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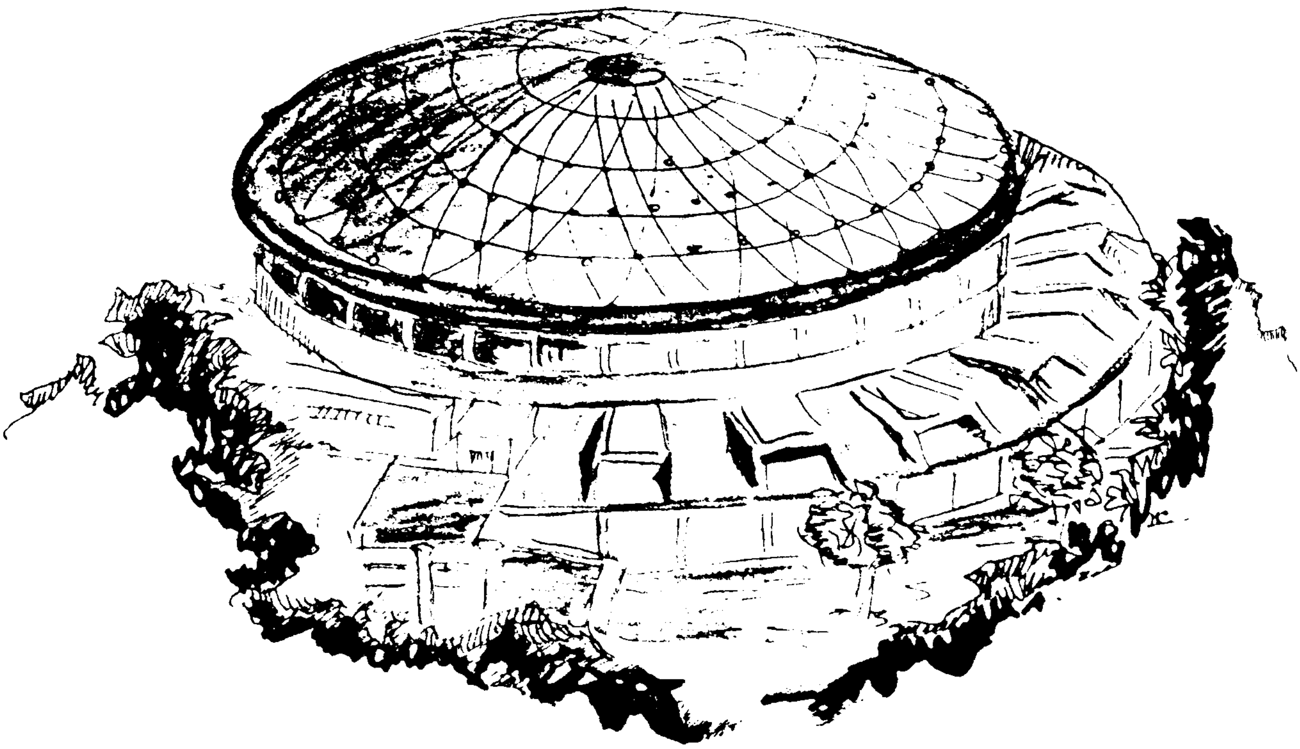
E. Coccia, L. Fraioli, G. Mazzitelli:

LECTURES IN ADVANCED PHYSICS: A PRIMER ON GENERAL RELATIVITY



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LECTURES IN ADVANCED PHYSICS:

A PRIMER on GENERAL RELATIVITY

Lecturer: E. Coccia

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Introduction

The gravitational force dominates our universe, binding matter into stars, stars into galaxies and these into clusters. The classical theory is based on the Newton's law of gravitation, which states that two masses m and M separated by a distance r feel a gravitational attraction mM/r^2. Calculations based on this equation have been used to predict with astonishing accuracy the motion of the planets. For instance, it was straightforward to calculate the orbit that has taken Voyager 2 towards Jupiter, Saturn and Uranus. But in spite of its success Newton's law is fundamentally flawed. First of all it contains no time dependence so that gravitational force need to act instantaneusly to all distances. This is in contradiction to the special theory of relativity (SR) which requires that no signal can travel faster than light. Moreover the Newtonian mechanics contains an inherent epistemological defect in the concept of inertial reference system. One of the aims of Einstein was therefore to find a set of equations to describe gravitation that would be consistent with SR. In doing that, the interplay of space-time and matter was revealed. The presence of matter causes curvature of both space and time and in turn this curvature affects the paths of material particles and of light.

But let us start from the beginning. In order understand what convinced Einstein of the necessity of a new theory of Gravitation we have to put in evidence where the Newton's mechanics fails. We have to underline that Newton too wasn't completely satisfied with his own theory. He had some difficulties in understanding why the field propagates with infinite velocity.

A second weak point of the classical theory is related to the second law of mechanics:

$$F = m_i a$$

this relation shows that the force necessary to move a body with an acceleration a is proportional to an intrinsic quantity of the body, *the inertial mass*. Furthermore, for a body in a gravitational field, subject to a force,

$$F = m_g g$$

where g is the gravity acceleration and m_g is the mass of the body, the acceleration at a given point is:

$$a = \left(\frac{m_g}{m_i}\right) g$$

The fundamental problem is to understand whether the ratio $\frac{m_g}{m_i}$ is the same for all the bodies or it depends on the bodies' internal composition. Galileo's and Newton's experiments (and also recent ones) show that the ratio is close to one (to one part on 10^{12}), suggesting the equivalence between inertial and gravitational mass. This remarkable result isn't explained by the newtonian theory.

Newtonian mechanics, as also the special theory of relativity, defines a family of reference frame, *the inertial frames*, within which the laws of nature take the same form. In other words, if a system of co-ordinates K is chosen so that, in relation to it, physical laws have a well defined form, the same laws must have the same form in any other system of co-ordinate K' moving in uniform translation relatively to K; this is known as *special principle of relativity*. Einstein remarks that "...special relativity does not depart from classical mechanics through the postulate of relativity, but through the postulate of the constancy of the velocity of light in vacuo...".

The word special is the first sintomatic evidence of the limits of classical mechanics and special relativity. Special means that we have chosen a ***special class*** of reference frames, the frames that are correlated by a uniform motion one to each other, or inertial frames. To define which reference frames are inertial frames we have to choose a frame that must be taken as foundamental or preferred. Newton said that an absolute space must exist , and a frame is an inertial one when it's at rest in absolute space, or in a state of uniform motion with respect to absolute space. So we have to find experimental evidence of an absolute space and to answer why the laws of physics must be the same only in a special class of reference frame.

Newton described and made several experiments that demostrated the existence of absolute frame. The most famous is the rotating bucket: "If a bucket, suspended by a long cord, is so often turned about that finally the cord is strongly twisted, then is filled with water, and held at rest together whit the water; and afterwards by the action of a second force, it is suddenly set whirling about the contrary way, and continues, while the cord is untwisting itself, for same time in this motion; the surface of water will at the first be level, just as it was before the vessel began to move; bat subsequently the vessel, by gradually communicating its motion to the water, will make it begin sensibly to rotate, and the water will recede little by little from the middle and rise up at the sides of vessel; its surface assuming a concave form. (This experiment I have made myself)... At first, when the *relative* motion of the water in the vessel was greatest, that motion produced no tendecy whatever of recession from the axis, the water made no endeavor to move upwards towards the circonference, by rising at the side of the vessel, but remained level, and for that reason its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the rising of the water at the side of the vessel indicated an endeavor to recede from the axis; and this endeavor reveals the real circular motion of the water, continualy increasing till it had reached its greatest point, when relatively the water was at rest in the vessel...."

