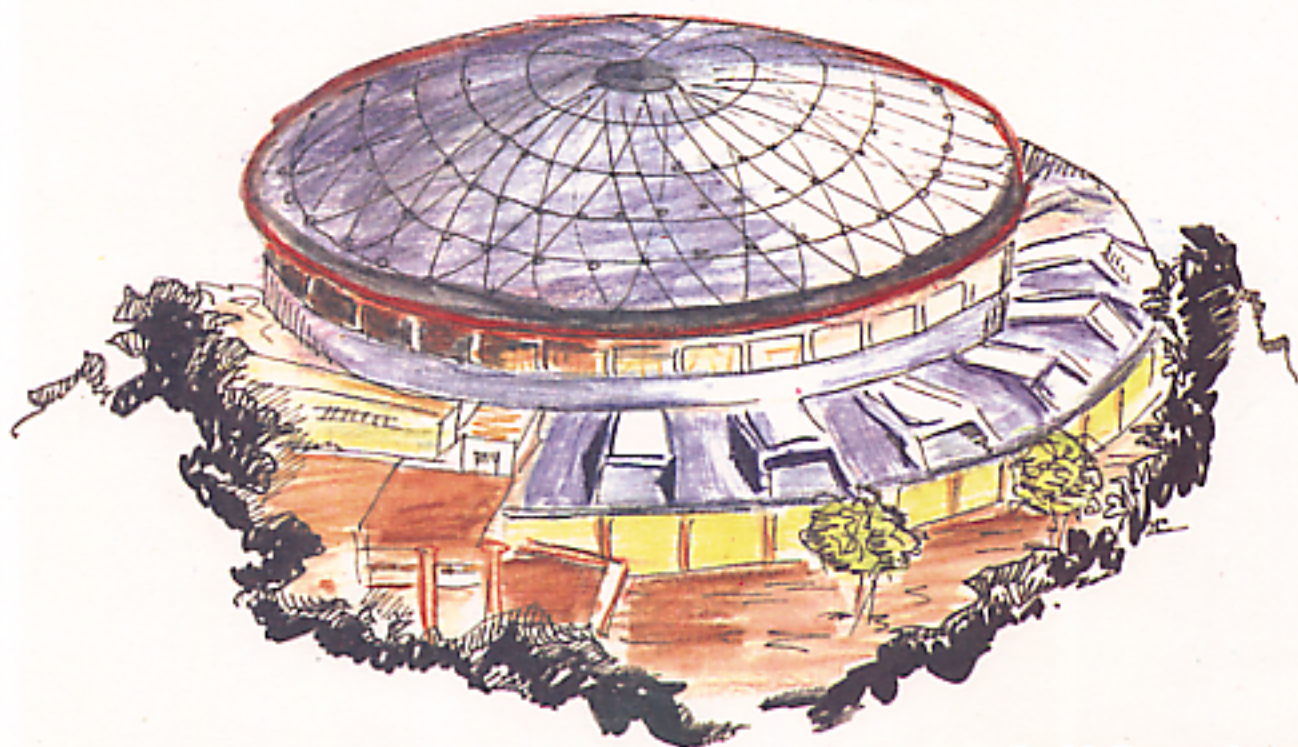
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NONLINEAR REALIZATIONS OF THE $W_3^{(2)}$ ALGEBRA

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Abstract

In this letter we consider the nonlinear realizations of the classical Polyakov's algebra $W_3^{(2)}$. The coset space method and the covariant reduction procedure allow us to deduce the Boussinesq equation with interchanged space and evolution coordinates. By adding one more space coordinate and introducing two copies of the $W_3^{(2)}$ algebra, the same method yields the $sl(3, R)$ Toda lattice equations.

1 Introduction

Recently it was realized [1-3] that the classical $W_N^{(l)}$ algebras, considered as second Hamiltonian structures, give rise to the integrable hierarchy resulting from the interchange of the x and t variables in the $sl(N)$ KdV one. In particular, the Boussinesq equation in x -evolution has been constructed along this line [1] starting from the classical $W_3^{(2)}$ algebra [4]. In view of the growing interest in theories with fractional spins, it is important to understand, from various points of view, the geometric origin of these $W_N^{(l)}$ symmetries.

