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Laboratori Nazionali di Frascati

LNF–93/071 (IR)
17 Novembre 1993

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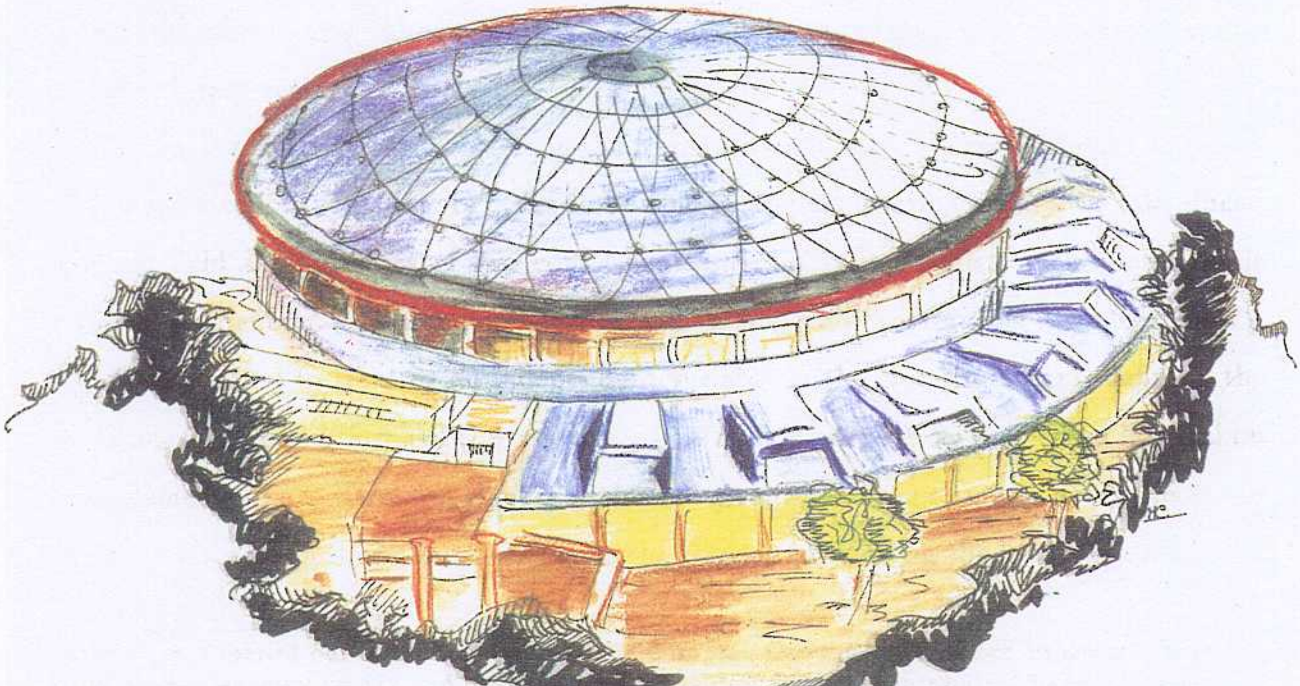
NILPOTENT COMMUTING SCALAR FIELDS AND RANDOM WALK

PACS.: 11.10.Gh



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NILPOTENT COMMUTING SCALAR FIELDS AND RANDOM WALK

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Abstract We show that the correlation functions of a free nilpotent commuting scalar field are equal to the correlation functions of a random walk where paths with odd number of crossings give a negative contribution. It follows that in a number of dimensions where the self-avoiding random walk is a free theory in the continuum limit, the nilpotent commuting scalar behaves as an ordinary scalar. We show how a nilpotent commuting scalar can be related to fermionic constituents and we discuss a model with two flavours and the coupling to an abelian gauge field.

The main purpose of the present paper is to evaluate the correlation functions of free nilpotent commuting scalar fields. We will show that they are simply related to the correlation functions of a random walk in such a way that in a number of dimensions where the self-avoiding random walk is a free theory in the continuum limit they describe a free ordinary scalar.