The appearance of the so called inertial or "fictictious" force must be related to an accelerated system, and Newton thoght that the existence of an absolute space made it possible to define an absolute accelaration. In the pail experiment he saw the existence of an absolute space; in fact centrifugal forces on the wather are not due to relative rotation respect to the pail, but to an absolute rotation.

This interpretation was criticized by Berkeley and Leibniz, which argued that there is no philosophical need for any concept of space apart from the relation of material object. This ideas were deepened by the Austrian philosopher Ernst Mach; he said: "Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to side of the vessel produce no noticeable centrifugal forces, but that such forces are produced by its relative motion with respect to the mass of the Earth and the other celestial bodies. No one is competent to say how the experiment would turn out if the side of the vessel increased in the thickness and mass until they were several leagues thick." To be precise, Mach suggested to imagine an experiment in which the water of the pail was at rest and all the fixed stars were moving around it. So it is the relative motion respect to the matter in the universe which produces those effects interpreted by Newton as inertial forces. Mach made a further step: the inertia itself of a body is due to the gravitational interaction with all the other matters in the universe. This is known as the *Mach principle*, not yet experimentally proved, neither theoreticaly clarified.

The ideas of Mach strongly influenced Einstein to postulate the general covariance of the physical laws: "In classical mechanics, and no less in the special theory of relativity, there is an inherent epistemological defect which was, perhaps for the first time, clearly pointed out by Ernst Mach. We will elucidate it by the following example: two fluid bodies of the same size and nature hover freely at so great distance from each other and from all other masses that only those gravitational forces need be taken in account which arise from the interaction of different parts of the same body. Let the distance between the two bodies be invariable, and in neither of the bodies let there be any relative movements of the parts with respect to one another. But let either mass, as judged by an observer at rest relatively to the other mass, rotate with constant angular velocity about the line joining the masses. This a verifiable relative motion of the two bodies. Now let us imagine that each of the bodies has been surveyed by means of measuring instruments at rest relatively to itself, and let the surface of S_1 prove to be a sphere, and that of S_2 an ellipsoid of revolution. Thereupon we put the question: What is the reason for this difference in the two bodies? No answer can be admitted as epistemologically satisfactory, unless the reason given is an *observable fact of experience*. The law of causality has not the significance of a statement as to the world of experience, except when *observable facts* ultimately appear as causes and effects.

Newtonian mechanics does not give a satisfactory answer to this question. It pronounces as follows: the laws of mechanics apply to the space R_1, in respect to which the body S_1 is at rest, but not to the space R_2, in respect to which the body S_2 is at rest. But the privileged space R_1 of Galileo, thus introduced is a merely factitious cause, and not a thing that can be observed. It is therefore clear that Newton's mechanics does not really satisfy the requirement of causality in the case under consideration, but only apparently does so, since it makes the factitious cause R_1 responsible for the observable difference in the bodies S_1 and S_2.

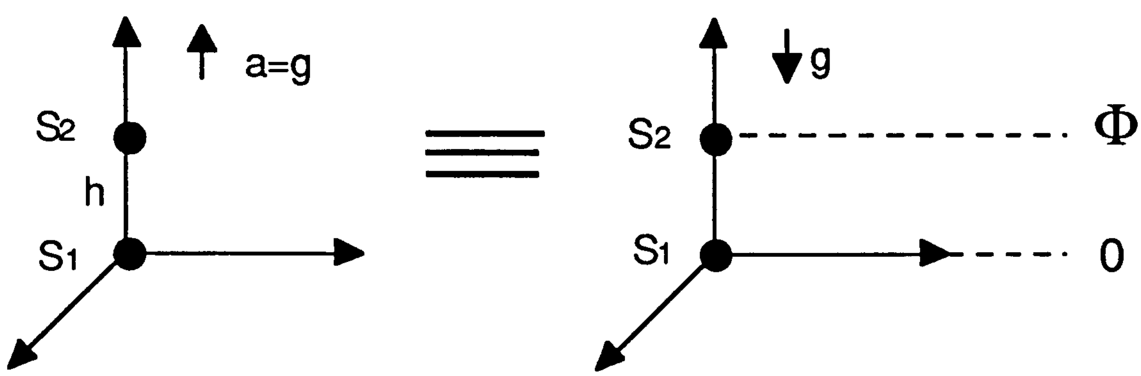
The satisfactory answer can only be that the physical system consisting of S_1 and S_2 reveals within itself no imaginable cause to which the differing behaviour of S_1 and S_2 can be referred. The cause must therefore lie *outside* this system. We have to take it that the general laws of motion, which in particular determine the shape of S_1 and S_2, must be such that the mechanical behaviour of S_1 and S_2, is partly conditioned, in quite essential respects, by distant masses which we have not included in the system under consideration. These distant masses and their motion relative to S_1 and S_2 must than be regarded as the seat of the causes (which must be susceptible to observaction) of the different behaviour of our two bodies S_1 and S_2. They take over the role of the fictitious cause R_1. Of all imaginable spaces R_1, R_2, etc., in any kind of motion relatively to one another, there is none which we may look upon as privileged *a priori* without reviving the above-mentined epistemoligical objection. *The laws of physics must be of such a*

nature that they apply to systems of reference in any kind of motion. In this way we arrive at an extension of the postulate of relativity."

Another fundamental result for the formulation of General Relativity (G.R.) is the equal fall of all bodies in the gravitational field. Einstein realized that a reference frame (X,Y,Z) at rest in a homegeneus, along -Z axis, gravitational field g and another reference frame moving with uniform acceleration g, along +Z axis, are physically exactly equivalent, and made of this equivalence a law of nature, ***the equivalence principle***: at every space-time point in an arbitrary gravitational field it is possible to choose a ***locally*** inertial frame such the laws of nature take the same form as in an unaccelerated cartesian frame in absence of gravitation. It's important to note the deep analogy with the Gauss axyom of non euclidean geometry: at any point on a curved surface we may erect a locally cartesian coordinate system in which distances obey the law of Pythagoras.

Two direct consequences of the equivalence principle are the gravitational spectral shift and the bending of light rays. Given a reference frame (X, Y, Z) moving with an accelarion g along the Z axis, we consider an eletromagnetic source at rest in the frame whith coordinate $(0, 0, h)$. An obsever, fixed in the origin, will see a different frequency from the emitted one:

$$\nu_1 = \nu_2 \left(1 + \frac{gh}{c^2}\right)$$



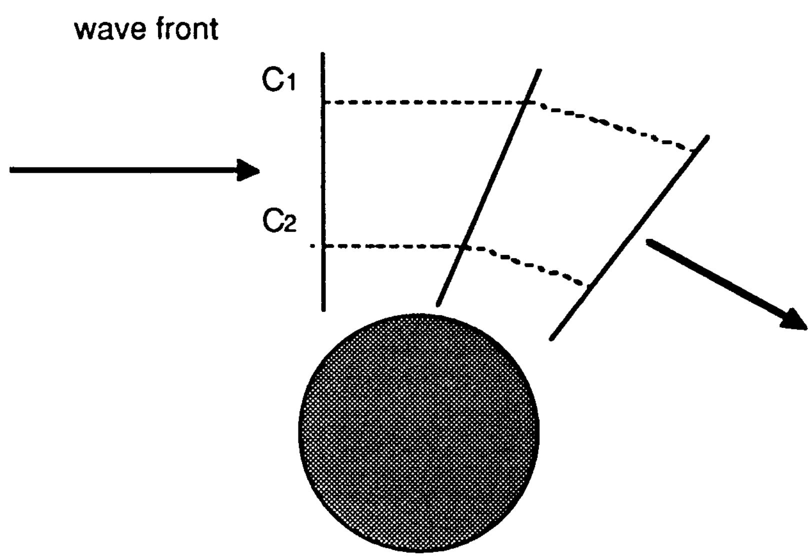
For the equivalence principle the reference frame (X, Y, Z) is equivalent to an unaccelerated reference frame (X', Y', Z') in presence of gravitational field g, along $-Z$ axis. So the new observer, in the origin of this second frame, must see the same shift of the observer in the previous frame, but in this frame $gh = \Phi$, where Φ is the gravitational potential at height h:

$$\nu_1 = \nu_2 \left(1 + \frac{\Phi}{c^2}\right)$$

$$\frac{\Delta\nu}{\nu} = \frac{\Phi}{c^2}$$

Obviously, the frequency shift is equivalent to a time delay of two clocks placed in different points with different gravitational potentials. The gravitational spectral shift was experimentally confirmed by Poud and Rebka in 1960.

From this point of view it's possible to explain the bending of light rays assuming that the velocity of light in the gravitational field is a function of position.



Experimentally confirmed in 1919, observing the apparent position of some stars during a solar eclipse.

Space-Time and Its Geometrical Interpretation.

In a gravitational field, or in a non inertial frame, space-time is distorted and the euclidean geometry is no longer valid; for instance if (x', y', z', ct') are the coordinates of a frame uniformally rotating respect to (x, y, z, ct), around the $z = z'$ axis the trasformations are:

$$\begin{cases} x = x' \cos\omega t - y' \sin\omega t \\ y = x' \sin\omega t + y' \cos\omega t \\ z = z' \end{cases}$$

and the line element becomes

$$ds^2 = \left[c^2 - \omega^2(x' + y')\right] dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\omega y' dx' dt - 2\omega x' dy' dt$$

so it is no longer given by the sum of the squares of four differentials of the coordinate. We can write a general quadratic form in the differentials of the coordinate $x^1, x^2, x^3, x^4 = ct$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where Greek indices run from 1 to 4 and we have used the index summation convention introduced by Einstein.

$g_{\mu\nu}$ determines completely the geometrical properties of the system and its symmetry $(g_{\mu\nu} = g_{\nu\mu})$ implies the resolution of ten independent equations.

In an inertial frame (or in absence of a gravitational field)

$$\begin{cases} g_{11} = g_{22} = g_{33} = -1 \\ g_{44} = 1 \\ g_{\mu\nu} = 0 \quad for \quad \mu \neq \nu \end{cases}$$

and the corresponding metric is the flat Minkowsky metric of special relativity

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

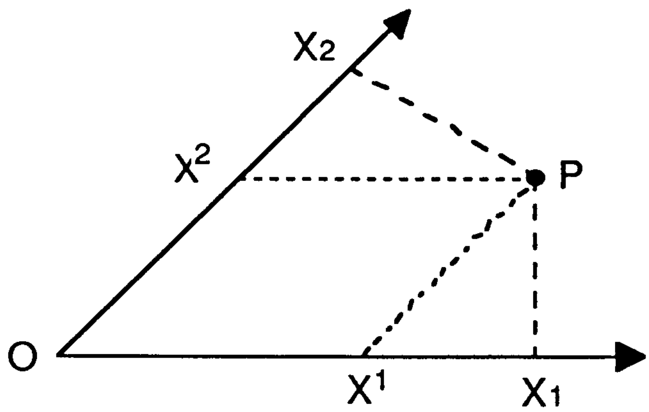
Gravitation is a deviation of the metric of space-time from the flat Minkowsky metric; its effects are described by $g_{\mu\nu}$. The space-time of G.R. is called Riemann space-time and consists of a patchwork of regions that are locally flat and match the Minkowsky space-time of special relativity. The validity of E.P. referred to our last sentencies implies that

- space-time is endowed whith a metric ($g_{\mu\nu}$);
- the world lines of test bodies are geodesic of that metric;
- in a local freely falling frame the laws of physics are those of special relativity.

Elements of Tensorial Algebra.

Covariant and Controvariant Vectors.

In a curved space-time the coordinate axis can be non-orthogonal; to understand the meaning of covariant and controvariant vectors, we consider as simple example a flat non-orthogonal frame. Here, there are two equally natural ways to measure distances along the axis:



So the covariant vectors are those which transform as the coordinates of the point P, (x_1, x_2), while the controvariant vectors trasform as the components of the vector $\overline{OP}$, (x^1, x^2). Covariant and controvariant vectors are related by laws of transformation; in our case,

$$\begin{cases} x_1 = x^1 + x^2 \cos\theta \\ x_2 = x^1 \cos\theta + x^2 \end{cases}$$

introducing the tensor

$$g_{ij} = \begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}$$

we can write

$$x_i = g_{i1}x^1 + g_{i2}x^2$$

The general definitions of covariant and controvariant vectors, in a four dimentional space, are related to their properties under coordinates transformation from x^μ to x'^μ or x_μ to x'_μ. Let us consider a controvariant vector A^μ; since scalar products are unaffected by the tranformation

$$A'^\mu dx'_\mu = A^\nu dx_\nu$$

$$A'^\mu = A^\nu \left(\frac{\partial x_\nu}{\partial x'_\mu}\right) = \Lambda^\mu_\nu A^\nu$$

where

$$\Lambda^\mu_\nu = \left(\frac{\partial x_\nu}{\partial x'_\mu}\right)$$

Similarly

$$ds^2 = dx_\nu dx^\nu = dx'_\mu dx'^\mu$$

so that

$$\Lambda^\mu_\nu = \left(\frac{\partial x_\nu}{\partial x'_\mu}\right) = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$$

(note that $\Lambda^\alpha_\nu \Lambda^n u_\beta = \delta_{\alpha\beta}$). For covariant vectors we have:

$$A'_\mu dx'^\mu = A_\nu dx^\nu$$

then

$$A'_\mu = \Lambda^\nu_\mu A_\nu$$

where

$$\Lambda^\nu_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right) = \left(\frac{\partial x'_\mu}{\partial x_\nu}\right)$$

The trasformation laws

$$A'^\mu = \Lambda^\mu_\nu A^\nu$$

$$A'_\mu = \Lambda^\nu_\mu A_\nu$$

define respectively the controvariant and covariant vectors; so covariant vectors transform like the differentials

$$dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu$$

while covariant ones like gradients, if ϕ is a scalar

$$\frac{\partial \phi}{\partial x^\mu} \rightarrow \frac{\partial \phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \phi}{\partial x^\nu}$$

We can demonstrate the relation

$$A^\mu = g^{\mu\nu} A_\nu$$

the metrics tensor raise and lower indicies and relate covariant components to controvariant ones.

Tensors.