The idea of a geometric set-up for classical infinite-dimensional algebras (covariant reduction method) goes back to Ref.[5]. There, it was shown that the Liouville equation is intimately related to the intrinsic geometry of the Virasoro algebra: it singles out a two-dimensional fully geodesic surface in the coset space of the Virasoro group. The extension of this method to the case of nonlinear W_N -type algebras has been proposed in [6], and sl_3 -Toda [6] and Boussinesq equation [7] have been constructed starting from the W_3 algebra. In this letter we apply the covariant reduction method to the simplest case of $W_3^{(2)}$ algebra [4], i.e. the first representative of the nonlinear algebras with generators having non-canonical spins.

This letter is organized as follows. In Sec. 2, the coset space method and the covariant reduction procedure, applied to the $W_3^{(2)}$ algebra, allow us to deduce the Boussinesq equation in x -evolution together with the Miura maps. In Sec. 3, by adding one more

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space coordinate and introducing two copies of the $W_3^{(2)}$ algebra, the same method yields the sl_3 Toda lattice equations. Finally, the conclusions of this work are drawn in Sec. 4.

2 Nonlinear Realizations of $W_3^{(2)}$ and x -Boussinesq equation

The main idea of extending the nonlinear realization method to the nonlinear W -type algebras[6] implies replacing the latter by the linear algebras W^∞ and then constructing the coset realizations of these symmetries. The original W symmetry appears as a particular field realization of W^∞ . In this Section we construct a set of nonlinear realizations of $W_3^{(2)\infty}$ and show that these realizations have a deep relation to the x -Boussinesq equation and Miura maps for it.

We start from the commutation relations of the $W_3^{(2)}$ algebra [4]

$$\begin{aligned} [L_n, L_m] &= (n - m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m}, \\ [L_n, G_r^\pm] &= \left(\frac{n}{2} - r\right) G_{n+r}^\pm, [L_n, U_m] = -m U_{n+m}, \\ [U_n, G_r^\pm] &= \pm G_{n+r}^\pm, [U_n, U_m] = -\frac{c}{9}n\delta_{n+m}, \\ [G_r^+, G_s^-] &= -\frac{c}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s} + \frac{3}{2}(r - s)U_{r+s} - L_{r+s} + \Lambda_{r+s}^{(2)}, \end{aligned} \quad (2.1)$$

where

$$\Lambda_n^{(2)} = -\frac{18}{c} \sum_m U_{n-m} U_m. \quad (2.2)$$

The indices m, n and r, s have integer and half integer values, respectively, and run from $-\infty$ to $+\infty$.

The algebra (2.1) is nonlinear. A useful idea [6] is to consider the composite $\Lambda_n^{(2)}$ as a new higher-spin generator. Commutators of $\Lambda_n^{(2)}$ with other generators give us new composites, which we will treat as additional generators and so on. The $W_3^{(2)\infty}$ algebra is then defined as the set of generators $L_n, G_r^\pm, U_n, \Lambda_n^{(2)}$ and all higher-spin composites $\Lambda_n^{(s)}$ ($s \geq 3$) which appear in the commutators of the lower-spin generators. Thus, the $W_3^{(2)\infty}$ algebra contains an infinite set of generators with increasing conformal weights. The commutation relations involving the higher-spin generators $\Lambda^{(s)}$ can be computed using the basic relations (2.1)-(2.2).

Following Refs.[6, 7], in constructing the nonlinear realization of $W_3^{(2)\infty}$ we deal with the following subalgebra of $W_3^{(2)\infty}$, rather than the whole algebra:¹

$$L_n, G_{n+\frac{1}{2}}^\pm, U_{n+1}, \Lambda_{n+2-s}^{(s)}, (n \geq -1). \quad (2.3)$$

It is interesting to notice that the wedge algebra W_\wedge defined as the subalgebra of (2.3) with indices varying from $-(s-1)$ to $(s-1)$ for each spin s contains, in close analogy with the W_3 case[6], the $sl(3, R)$ factor algebra spanned by the generators

$$\{L_{-1}, L_0, L_1, G_{-1/2}^\pm, G_{1/2}^\pm, U_0\} \sim sl(3, R). \quad (2.4)$$

¹In what follows we deal only with this subalgebra, so we use for it the same notation $W_3^{(2)\infty}$.

We will explore this factor algebra in the next Section, where we consider the Toda-type nonlinear realization of $W_3^{(2)\infty}$.

Any nonlinear realization is specified by the choice of the stability subgroup H or, equivalently, its stability subalgebra \mathcal{H} . There are many different subalgebras in the $W_3^{(2)\infty}$ algebra. Next, let us discuss the main peculiarities of the subalgebras we are looking for.