We were led to study this problem in the framework of a model of gauge fields composite of fermionic constituents [1]. In order to be able to do perturbation theory, we introduced a scalar field as a product of two constituent anticommuting fields. Such a product is an even element of a Grassmann algebra, i.e. a nilpotent commuting variable (NCV), which we wanted to treat as an independent variable in the Berezin integral defining the partition function in terms of the constituents. For this purpose we defined an integral on even elements of a Grassmann algebra such as to give, when even elements are expressed in terms of the odd ones, the same results as the Berezin integral on the latters. It remained

[1]This work is carried out in the framework of the European Community Research Program"Gauge theories, applied supersymmetry and quantum gravity" with a financial contribution under contract SC1-CT92-0789 .

to evaluate the propagator of the nilpotent scalar field according to such an integration rule.

There are additional motivations to study NCV. Before discussing them we should mention that even elements of a Grassmann algebra are characterized by the order of nilpotency, which is the smallest integer n^* such that $(NCV)^n = 0$ for $n > n^*$. The order of nilpotency is changed by a linear transformation. If a_1, a_2, are NCV of order 1, for instance, their sum is of order 2. Therefore if a field is nilpotent of order 1, its Fourier components are nilpotent of infinite order, or of order equal to the number of lattice sites if the field is defined on a lattice. And vice versa.

Let us now come back to the additional motivations to study nilpotent commuting fields. Once we have a convenient formalism, we can consider models involving such fields independently from the way they might be related to fermionic constituents. For instance we know that the ϕ^4 theory with negative coupling is perturbatively asymptotically free, but its euclidean action is unbounded from below, so that its partition function is undefined [2]. This obstruction does not exist if the ϕ-field is a NCV. If the perturbative behaviour of the model with the nilpotent field turns out to be similar to that of the ordinary model, we have an asymptotically free theory, whose field can be associated to fermionic constituents. Such a possibility is under investigation for a ϕ-field whose Fourier components are complex NCV of order 1 [3], in which case the propagator can be easily evaluated.

In the present paper we investigate the complementary case where the ϕ-field itself is a complex NCV of order 1. The propagator turns out to be equal to the correlation function of a random walk where paths with an odd number of crossings give a negative contribution. As a consequence, in a number of dimensions where the self-avoiding random walk is a free theory in the continuum limit, the propagator of nilpotent commuting scalars is that of an ordinary scalar. This happens for dimension greater than 4 and conjecturally also for dimension 4 [4]. When the propagator is that of a free particle we can study interactions of these fields independently from the way they are related to fermionic constituents. Such a study is also preparatory to the investigation of the more complicated composite model of gauge fields. We will further comment on this point at the end of the paper.

Obviously a local selfinteraction requires a higher order of nilpotency, so that we have to postpone the study of a ϕ^4 model till when the propagator of a complex nilpotent field of order 2 (equivalent to a real one of order 4) will be determined. We therefore consider two other possibilities, namely a model with two flavours and the coupling to an abelian field. Both models turn out to be nonrenormalizable due to the impossibility of writing the necessary counterterms. A necessary condition for renormalizable interactions is therefore a higher order of nilpotency.

Let us start by reporting the definition of integral. For a single complex NCV of order 1

$$a: \quad a^2 = 0, \quad a\, a^* = a^* a \tag{1}$$

the integral is defined according to

$$\int da^* da\; a^* a = 1 \tag{2}$$

all other integrals vanishing.If

$$a = c_1 c_2, \quad a^* = c_2^* c_1^* \tag{3}$$

the c_i' being odd Grassmann variables, the definition (1) gives the same result as Berezin integration over the c_i's

$$\int dc_1^* dc_1 dc_2^* dc_2 c_2^* c_1^* c_1 c_2 = 1. \tag{4}$$

Notice that according to such a definition

$$\int da^* da \; exp(a^* a) = 1 \tag{5}$$

with a plus sign in the exponent.