In the same way, the tensor $A^{\mu\nu}$ is controvariant if

$$A'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} A^{\alpha\beta}$$

while $A_{\mu\nu}$ is a covariant tensor if

$$A'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu A_{\alpha\beta}}$$

In the tensorial algebra a tensor of 0 rank is a scalar, a tensor of 1 rank is a usual vector while the metric tensor $g_{\mu\nu}$ is an example of 2 rank tensor.

The use of tensors rather than their components is very powerful because once a tensor equality has been prooved in one frame, it's automatically true in all frames; the proof of such an identity can be made using a frame in which the proof is simple, for instance a freely falling frame where the physics is that of special relativity. The result will apply equally well in accelerating frames.

A usefull property of tensors of any rank is that if

$$A^{\mu\nu} = B^{\mu\nu}$$

then

$$A_{\mu\nu} = B_{\mu\nu}$$

Covariant Derivation.

We want to write physical laws so that they apply to curved space-time; a simple procedure allows us to convert all the relationships which are valid in flat space-time so that they became valid in G.R.: to replace standard space-time derivatives by covariant derivatives.

In a curvilinear system of coordinates the differential of a vector isn't a vector; in fact

$$A^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} A'^{\nu}$$

$$dA^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} dA'^{\nu} + A'^{\nu} d\frac{\partial x^{\mu}}{\partial x'^{\nu}} =$$

$$= \frac{\partial x^{\mu}}{\partial x'^{\nu}} dA'^{\nu} + A'^{\nu} \frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\alpha}} dx'^{\alpha}$$

so dA^{μ} would be a vector if $\frac{\partial^2 x^{\mu}}{\partial x'^{\nu} \partial x'^{\alpha}}$ was zero. This is due to the fact that dA^{μ} is computed by taking the difference between two vectors lying at two different points of the space-time, and the transformation laws of vectors are position dipendent.

To perfom the difference between vectors at the same point of space-time, it is necessary to define a transport operation.

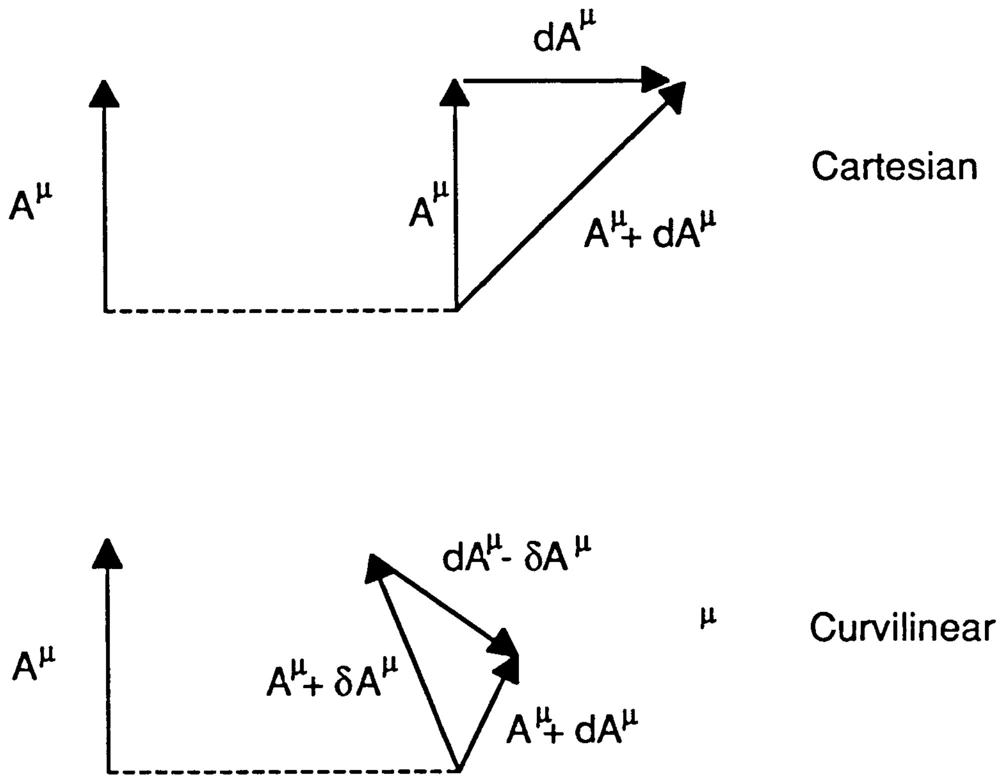
In figure the components undergoing parallel transport are however in general changed; after the displacement, the difference between the two vectors is

$$DA^{\mu} = dA^{\mu} - \delta A^{\mu}$$

We make the assumption that δA^{μ} linearly depends on dx^{μ} and A^{μ}

$$\delta A^{\mu} = -\Gamma^{\mu}_{\beta\alpha} A^{\alpha} dx^{\beta}$$

the quantities $\Gamma^{\mu}_{\beta\alpha}$ are called metric connections or Christoffel's symbols. It's to notice that they are not the components of a tensor, in fact they depend on the choice of the coordinates systems; Christoffel's symbols also satisfied the relation $\Gamma^{\mu}_{\beta\alpha} = \Gamma^{\mu}_{\alpha\beta}$.



Contrary to what we said before about dA^{μ}, the quantity DA^{μ} is computed by taking the difference between two vectors lying at the same point of the space time; so

$$DA^{\mu} = dA^{\mu} + \Gamma^{\mu}_{\beta\alpha} A^{\alpha} dx^{\beta}$$

is a vector and may be demostrated that the trasfomation rule is:

$$DA^{\mu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} (DA^{\nu})'$$

For a covariant vector A_{μ} we will write:

$$DA_{\mu} = dA_{\mu} - \delta A_{\mu} = dA_{\mu} - \Gamma^{\alpha}_{\beta\mu} A_{\alpha} dx^{\beta}$$

So we can generalize the derivation operator in the following way:

$$\frac{DA^\mu}{Dx^\beta} = A^\mu_{;\beta} = \frac{\partial A^\mu}{\partial x^\beta} + \Gamma^\mu_{\beta\alpha} A^\alpha.$$

Relation between metrics and Christoffel's symbols.

It's easy to demonstrate the relation betwen the metric $g_{\mu\nu}$ and the Christoffel's symbols $\Gamma\nu\mu\rho$.

It's possible to generalize the covariant derivation to a tensor $T_{\mu\nu} = A_\mu B_\nu$

$$T_{\mu\nu;\rho} = B_\nu A_{\mu;\rho} + A_\mu B_{\nu;\rho} = \frac{\partial T_{\mu\nu}}{\partial x^\rho} - \Gamma^\gamma_{\mu\rho} T_{\gamma\nu} - \Gamma^\gamma_{\nu\rho} T_{\mu\gamma}.$$

We can write

$$A_{\mu\nu} = (g_{\mu\gamma} A^\gamma)_{;\nu} = g_{\mu\gamma;\nu} A^\gamma + g_{\mu\gamma} A^\gamma_{;\nu}$$

but $A_{\mu\nu}$ is a tensor and so $A_{\mu;\nu} = g_{\mu\gamma} A^\gamma_{;\nu}$; observing the last two equations that we have written we conclude that

$$g_{\mu\gamma;\nu} = \frac{\partial g_{\mu\gamma}}{\partial x^\nu} - \Gamma^\rho_{\mu\nu} g_{\rho\gamma} - \Gamma^\rho_{\gamma\nu} g_{\mu\rho} = 0$$

Solving this equation in Γ,

$$\Gamma_{\nu\mu\rho} = \frac{1}{2} (\partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho})$$

The metric connections are thus symmetric and unique (*foundamental theorem of Riemann Geometry*).