First of all, as in Refs.[6, 7], we always put all higher-spin generators in the stability subalgebra. Let us remark that all higher-spin generators, by construction, are at least quadratic in the basic generators L, G^\pm, U . Therefore, under the classical commutation relations (2.1) they form themselves a subalgebra \mathcal{HS} .

Secondly, let us remind that in the nonlinear realizations approach [8], the parameters associated with the coset generators have the meaning of coordinates and/or fields. Among the remaining generators U_n, L_n, G_r^\pm in the $W_3^{(2)\infty}$ algebra (2.3) three possess a negative dimension: $L_{-1}(cm^{-1}), G_{-1/2}^\pm(cm^{-1/2})$. Following the W_3 case [6, 7], it is natural to associate the coordinate x with the generator L_{-1} , so we will keep this generator in the coset. As for the $G_{-1/2}^\pm$ generators, the dimension of the corresponding coset parameters ($cm^{1/2}$) is unsuitable for treating them as fields. On the other hand, we cannot put both $G_{-1/2}^+$ and $G_{-1/2}^-$ in the stability subgroup, because their commutator yields L_{-1} . So, when both $G_{-1/2}^+$ and $G_{-1/2}^-$ are present in the coset (and the stability subgroup coincides with \mathcal{HS}), we have to introduce two additional coordinates t^\pm with dimension ($cm^{1/2}$). We postpone the discussion of this 3-dimensional case for the future, limiting our consideration here to the following three possibilities $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ for the stability subgroup:

$$\mathcal{H} = \{G_{-1/2}^-, L_0, U_0, G_{1/2}^\pm, L_1, \mathcal{HS}\}, \quad (2.5)$$

$$\mathcal{H}_1 = \{G_{-1/2}^-, L_0, U_0, \mathcal{HS}\}, \quad (2.6)$$

$$\mathcal{H}_2 = \{G_{-1/2}^-, \mathcal{HS}\}. \quad (2.7)$$

In all these cases we associate a "time" coordinate t with the linear combination $G_{-1/2}^+ + G_{-1/2}^-$, keeping the latter in the coset. All other coset generators L_n, G_r^\pm, U_m for the stability subgroups $\mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ have growing positive dimensions, so the corresponding parameters can be associated with fields that depend on the coordinates x and t .

Keeping these facts in the mind, let us turn to the construction of the nonlinear realization of the $W_3^{(2)\infty}$ algebra (2.1)-(2.3).

The minimal coset space corresponds to the maximal subgroup \mathcal{H} (2.5) and contains the generators $\{L_{-1}, G_{-1/2}^+ + G_{-1/2}^-, U_n, G_{n+1/2}^\pm, L_{n+1}\}$ with $n \geq 1$. It can be parametrized in the following form:

$$g = e^{xL_{-1}} e^{t(G_{-1/2}^+ + G_{-1/2}^-)} \left(\prod_{n \geq 3} e^{\xi_{n-1/2}^\pm G_{n-1/2}^\pm} e^{u_n L_n} e^{\phi_{n-1} U_{n-1}} \right) e^{\phi_1 U_1} e^{\xi^+ G_{3/2}^+ + \xi^- G_{3/2}^-} e^{u_2 L_2}. \quad (2.8)$$

The parameters u_n, ϕ_n, ξ_r^\pm depend on x and t . As usual in the nonlinear realization approach [8], the group $W_3^{(2)\infty}$ acts on the coset (2.8) as the left multiplication

$$g_0 \cdot g = g' \cdot h, \quad (2.9)$$

where g_0 is an arbitrary element of $W_3^{(2)\infty}$ group and h belongs to the stability group H . This provides the realization of $W_3^{(2)\infty}$ in the space spanned by the coordinates x, t and the infinite number of fields u_n, ϕ_n, ξ_r^\pm .

To reduce the number of fields we need to invoke the inverse Higgs effect [9] by putting some appropriate covariant constraints on the Cartan forms, which are defined in the standard way

$$\Omega = g^{-1}dg = \sum \omega_i A_i + \sum \tilde{\omega}_k V_k \quad . \quad (2.10)$$

Here we denote all generators from the stability subalgebra \mathcal{H} (2.5) and coset (2.8) as V_k and A_i respectively. It is well known [8], that the forms ω_i associated with the coset generators A_i transform homogeneously under the $W_3^{(2)\infty}$ transformations (2.9). So, we may put some of them to zero. These constraints will be covariant if the remaining non-zero forms belong to the generators spanning a subalgebra together with all stability subalgebra generators V_k . In the case at hand, we may extend the stability subalgebra \mathcal{H} (2.5) only by the generators L_1 and $G_{-1/2}^+ + G_{-1/2}^-$. Thus, the corresponding unique set of covariant constraints reads

$$\omega_n^L = \omega_{n-\frac{1}{2}}^\pm = \omega_{n-1}^U = 0 \quad , \quad \text{with } n \geq 2. \quad (2.11)$$