The generalization to more degrees of freedom

$$a_h: \quad a_h^2 = 0, \quad a_h a_k = a_k a_h, \;\; a_h^* a_k = a_k a_h^* \tag{6}$$

is obvious and the integral is defined according to

$$\int \prod_h da_h^* da_h a_h^* a_h = 1 \tag{7}$$

all other integrals vanishing. It is then easy to see that

$$\int [da^* da] \exp \sum_{h,k} a_h^* A_{h,k} a_k = per(A) \tag{8}$$

where

$$[da^* da] = \prod_h da_h^* da_h \tag{9}$$

and per(A) is the permanent of the matrix A.

Note that the integration measure is invariant under gauge transformations

$$a_h \to e^{i\theta_h} a_h, \tag{10}$$

which enables us to construct the coupling to a gauge field. But it is not invariant under orthogonal transformations, so that the permanent and therefore the propagator cannot be evaluated by diagonalization.

We can now consider the free theory of a complex scalar field $\phi(x)$ which is a NCV of order 1. To take into account nilpotency it is convenient that the arguments of the nilpotent variables be discrete. Therefore we define our system on a lattice. We will work in euclidean space.

The partition function is defined according to the above integration rule

$$Z_0 = \int [d\phi^* d\phi] e^{S_0}. \tag{11}$$

Note the plus sign in the exponent, which, as we will see below, is necessary to have the right propagator. The action

$$S_0 = S_H + S_{\tilde{M}} \tag{12}$$

is split into the usual hopping term

$$S_H = a^2 \sum_x \sum_\mu \phi^*(x)[\phi(x+\mu) + \phi(x-\mu)], \quad \mu_\nu = \delta_{\mu,\nu} \tag{13}$$

and a "mass" term

$$S_{\tilde{M}} = a^4 \sum_x \tilde{M}^2 \phi^* \phi. \tag{14}$$

In the above equations a is the lattice spacing, and the sum over x extends on the N^4 sites of a cubic lattice of edge L. It is easy to verify that such a partition function satisfies reflection positivity. This follows from the locality of the action in the same way as for the ordinary theory [5].

Free propagators are defined as

$$G_0(x-y) = < \phi^*(x)\phi(y) >_0 = \frac{1}{Z_0} \int [d\phi^* d\phi] \phi^*(x)\phi(y) \exp S_0. \tag{15}$$

Since there is no systematic way of evaluating the permanent of the wave operator, we make recourse to the hopping expansion

$$< \phi^*(x)\phi(y) >_0 = \frac{1}{Z_0}\sum_{\tau=0}^{\infty}\frac{1}{\tau!} << \phi^*(x)\phi(y)(S_H)^\tau >>, \tag{16}$$

$$Z_0 = \sum_{\tau=0}^{\infty}\frac{1}{\tau!} << (S_H)^\tau >>, \tag{17}$$

where

$$<< \mathcal{O} >> = \int [d\phi^* d\phi]\mathcal{O}\exp S_{\tilde{M}}. \tag{18}$$