In a frame in free fall, we locally have

$$g_{\mu\nu} = \eta_{\mu\nu} \qquad \partial_\rho g_{\mu\nu} = 0$$

and

$$\Gamma_{\nu\mu\rho} \equiv 0$$

while, in another frame, in general

$$\Gamma_{\nu\mu\rho} \neq 0$$

this confirms the non tensorial nature of the metric connections.

The Geodesic Equation.

Now we want to write the equation of motion of a free particle in curved space-time; the starting point is the variational principle

$$\delta \int ds = 0$$

So

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$\delta \int ds = \delta \int (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$$

In analogy to special relativity in which the variational principle implies that $dv^\mu = 0$ now we have $Dv^\mu = 0$, where v^μ is the velocity of the particle

$$dv^\mu + \Gamma^\mu_{\beta\alpha} v^\alpha dx^\beta = 0$$

and then the covariant law of motion is:

$$\frac{dv^\mu}{ds} + \Gamma^\mu_{\beta\alpha} v^\alpha v^\beta = \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\beta\alpha} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0$$

We can better understand the meaning of the Christoffel's symbols writing the equation for the forces:

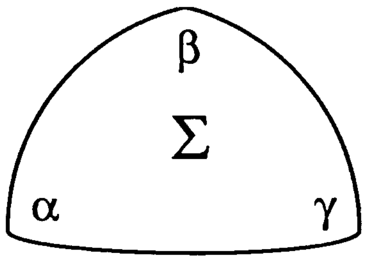
$$m \frac{d^2 x^\mu}{ds^2} = -m \Gamma^\mu_{\beta\alpha} v^\alpha v^\beta$$

where the second term plays the rule of gravitational force. So we can associate the force to the Γ and the gravitational potential to the tensor $g_{\mu\nu}$ remembering that the metric connections are proportional to the partial derivation of the metric tensor.
We estabilished how to calculate the motion of bodies in curved space-time and how to calculate the variation of vectors along paths in curved space-time; it remains to obtain a connection

$$mass/energy \quad \leftrightarrow \quad space - time\ curvature$$

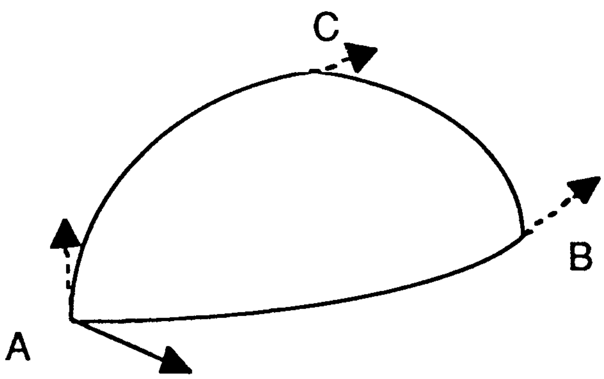
Curvature.

The intrinsic geometry of any differentiable surface is locally describable by means of a single parameter: *the gaussian curvature k.*



$$k = \lim_{\Sigma \to 0} \frac{\alpha + \beta + \gamma - \pi}{\Sigma}$$

The Curvature Tensor.



We now consider the tangent vector to the geodesic curve $x^\mu(s)$:

$$v^\mu = \frac{dx^\mu}{ds}$$

since it's a geodesic, we have

$$Dv^\mu = 0$$

and therefore the parallely displaced tangent vector is:

$$v^\mu + \delta v^\mu = v^\mu + dv^\mu$$

and coincides with the tangent in $x^\mu + dx^\mu$. In other words, transporting a vector along a geodesic, the angle between the vector and the tangent is constant.
The variation of a vector A_μ during its parallel displacement along the infinitesimal contour γ is

$$\Delta A_\mu = \int_\gamma \delta A_\mu = \int_\gamma \Gamma^\beta_{\alpha\mu} A_\beta dx^\alpha$$

Using the Stoches theorem for a vector V_μ

$$\int_\gamma V_\mu dx^\mu = \int_\Sigma d\Sigma^{\mu\nu} \partial_\mu V_\nu = \frac{1}{2} \int_\Sigma d\Sigma^{\mu\nu} \left(\partial_\mu V_\nu - \partial_\nu V_\mu \right)$$

from which

$$\Delta A_\mu = \frac{1}{2} \int d\Sigma^{\alpha\nu} \left[\partial_\alpha \left(\Gamma^\beta_{\nu\mu} A_\beta \right) - \partial_\nu \left(\Gamma^\beta_{\alpha\mu} A_\beta \right) \right] =$$

$$\frac{1}{2} \int \left[\partial_\alpha \Gamma^\beta_{\nu\mu} A_\beta - \partial_\nu \Gamma^\beta_{\alpha\mu} A_\beta + \Gamma^\beta_{\mu\nu} \partial_\alpha A_\beta - \Gamma^\beta_{\alpha\mu} \partial_\nu A_\beta \right] d\Sigma^{\alpha\nu}$$

but along a geodesic $DA_\mu = 0$ i.e. $\partial_\mu A_\nu = \Gamma^\alpha_{\mu\nu} A_\alpha$ and so

$$\frac{1}{2} \int \left[\partial_\alpha \Gamma^\beta_{\nu\mu} - \partial_\nu \Gamma^\beta_{\alpha\mu} + \Gamma^\beta_{\nu\mu} \Gamma^\beta_{\alpha\mu} - \Gamma^\rho_{\alpha\mu} \Gamma^\beta_{\nu\rho} \right] A_\beta d\Sigma^{\alpha\nu}$$

We can evaluate this integral replacing the integrand by its value at some points inside the infinitesimal contour γ.
We finally obtain

$$\Delta A_\mu = \frac{1}{2} R^\beta_{\alpha\nu\mu} A_\beta \Delta\Sigma^{\alpha\nu}$$

where

$$R^\beta_{\alpha\nu\mu} = \partial_\alpha \Gamma^\beta_{\nu\mu} - \partial_\nu \Gamma^\beta_{\alpha\nu} + \Gamma^\beta_{\alpha\rho} \Gamma^\rho_{\nu\mu} - \Gamma^\beta_{\nu\rho} \Gamma^\rho_{\alpha\mu}$$

is the *Riemann's curvature tensor* and it contains a full description of the space-time curvature. We note that it has $4^4 = 256$ components but only 20 are independent.
In an euclidean space we can eliminate the Γ, writing

$$R^\beta_{\alpha\nu\mu} = 0$$

We have to remark that this relation tells us that the space is flat everywhere and not only locally, because $R^\beta_{\alpha\nu\mu}$ is a tensor. If the space-time is not flat the Riemann's tensor cannot be vanished, even if, locally we can obtain $\Gamma^{\alpha}_{\mu\nu} = 0$ by suitable transformation of coordinates, because the derivative $\partial_\beta \Gamma^{\alpha}_{\mu\nu} \neq 0$ and then

$$R^{\beta}_{\alpha\nu\mu} = \partial_\alpha \Gamma^{\beta}_{\nu\mu} - \partial_\nu \Gamma^{\beta}_{\alpha\mu} \neq 0$$

Useful contraction of Riemann's tensor are the *Ricci's tensor*

$$R_{\mu\nu} = g^{\beta}_{\alpha} R^{\alpha}_{\alpha\nu\mu} = R_{\nu\mu}$$

and the *curvature scalar*

$$R = g_{\mu\nu} R^{\mu\nu}$$

that in a two dimensional space is proportional to the Gauss' curvature

$$R = 2k$$

Einstein's field equation.