Let us present explicitly the first forms and the corresponding equations

$$\omega_1^U = d\phi_1 - 3(\xi^+ + \xi^-)dt - 2\phi_2 dx = 0 : \begin{cases} \phi_1' - 2\phi_2 = 0, \\ \dot{\phi}_1 - 3(\xi^+ + \xi^-) = 0, \end{cases} \quad (2.12)$$

$$\omega_{\frac{3}{2}}^+ = d\xi^+ - \xi^+ \phi_1 dx + \left(\frac{1}{2}\phi_1^2 - \phi_2 - \frac{3}{2}u_2\right)dt - 3\xi_{\frac{5}{2}}^+ dx = 0 : \begin{cases} \xi^{+'} - \xi^+ \phi_1 - 3\xi_{\frac{5}{2}}^+ = 0, \\ \dot{\xi}^+ + \frac{1}{2}\phi_1^2 - \phi_2 - \frac{3}{2}u_2 = 0, \end{cases} \quad (2.13)$$

$$\omega_{\frac{3}{2}}^- = d\xi^- + \xi^- \phi_1 dx + \left(\frac{1}{2}\phi_1^2 + \phi_2 - \frac{3}{2}u_2\right)dt - 3\xi_{\frac{5}{2}}^- dx = 0 : \begin{cases} \xi^{-'} + \xi^- \phi_1 - 3\xi_{\frac{5}{2}}^- = 0, \\ \dot{\xi}^- + \frac{1}{2}\phi_1^2 + \phi_2 - \frac{3}{2}u_2 = 0, \end{cases} \quad (2.14)$$

$$\omega_2^L = du_2 + \left(\xi_{\frac{5}{2}}^+ - \xi_{\frac{5}{2}}^- + \xi^+ \phi_1 + \xi^- \phi_1\right)dt - 4u_3 dx = 0 : \begin{cases} u_2' - 4u_3 = 0, \\ \dot{u}_2 + \phi_1(\xi^+ + \xi^-) + \xi_{\frac{5}{2}}^+ - \xi_{\frac{5}{2}}^- = 0, \end{cases} \quad (2.15)$$

where we denote by prime and dot the x and t derivatives, respectively.

The first equations in (2.12)-(2.15) express the higher coset fields in terms of the lower ones u_2, ϕ_1, ξ^\pm :

$$\begin{aligned} u_3 &= \frac{1}{4}u_2', \quad \phi_2 = \frac{1}{2}\phi_1', \\ \xi_{\frac{5}{2}}^+ &= \frac{1}{3}(\xi^{+'} - \xi^+ \phi_1), \quad \xi_{\frac{5}{2}}^- = \frac{1}{3}(\xi^{-'} + \xi^- \phi_1). \end{aligned} \quad (2.16)$$

The meaning of these equations is simple: they claim that the transformation properties of the fields $u_3, \phi_2, \xi_{5/2}^\pm$ under the $W_3^{(2)\infty}$ group (2.9) coincide with the transformations of the right hand side expressions in Eqs. (2.16). Thus, in all calculations we may ignore the fields $u_3, \phi_2, \xi_{5/2}^\pm$ substituting their expressions in terms of the essential ones u_2, ϕ_1, ξ^\pm .

The second equations in (2.12)-(2.15) give the equations of motion for the essential fields (after using (2.16)):

$$\begin{aligned}\dot{\phi}_1 &= 3(\xi^+ + \xi^-), \\ \dot{\xi}^+ &= \frac{3}{2}u_2 + \frac{1}{2}\phi_1' - \frac{1}{2}\phi_1^2, \\ \dot{\xi}^- &= \frac{3}{2}u_2 - \frac{1}{2}\phi_1' - \frac{1}{2}\phi_1^2, \\ \dot{u}_2 &= \frac{1}{3}(\xi^{-'} - \xi^{+'}) - \frac{2}{3}\phi_1(\xi^+ + \xi^-).\end{aligned}\tag{2.17}$$

The x -Boussinesq equation

$$\phi_1'' - \frac{1}{3}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2}{\partial t^2}\phi_1 + 4\phi_1^2\right) = 0\tag{2.18}$$

arises as a consistency condition for this system.

It can be checked that the remaining infinitely many equations in (2.11) express the higher fields u_n, ϕ_n, ξ_r^\pm in terms of u_2, ϕ_1, ξ^\pm , and do not put any additional dynamical constraints on the essential fields, besides (2.17). Thus, we deduced the x -Boussinesq equation in a purely geometrical way, starting from the nonlinear realization of $W_3^{(2)\infty}$ together with the constraints (2.11).

What happens if we consider the narrow stability subalgebra \mathcal{H}_1 ? Let us take the following parametrizations for the coset:

$$g_1 = g e^{\eta^+ G_{1/2}^+ + \eta^- G_{1/2}^-} e^{u_1 L_1}.\tag{2.19}$$

It can be checked explicitly, that using the previous constraints (2.11) gives the same equations (2.12)-(2.15). However, owing to the presence of new generators in the coset (2.19), we could extend our constraints (2.11) by the following additional ones:

$$\omega_1^L = \omega_{\frac{1}{2}}^\pm = 0.\tag{2.20}$$

Without going into the details, let us write the new equations, besides (2.16), (2.17)

$$\begin{aligned}\xi^+ &= \frac{1}{2}\left(\eta^{+'} + \frac{1}{2}\eta^+\eta^+\eta^- + (u_1 - \phi_1)\eta^+\right), \\ \xi^- &= \frac{1}{2}\left(\eta^{-'} - \frac{1}{2}\eta^-\eta^-\eta^+ + (u_1 + \phi_1)\eta^-\right), \\ u_2 &= \frac{1}{3}\left(u_1' + u_1^2 + \frac{3}{2}(\eta^+\eta^{-'} - \eta^-\eta^{+'}) - \eta^+\eta^-(\phi_1 - \frac{3}{4}\eta^+\eta^-)\right),\end{aligned}\tag{2.21}$$

and

$$\begin{aligned}\dot{u}_1 &= \frac{1}{2}\left(\eta^-\dot{\eta}^+ - \eta^+\dot{\eta}^- + \eta^{-'} - \eta^{+'} + u_1(\eta^+ - \eta^-) - (\eta^+ + \eta^-)(\phi_1 + \frac{3}{2}\eta^+\eta^-)\right), \\ \dot{\eta}^+ &= \eta^+\left(\eta^+ + \frac{1}{2}\eta^-\right) + u_1 + \phi_1, \\ \dot{\eta}^- &= -\eta^-\left(\eta^- + \frac{1}{2}\eta^+\right) + u_1 - \phi_1.\end{aligned}\tag{2.22}$$

The equations (2.21) express the fields u_2, ξ^\pm with spins 2, 3/2, respectively, in terms of the new fields u_1, ϕ_1, η^\pm with spins 1, 1, 1/2, which obey the dynamical equations (2.22). Thus, we recognize the equations (2.21)-(2.22) as Miura transformations for the system (2.17).

Analogous calculations for the stability subalgebra \mathcal{H}_2 with the following parametrization of the coset g_2 :

$$g_2 = g_1 e^{u_0 L_0} e^{\phi_0 U_0} , \quad (2.23)$$

and the additional constraints:

$$\omega_0^L = \omega_0^U = 0 \quad (2.24)$$

give the next Miura transformations

$$\phi_1 = \phi'_0 + \frac{3}{2} \eta^+ \eta^- , \quad u_1 = \frac{1}{2} u'_0 \quad (2.25)$$

and

$$\dot{\phi}_0 = \frac{3}{2} (\eta^+ + \eta^-) , \quad \dot{u}_0 = \eta^- - \eta^+ . \quad (2.26)$$

Thus, passing from the stability subgroup \mathcal{H} to \mathcal{H}_1 and \mathcal{H}_2 and putting the corresponding constraints on the Cartan forms, we obtain the Miura transformations, expressing the higher-spin fields in terms of lower-spin ones, as well as the equations of motion for the latter. It can be checked, that all these systems (2.17), (2.21), (2.22), (2.25), (2.26) are self-consistent, i.e. starting from the lowest spin fields u_0, ϕ_0, η^\pm and their equations of motion one could recover the equations of motion for the higher-spin fields u_1, ϕ_1, η^\pm and u_2, ϕ_1, ξ^\pm defined through the Miura transformations (2.25) and (2.21).

Let us finish this Section with some comments.