In the theory of Newton, the gravitational scalar potential ϕ satisfies the Poisson equation

$$\nabla^2 \phi = 4\pi G \rho$$

The relativistic generalization of the matter energy density is the stress tensor $T_{\mu\nu}$; it's the flow of μ-component of the 4-momentum across a surface perpendicular to the ν-direction:

$T_{44} =$ energy density;

$T_{4i} =$ energy flow across i-plane;

$T_{ii} =$ pressure across i-plane;

$T_{ji} =$ flow of j-component of momentum across i-plane;

$T_{i4} =$ density of the i-component of momentum across.

To maintain the analogy with the Poisson's equation, the first term must contain no derivatives of $g_{\mu\nu}$ of order higher than 2 and it must be linear in the second derivative of $g_{\mu\nu}$.

We can write the conservation laws of 4-momentum in special relativity:

$$\partial_\mu T^{\mu\nu} = 0$$

In G.R.

$$T^{\mu\nu}_{;\nu} = 0$$

So, the matter energy density tensor has the following properties:

- it vanishes in absence of matter;
- it is of second rank;
- its divergence vanishes everywhere;
- it is symmetric, $T^{\mu\nu} = T^{\nu\mu}$.

Einstein identified the stress-energy tensor as the source of space-time curvature, and suggested the simplest possible relationship

$$T_{\mu\nu} \propto G_{\mu\nu}$$

where $G_{\mu\nu}$, the *Einstein tensor*, should be a symmetric, divergenceless, second rank tensor and related with the Riemann tensor which unique contraction is the Ricci tensor $R_{\mu\nu}$. It has a non-zero divergence that can be removed by a simple subtraction:

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = G_{\mu\nu}$$

To determine the proportionality costant χ between $G_{\mu\nu}$ and $T_{\mu\nu}$

$$G_{\mu\nu} = \chi T_{\mu\nu}$$

we require that, in the nonrelativistic limit, the Poisson equation holds.

Before continuing, we want to remark the analogy between the previous equation and the *Hooke's law*

$$\sigma = k \frac{\Delta x}{x}$$

that relates the stress σ of a body to its relative deformation $\frac{\Delta x}{x}$ by the module of elasticity k.

Returning to the main problem we start from the geodesic equation:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\beta\alpha}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

that in this limit, in which $ds^2 \simeq c^2dt^2 = (dx^4)^2$, becomes:

$$\frac{d^2x^\mu}{dt^2} = -\Gamma^\mu_{44}\left(\frac{dx^4}{dt}\right)^2 = -c^2\Gamma^\mu_{44}$$

In case of weak gravitational field ($|h_{\mu\nu}| \ll 1$)

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

and static field

$$\partial_4 g_{\mu\nu} = 0$$

we can write

$$\Gamma^k_{44} = \frac{1}{2}g^{km}\left(-\partial_m g_{44}\right) = \frac{1}{2}\partial_k h_{44}$$

Substituting the last result in the geodesic equation we obtain

$$\frac{d^2\vec{x}}{dt^2} = -\frac{c^2}{2}\vec{\nabla}h_{44}$$

that concides with Newton's

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi$$

if

$$h_{44} = \frac{2}{c^2}\phi$$

In case of very slow motions, the leading term in $T_{\mu\nu}$ is

$$T_{44} = \rho c^2$$

Inverting the relation between the Riemann tensor and the Einstein's one, we have:

$$R_{\mu\nu} = \chi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

and so

$$R_{44} = \frac{1}{2}\chi\rho c^2$$

From the definition of $R_{\mu\nu}$, neglecting temporal derivatives and Γ^2 terms,

$$R_{44} = \partial_k \Gamma^k_{44} = \frac{1}{2}\nabla^2 h_{44} = \frac{1}{c^2}\nabla^2 \phi$$

Combinig the last two equation with the Poisson equations we obtain the value of the costant χ

$$\chi = \frac{8\pi G}{c^4}$$

that we can introduce in the gravitational field equation

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$$

that is nonlinear and that contains the conservation equation $T^{\mu\nu}_{;\nu} = 0$.
At last, remembering the analogy proposed before, we suggest to interpretate the costant

$$\chi = 2.073 \times 10^{-48}$$

as the module of elasticity of the space-time.
We have written ten, non linear, second order, partial differential equations for the ten components of $g_{\mu\nu}$; it's important to remark that:
- superposition principle is no longer valid;
- the conservation equation $T^{\mu\nu}_{;\nu} = 0$ follows from the field equations themselves.

The Einstein's field equation suggests the following comments on the nature of the gravitational fields. They carry energy and momentum and must therefore contribute to the curvature itself. Maxwell's equations are linear because the electromagnetic field doesn't itself carry charge; so, the non linearity of Einstein's equation represents the effect of gravitation on itself.
In the electromagnetic theory, from the Maxwell field equations, one can deduce the conservation of the electromagnetic current, but not the equations of motions of charges, i.e. the Lorentz equations. In the gravitational case, on the contrary, the field equations

contain also the equations of motion for the matter producing the field and so the matter distribution and its motion cannot be described independently from the gravitational field produced by them.

This characteristic aspect of gravitation is to be ascribed to the fact that the energy plays the double role of gravitational source and inertial mass.

Einstein's equation solution.

We have ten equations and ten unknown $g_{\mu\nu}$; according to the usual Cauchy problem, one should expect then that once the values of $g_{\mu\nu}$ and $\partial_t g_{\mu\nu}$ are assigned on the space initial hypersurface $t = t_0$, the temporal evolution of $g_{\mu\nu}$ is fixed, and then the metric can be calculated for each value of $\vec{x}$ and t.

But physically we know that $g_{\mu\nu}$ never may be determined univocally as it can be always subject to an arbitrary change of coordinates.

Mathematically, the ten equations $G_{\mu\nu} = \chi T_{\mu\nu}$ are related by four conditions following from the Bianchi identities $G^{\nu}_{\mu;} = 0$; then only six independent equations are left to determine the ten unknowns $g_{\mu\nu}$. So, we need four "gauge" conditions to fix the metric tensor.

The Schwarzschild solution.

We are going to expose the first soluton of the Einstein equations, obtained for a static and spherically symmetric field.

Starting with the most general spherically symmetric line element $(r,\ \theta,\ \psi,\ t)$

$$ds^2 = W(r)dt^2 - U(r)dr^2 - V(r)\left(r^2 d\theta^2 + r^2 \sin^2\theta d\psi\right)$$

rewritten as

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\psi^2$$

we get a metric tensor

$$g_{44} = e^{\nu(r)}$$

$$g_{11} = -e^{\lambda(r)}$$

$$g_{22} = -r^2 \sin^2\theta$$

$$g_{33} = e^{\nu(r)}$$

$$g_{\mu\nu} = 0 \quad \mu \neq \nu$$

In order to find $\lambda(r)$ and $\nu(r)$, one must calculate explicitely the Christoffel symbols Γ and then the Ricci tensor $R_{\mu\nu}$, putting in vacuum $R_{\mu\nu} = 0$.