First, it is clear that due to presence of the generator $G_{-1/2}^-$ in all stability subalgebras \mathcal{H} , \mathcal{H}_1 and \mathcal{H}_2 (2.5)-(2.7), we are free to associate the t -coordinate with the generator $G_{-1/2}^+ + \beta G_{-1/2}^-$:

$$\tilde{g} = e^{xL_{-1}} e^{t(G_{-1/2}^+ + \beta G_{-1/2}^-)} \dots , \quad (2.27)$$

where β is an arbitrary parameter. All values of β are equivalent up to the corresponding renormalizations, except the case $\beta = 0$. In this case we get the following equations of motion instead of (2.17):

$$\begin{aligned} \dot{\tilde{\phi}}_1 &= 3\tilde{\xi}^- , \\ \dot{\tilde{\xi}}^+ &= \frac{3}{2}\tilde{u}_2 + \frac{1}{2}\tilde{\phi}'_1 - \frac{1}{2}\tilde{\phi}_1^2 , \\ \dot{\tilde{\xi}}^- &= 0 , \\ \dot{\tilde{u}}_2 &= \frac{1}{3}\tilde{\xi}^{-\prime} - \frac{2}{3}\tilde{\phi}_1\tilde{\xi}^- . \end{aligned} \quad (2.28)$$

Hence the field $\tilde{\phi}_1$ obeys the free equation of motion

$$\ddot{\tilde{\phi}}_1 = 0 \quad (2.29)$$

rather than the x -Boussinesq one (2.18). Close inspection shows that the equations (2.28) can be explicitly integrated. But in the nonlinear realization framework both choices

of the coset element (2.8) and (2.27) are equivalent and are connected through a right transformation by the stability subgroup

$$g = \tilde{g} h, \quad h \in H. \quad (2.30)$$

In principle, one could find this transformation explicitly in a purely algebraic way, although of course in practice the calculations will rapidly become unmanageable. Thus, the transformation (2.30) to this new coset parametrization (2.27) gives the action-angle variables which allow to rewrite the equations of motion (2.17) and the x -Boussinesq equation (2.18) in the trivial form (2.28) and (2.29), respectively.

Second, in the proposed approach the invariance of the equations (2.17), (2.22), (2.26) with respect to $W_3^{(2)\infty}$ transformations is evident due to the covariance of the constraints (2.11), (2.20), (2.24). The explicit transformation laws for the coordinates and fields can be easily found from the general formula (2.9).

Finally, let us remark that in accordance with the inhomogeneous transformation laws of the Goldstone fields we can associate the fields ϕ_1, ξ^\pm, u_2 with the currents U, G^\pm, T that generate the $W_3^{(2)}$ algebra [4] under appropriate Poisson brackets. Then, the Miura maps (2.21), (2.25) give the free fields representation for the currents, in close analogy with the W_3 case [7].

3 $SL(3, R)$ -Toda lattice

We have seen in the previous Section how the x -Boussinesq equation is connected with the nonlinear realization of the algebra $W_3^{(2)\infty}$ (2.1). All fields in the coset space are functions of the two coordinates x and t , whose dimensionalities are, respectively, 1 and $\frac{1}{2}$. In this Section we show that 2D sl_3 Toda equations of motion also result from a particular coset realization of $W_3^{(2)\infty}$.

In order to have manifest 2D Lorentz symmetry, we start from the product of two commuting copies $W_3^{(2)(\pm)}$ of the algebra. As a next step we need to specify the stability subalgebra \mathcal{H} . First of all, we put, as in the previous Section, all higher-spin composite generators into the stability subalgebra and neglect them in what follows. Secondly, in order to have a real two-dimensional space with coordinates of the same dimensionality, we need to keep in the coset only two generators with negative dimension and associate with them the coordinates x^\pm . Let us remind, that six of the generators of the two copies of $W_3^{(2)\infty}$ algebras (2.3) possess a negative dimension, i.e. $L_{-1}^{(\pm)}, G_{-1/2}^{\pm(\pm)}$. The only possibility to have two commuting x^\pm coordinates is to associate them with the generators $G_{-1/2}^{+(\pm)} + G_{-1/2}^{-(\pm)}$. Finally, we put the generator of the Lorentz rotation $L_0^{(+)} - L_0^{(-)}$ in the stability subalgebra, with the Liouville field $u(x)$ as the parameter corresponding to the coset generator $L_0^{(+)} + L_0^{(-)}$. But we need also to introduce the second Toda field $\phi(x)$ as a coset parameter. The only appropriate generators are U_0^\pm , so we include a linear combination of them $U_0^{(+)} + U_0^{(-)}$ into the coset.

All these requirements are satisfied from the two-parameter family of the stability subgroups generated by

$$\mathcal{H} = \left\{ L_{-1}^{(\pm)} - m_1 m_2 L_1^{(\mp)}, G_{-1/2}^{(+)} + m_1 G_{1/2}^{(-)}, G_{-1/2}^{(-)} + m_2 G_{1/2}^{(+)}, \right. \\ \left. L_0^{(+)} - L_0^{(-)}, U_0^{(+)} - U_0^{(-)}, \mathcal{HS}^{(\pm)} \right\}. \quad (3.1)$$

Let us note that the combinations of generators L, G and U in (3.1) form, in close analogy with the W_3 case [6], the Borel subalgebra of the diagonal $sl(3, R)$ in the sum of the two commuting factor-algebras $sl(3, R)^\pm$ (2.4) in $W_3^{(2)\infty(+)}$ and $W_3^{(2)\infty(-)}$. We parametrize the resulting coset space as follows:

$$g = e^{x+(G_{-1/2}^{+(+)}+G_{-1/2}^{-(+)})} e^{x-(G_{-1/2}^{+(-)}+G_{-1/2}^{-(-)})} \prod_{n=1}^{\infty} \left(e^{\xi_{n-1/2}^{+(\pm)} G_{n-1/2}^{+(\pm)} + \xi_{n-1/2}^{-(-)} G_{n-1/2}^{-(-)} e^{u_n^{(\pm)} L_n^{(\pm)} + \phi_n^{(\pm)} U_n^{(\pm)}} \right) e^{u(L_0^{(+)}+L_0^{(-)})} e^{\phi(U_0^+ + U_0^-)}. \quad (3.2)$$

Next, we show that after imposing appropriate covariant constraints one can express the higher fields in terms of $u(x)$ and $\phi(x)$ which obey the sl_3 Toda equations. Defining Cartan forms in the standard way (2.10), one constrains them as follows:

$$g^{-1} dg = \sum \omega_n^{L^{(\pm)}} L_n^{(\pm)} + \sum \omega_r^{G^{\pm(\pm)}} G_r^{\pm(\pm)} + \sum \omega_n^{U^{(\pm)}} U_n^{(\pm)} + \dots = \tilde{g}^{-1} d\tilde{g} \in \tilde{\mathcal{H}}, \quad (3.3)$$

where $\tilde{\mathcal{H}}$ is some subalgebra containing the stability subalgebra \mathcal{H} (3.1). In other words, we put all forms to zero except those which belong to the generators of the subalgebra $\tilde{\mathcal{H}}$. It is natural to choose the following subalgebra $\tilde{\mathcal{H}}$:

$$\begin{aligned} \tilde{\mathcal{H}} = \{ & L_- = L_{-1}^- - m_1 m_2 L_1^+, \quad L_+ = L_{-1}^+ - m_1 m_2 L_1^-, \\ & G^{--} = G_{-\frac{1}{2}}^{-(-)} + m_2 G_{\frac{1}{2}}^{+(+)}, \quad G^{-+} = G_{-\frac{1}{2}}^{-(+)} + m_2 G_{\frac{1}{2}}^{+(-)}, \\ & G^{+-} = G_{-\frac{1}{2}}^{+(-)} + m_1 G_{\frac{1}{2}}^{-(+)}, \quad G^{++} = G_{-\frac{1}{2}}^{+(+)} + m_1 G_{\frac{1}{2}}^{-(-)}, \\ & L = L_0^+ - L_0^-, \quad U = U_0^+ - U_0^-, \mathcal{H} S^\pm \}. \end{aligned} \quad (3.4)$$

This algebra is an extension of the stability subalgebra \mathcal{H} (3.1), obtained by including the generators G^{+-} and G^{-+} . Let us stress that all higher-spin generators still form an ideal in (3.4) and the factor-algebra of $\tilde{\mathcal{H}}$ over this ideal is the diagonal $sl(3, R)$. The parameters m_1, m_2 correspond to the freedom in extracting the diagonal $sl(3, R)$ from $sl(3, R)^{(+)} \times sl(3, R)^{(-)}$.