We obtain

$$R_{44} = e^{\nu-\lambda}\left(-\frac{\nu''}{2} - \frac{\nu'}{r} - \frac{\nu'^2}{4} + \frac{\lambda'\nu'}{4}\right) = 0$$

$$R_{11} = -\frac{\lambda'}{r} - \frac{\lambda'\nu'}{4} + \frac{\nu''}{2} + \frac{\nu'^2}{4} = 0$$

$$R_{22} = e^{-\lambda}\left[1 + \frac{r}{2}(\nu' - \lambda')\right] - 1 = 0$$

$$R_{33} = \sin^2\theta e^{-\lambda}\left[1 + \frac{r}{2}(\nu' - \lambda')\right] - \sin^2\theta = 0$$

The last two equations are equivalent, so the following three equations are left :

$$\frac{\nu''}{2} + \frac{\nu'}{r} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} = 0$$

$$\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'}{r} - \frac{\lambda'\nu'}{4} = 0$$

$$e^{-\lambda}\left[1 + \frac{r}{2}(\nu' - \lambda')\right] - 1 = 0$$

The difference between the first two equations gives

$$\nu' + \lambda' = 0$$

so that

$$\nu + \lambda = const.$$

Imposing $\nu(\infty) + \lambda(\infty) = 0$, i.e. $g_{\mu\nu} = \eta_{\mu\nu}$ at $r = \infty$, we obtain

$$\lambda = -\nu$$

so the third relation becomes

$$e^{\nu}(1 + r\nu') = 1$$

or

$$\left(e^{\nu}r\right)' = 1$$

Its integration

$$e^{\nu}r = r + const.$$

so, calling the constant $-2m$, the result is

$$e^{\nu} = e^{-\lambda} = 1 - \frac{2m}{r}$$

and ds^2 is

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2\left(d\theta^2 + \sin^2\theta d\psi^2\right)$$

This is the Schwarzschild metric and describes the gravitational field outside a central, spherically symmetric body.

To understand the meaning of constant m, we can consider the non relativistic limit

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \simeq c^2 dt^2$$

and a weak gravitational field

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad |h_{\mu\nu}| \ll 1$$

in which

$$h_{44} = \frac{2}{c^2}\phi$$

Since now

$$g_{44} = 1 + h_{44} = e^{\nu} = 1 - \frac{2m}{r}$$

that implies

$$\frac{2m}{r} = -\frac{2}{c^2}\phi$$

but, for a body of mass M, $\phi = -\frac{GM}{r}$ and so

$$m = \frac{GM}{c^2}$$

is the gravitational mass of the source in the relativistic units; it is a length and for this reason

$$R_S = 2m$$

is usually called *gravitational radius* or *Schwarzschild radius*.

Body	R_S *cm*
Sun	1.475×10^5
Earth	0.5
Proton	10^{-50}

It's important to note that when $r = R_S = 2m$ there is a singularity of the metric.

Classic Tests of General Relativity.

Einstein himself suggested three tests to prove the validity of G.R.: the precession of perihelia, the deflection of light rays and the gravitational redshift. All of them are carried out in empty space and in static and spherically symmetric gravitational fields.

Precession of perihelia.

The motion of a free particle is described by the geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

taking

$$x_1 = r \qquad x^2 = \theta \qquad x^3 = \varphi \qquad x^4 = t$$

and

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\nu,\mu} + g_{\beta\mu,\nu} - g_{\mu\nu,\beta})$$

Chosing for simplicity the initial conditions $\theta = \frac{\pi}{2}$ and $\frac{d\theta}{ds} = 0$, that is equivalent to confine the motion on a plain, we find the two solutions

$$r^2\frac{d\varphi}{ds} = h$$

$$\frac{dt}{ds} = ke^{-\nu} = \frac{k}{\gamma}$$

where h and k are two integration constants, and we have put $e^{\nu} = \gamma = 1 - \frac{2m}{r}$ and m is a constant related to the mass of the central source as we will see later.
The equation for the radial coordinate is

$$\left(\frac{du}{d\varphi}\right)^2 + u^2 = \frac{k^2 - 1}{h^2} + \frac{2m}{h^2}u + 2mu^3$$

in which $u = \frac{1}{r}$. Differentiating with respect to φ, we obtain two solution: one is $\frac{du}{d\varphi} = 0$, corresponding to $r = const.$, i.e. to a circular orbit. The other case $\frac{du}{ds} \neq 0$, corresponds to an orbit described by the following differential equation

$$\frac{d^2u}{d\varphi^2} + u = \frac{m}{h^2} + 3mu^2$$

to be compared whith the Newtonian equation for a test particle of unit mass that moves around a body of mass M,

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{c^2h^2}$$

This equation differs from the general relativistic equation of the orbit by nonlinear term $3mu^2$. Comparing the two equations we obtain again $m = \frac{GM}{c^2}$. The general relativistic correction is very small; in fact, the ratio between $3mu^2$ and $\frac{m}{h^2}$ is

$$3h^2u^2 = 3u^2r^4\frac{\dot{\varphi}^2}{c^2} = 3\left(r\frac{\dot{\varphi}}{c}\right)^2$$

and represents three times the square of the transverse velocity of the planet measured in units of c. For example the transverse velocity of the Earth is $v_t \simeq 30 \ \frac{Km}{s}$ and so the relativistic corretion to the orbit is $\sim 10^{-8}$. We can see that after a full revolution of the planet, the perihelion has advanced for an angle equal to

$$\Delta\varphi_0 = 6\pi\frac{m^2}{h^2} = 6\pi\frac{m}{L} = \frac{6\pi GM}{c^2a\left(1 - e^2\right)}$$

where e is the eccentricity of the orbit, $L = \frac{m}{h^2}$ is the semilatus rectum and a the major semiaxis.

Introducing in this equation the value of m for the Sun, $m = 1.475 \ \ Km$, and the value of L for Mercury, $L = 55.3 \times 10^6 \ \ Km$, we obtain $\Delta\varphi_0 \simeq 0.1''$. But this is a secular effect that increases with the number of revolution; after 100 years ($\simeq$ 400 revolutions) $\Delta\varphi \simeq 43''$. This value coincides, up to 1%, with the residue in the motion of the perihelion of Mercury left unexplained in the Newton theory of gravitation, after that all the perturbations caused by the other planets are taken into account.

Deflection of Light Rays.

We now regard a light ray as a beam composed of a large number of photons, and we'll study the path of the test particles moving with a speed c and a vanishing rest mass. Such a kind of particles follows, like in special relativity, a null geodesic, $ds^2 = 0$. In the case of the Schwarzschild metric we obtain:

$$\frac{d^2u}{d\varphi^2} + u = 3mu^2$$

To the first order, the total deviation angle from a stright line, is:

$$\delta = 4\frac{m}{R} = 4\frac{GM}{Rc^2}$$

were R is the distance between the body centre and the light ray undeflected. For a light ray coming from a distant star and passing just outside the Sun surface, we have $R \simeq 7 \times 10^{10} \ \ cm$ and then a maximum deflection angle

$$\delta = 1.75''$$

The Shift of Spectral Line.