Each of the infinite number of equations (3.3) gives rise to two equations for the dx^+ and dx^- projection of forms. We quote the first few equations explicitly

$$\begin{aligned} 2\partial_+ u + \xi^{+(+)} - \xi^{-(+)} &= 0, \quad 2\partial_- u + \xi^{+(-)} - \xi^{-(-)} = 0, \\ 2\partial_+ \phi - \frac{3}{2}(\xi^{+(+)} + \xi^{-(+)} &= 0, \quad 2\partial_- \phi - \frac{3}{2}(\xi^{+(-)} + \xi^{-(-)}) = 0, \\ \partial_+ \xi^{+(+)} - \phi_1^{(+)} - u_1^{(+)} - \frac{1}{2}\xi^{+(+)}\xi^{-(+)} - \xi^{+(+)}\xi^{+(+)} &= 0, \\ \partial_+ \xi^{-(+)} + \phi_1^{(+)} - u_1^{(+)} + \frac{1}{2}\xi^{+(+)}\xi^{-(+)} + \xi^{-(+)}\xi^{-(+)} &= 0, \\ \partial_- \xi^{+(-)} - \phi_1^{(-)} - u_1^{(-)} - \frac{1}{2}\xi^{+(-)}\xi^{-(-)} - \xi^{+(-)}\xi^{+(-)} &= 0, \\ \partial_- \xi^{-(-)} + \phi_1^{(-)} - u_1^{(-)} + \frac{1}{2}\xi^{+(-)}\xi^{-(-)} + \xi^{-(-)}\xi^{-(-)} &= 0 \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \partial_- \xi^{+(+)} - m_- e^{-u+2\phi} &= 0, \quad \partial_- \xi^{-(+)} - m_+ e^{-u-2\phi} = 0, \\ \partial_+ \xi^{+(-)} - m_- e^{-u+2\phi} &= 0, \quad \partial_+ \xi^{-(-)} - m_+ e^{-u-2\phi} = 0. \end{aligned} \quad (3.6)$$

We use the notations $\partial_{\pm} = \partial/\partial x^{\pm}$.

The equations (3.5) are purely algebraic and allow to express the higher-spin coset fields in terms of $u(x)$ and $\phi(x)$. Let us stress that all coset fields can be eliminated in this manner, e.g.:

$$\begin{aligned}\xi^{-(+)} &= \partial_+ u + \frac{2}{3} \partial_+ \phi, \quad \xi^{+(+)} = -\partial_+ u + \frac{2}{3} \partial_+ \phi, \\ \xi^{-(-)} &= \partial_- u + \frac{2}{3} \partial_- \phi, \quad \xi^{+(-)} = -\partial_- u + \frac{2}{3} \partial_- \phi \text{ etc.}\end{aligned}\tag{3.7}$$

The remaining equations (3.6) constrain $u(x), \phi(x)$ to obey the sl_3 Toda equation

$$\begin{aligned}\partial_{+-} u &= \frac{1}{2} (m_1 e^{-u-2\phi} - m_2 e^{-u+2\phi}), \\ \partial_{+-} \phi &= \frac{3}{4} (m_1 e^{-u-2\phi} + m_2 e^{-u+2\phi}).\end{aligned}\tag{3.8}$$

Thus, we succeeded in constructing the sl_3 Toda lattice equations starting from a purely geometric coset realization of $W_3^{(2)\infty(+)} \times W_3^{(2)\infty(-)}$ symmetry. It is worth noticing that the dimensionality of the coordinates in (3.8) is $cm^{1/2}$ in contrast with the ordinary 2D Minkowski space coordinates. However, we are free to renormalize this coordinates at our wishes. Finally, let us stress that the sl_3 Toda lattice equations are *by construction* invariant with respect to the $W_3^{(2)+} \times W_3^{(2)-}$ symmetry realized as left shifts of the coset (3.2).

4 Conclusion

The abovementioned recent literature elucidates many facets of the intimate relationship between the W -type algebras, on the one hand, and the conformal field theory and integrable systems in 1+1 dimensions, on the other hand. Such close connection has far-reaching implications for two-dimensional gravity and string theories.

In this letter we have demonstrated, in the simplest example of the $W_3^{(2)}$ symmetry, that the covariant reduction approach can be applied also to the nonlinear algebras with non-canonical spin generators. We have shown that the Boussinesq equation in x -evolution together with the Miura maps for it and the sl_3 Toda system have a clear geometric origin: they can be derived from a coset realization of the linear algebra $W_3^{(2)\infty}$.

Actually, this work is part of a more ambitious project whose goal is to obtain (and maybe classify) all 2D integrable systems from the nonlinear realizations of some infinite-dimensional (nonlinear) algebras. The advantage of such approach, apart from its clear geometric meaning, is that it yields directly the equations of motion, the Miura maps, as well as the transformation properties of the fields and coordinates for the given algebra. In the framework of the proposed approach, one of the most intriguing questions is how to establish a relation between W_3 and $W_3^{(2)}$ algebras which give rise, in essence, to the same integrable systems. This issue is currently under consideration and further results will be reported elsewhere.

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