Consider the static gravitational line element

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)} - r^2 \left(d\theta^2 + \sin^2\theta d\psi^2\right)$$

The proper time $d\tau = \frac{ds}{c}$ is defined as the time interval between two events whose spatial separation is vanishing, $dr = d\theta = d\varphi = 0$. It is related to coordinate time dt by

$$d\tau = \sqrt{g_{44}}dt = \sqrt{1 - \frac{2m}{r}}dt$$

Given an electromagnetic field $E = ae^{i\psi}$, the frequency of the wave can be expressed as the derivative of ψ with respect to the time, and one has a coordinate frequency $\omega_0 = \frac{\partial \psi}{\partial t}$, and a proper frequency $\omega_0 = \frac{\partial \psi}{\partial \tau}$.
If the wave is emitted at a point P_1 by an atom with a proper frequency ω_1, then at another point P_2, with a different gravitational field, so that $(g_{44})_1 \neq (g_{44})_2$, one will observe a difference proper frequency ω_2, such that

$$\frac{\omega_2}{\omega_1} = \frac{\left(\frac{\partial \psi}{\partial \tau}\right)_2}{\left(\frac{\partial \psi}{\partial \tau}\right)_1} = \frac{\frac{\partial \psi}{\partial t}\left(\frac{\partial t}{\partial \tau}\right)_2}{\frac{\partial \psi}{\partial t}\left(\frac{\partial t}{\partial \tau}\right)_1} = \frac{\left(\sqrt{g_{44}}\right)_1}{\left(\sqrt{g_{44}}\right)_2}$$

Putting $g_{44} = 1 + \frac{2\phi}{c^2}$, we have, for $\phi_{1,2} \ll c^2$

$$\omega_2 \omega_1 = \frac{\sqrt{1 + \frac{2\phi_1}{c^2}}}{\sqrt{1 + \frac{2\phi_2}{c^2}}} \simeq \left(1 + \frac{\phi_1}{c^2}\right)\left(1 - \frac{\phi_2}{c^2}\right) = 1 + \frac{(\phi_1 - \phi_2)}{c^2}$$

All these three classical tests of G.R. have been confirmed by experimental data and we here report a table containig the experiments and theories realized between 1960 and 1980.

Other Metrics Theories.

We have seen before that matter responds to metric and that the matter itself and possibly other gravitational fields generate the metric. The comparison of metric theories with each other and with experiments is simple in slow motion-weak field limit; this approximation in known as the Post Newtonian Limit.
Let us expand as power series of $\frac{GM}{c^2 r}$ the spherically simmetric line element:

$$ds^2 = \left(1 - 2\alpha \frac{GM}{c^2 r} + 2\beta \left(\frac{GM}{c^2 r}\right)^2 + \dots\right) dt^2 -$$

$$\left(1 + 2\gamma \frac{GM}{c^2 r} + \dots\right)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right)$$

with α, β, and γ unknown dimensionless parameters. In the more general case there are ten parameters, and each metric theory is caracterized by particular values. This is known as the Parametrized Post-Newtonian formalism; it provides the framework for discussing the theories and analysing the sperimental result.

Gravitational Waves.

One of the most interesting problems associated with the field equation of G.R. is that of the possible existence of gravitational waves. It was first investigated by Einstein, using approximated solutions of the linearized field equations, in the so called weak field approximation. In the weak field limit the metric tensor $g_{\mu\nu}$ is not very much different from the Minkowski metric $\eta_{\mu\nu}$, and it can be written

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ represents the small corrections to the flat space-time metric. Since $|h_{\mu\nu}| \ll 1$ the terms of order higher then the first in $h_{\mu\nu}$ can be neglected in the field equation. These, written for a vacuum space and with the use of a particular gauge, assume the form of D'Alambert equation

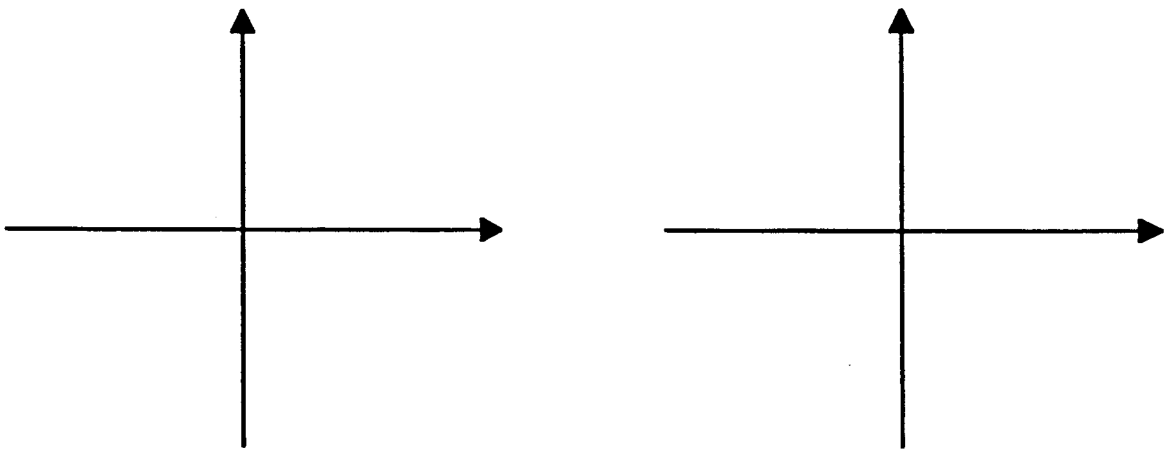
$$\nabla^2 h_{\mu\nu} = 0$$

that describes a weak gravitational perturbation propagating in vacuum with the velocity of light. The solution can be written, in general, as

$$h_{\mu\nu} = Re\left[A_{\mu\nu} e^{ik_\alpha x_\alpha}\right]$$

We report briefly the main features of the gravitational waves:

- Since they propagate at the speed of light the mass of the particles associated to the radiation, *the graviton*, vanishes.
- They are transversal waves.
- They have two states of polarization that implies a spin two for the graviton.



• There isn't dipole radiation, due to the coservation laws; in fact the expression for radiated power is

$$W = \frac{2}{3c^3}\ddot{d} + \frac{G}{5c^5}\dddot{q}_{ik} + \ldots$$

where $d = \Sigma m_i x_i$ is the dipole momentum and so $\dot{d} = const.$ is the momentum. Then the first nonvanishing multipole is the quadrupole.

• The amplitude of a gravitational wave is

$$h(r,t) = \frac{2}{r}\frac{G}{c^4}\left[\ddot{q}\left(t - \frac{r}{c}\right)\right]$$

• The energy transported is

$$W = \frac{c^3}{16\pi G}\left(\dot{h}_+^2 + \dot{h}_\times^2\right) \qquad \left[\frac{J}{m^2 s}\right]$$

To remark the impraticability of the generation of gravitational waves in a laboratory, we report the famous Einstein's experiment: consider a mass $M = 100$ *tons* rotating with a frequency $\nu = 4.5$ Hz; the power emitted as gravitational waves is

$$W \sim 10^{-30} \ \frac{J}{s}$$

that means an amplitude

$$h \sim 10^{-34}$$

The last value represents the percentual displacement of a body knocked by the wave. So to attempt at the detection of gravitational waves we have to look at astrophisical events, as supernova bursts, in which the energy trasported in a millisecond can be of $10^{46}\quad J$ implying

$$h \sim 10^{-17} \frac{1000\ pc}{R} \sqrt{\frac{M_{G.W.}}{10^{-2} m_{\odot}}}$$

where R is the distance of the supernova in pc and $M_{G.W.}$ is the mass of the star converted to gravitational waves during the explosion.