In such an expansion factorization of non connected diagrams is precluded by nilpotency, which forbids nodes with occupation larger than 2. From this point of view the situation is analogous to that of a spinless fermionic field, and we are going to exploit this analogy to obtain an expansion in connected paths. Let us consider the correlation function (15) for a fermionic field. All the above formulae are valid for this field as well, if the integral on even elements of the Grassmann algebra is replaced by the Berezin integral. Now the expansion of the correlation function can be arranged into a sum of connected paths which is obviously equal to the one defining the random walk. The decomposition of diagrams into paths is as follows. Connected self-avoiding diagrams are identified with connected self-avoiding paths. Non connected diagrams are identified with non connected paths. Diagrams with nodes with occupancy higher than 2 are decomposed into non connected paths plus connected paths with crossings, in such a way that their sum is zero. Precisely, a path joining two lattice sites by τ links has an absolute value equal to $a^{-2}(a\tilde{M})^{-2(\tau+1)}$, and its contribution is positive/negative if it contains an even/odd number of loops. It is essential for us that the paths originating from a diagram which violates the exclusion constraint can also be classified according to the number of crossings. The paths with even number of crossings have opposite sign w.r. to those whith an odd number. This follows from the fact that paths which differ by one loop also differ by one crossing. We assume the same decomposition of diagrams into paths for a NCV of order 1, but define the sign of the contribution of a path according to the number of its crossings, positive/negative if this number is even/odd. This rule is necessary to agree with the sign of non connected paths originating from diagrams non violating the exclusion constraint, which are all positive irrespective of the number of loops. Therefore it is easy to see that

$$<< \phi^*(x)\phi(y)(S_K)^\tau >>= \tau!(a^2)^\tau(a^4\tilde{M}^2)^{-(\tau+1)}Z_0[r_\tau^{s.a.}(x,y) + r_\tau^e(x,y) - r_\tau^o(x,y)]. \quad (19)$$

In the above equation $r_\tau^{s.a.}(x,y)$ is the number of self-avoiding paths of τ links joining the sites x, y, while $r_\tau^{e/o}(x,y)$ is the number of paths whith an even/odd number of crossings. Collecting our results

$$< \phi^*(x)\phi(y) >_0= \frac{1}{a^2}\sum_{\tau=0}^{\infty}(a^2\tilde{M}^2)^{-(\tau+1)}[r_\tau^{s.a.}(x,y) + r_\tau^e(x,y) - r_\tau^o(x,y)]. \quad (20)$$

The above formula is easily generalized to the 2n-point correlation functions

$$< \phi^*(x_1)..\phi^*(x_n)\phi(y_1)..\phi(y_n) >_0= \frac{1}{a^{2n}}\sum_p \sum_{\tau_1..\tau_n=0}^{\infty} (a^2\tilde{M})^{-(\tau_1+..\tau_n+n)}$$
$$[r_{\tau_1..\tau_n}^{s.a.}(x_1..x_n, y_{p(1)}..y_{p(n)}) + r_{\tau_1..\tau_n}^e(x_1..x_n, y_{p(1)}..y_{p(n)}) - r_{\tau_1..\tau_n}^o(x_1..x_n, y_{p(1)}..y_{p(n)})](21)$$

where {p(i)} are the permutations of {i} and the meaning of the other symbols should be obvious. In the derivation of the above equation we have again exploited the analogy to the fermionic case, where now paths contribute with a sign which depends not only on the number of loops but also on the permutation p. Going to the nilpotent commuting scalar field we observe that paths which differ by a permutation of two end points also differ by the number of crossings.

If the self-avoiding random walk is a free theory in the continuum limit in 4 dimensions [4] (the paths with crossings do not contribute), so is the nilpotent commuting scalar field. It is essential for this conclusion that the paths with crossings have the same absolute value as in the random walk.

To put this paper in the perspective of the model of composite gauge fields, we express now the ϕ-field in terms of fermionic fields $\lambda_i(x), i = 1,2$

$$\phi(x) = \lambda_1(x)\lambda_2(x). \tag{22}$$

Performing this change of variables in the partition function we get

$$Z_0 = \int [d\lambda^* d\lambda] e^{-S_\lambda} \tag{23}$$

where

$$S_\lambda = -a^4 \sum_x \tilde{M}^2 \lambda_2^* \lambda_1^* \lambda_1 \lambda_2 - S_H. \tag{24}$$

In the above equation S_H must be understood as a function of the fermionic fields. The action S_λ gives for the composite ϕ-field the same correlation functions as the action

$$S'_\lambda = a^2 \sum_x \tilde{M}[\lambda_1^* \lambda_1 + \lambda_2^* \lambda_2] - S_H. \tag{25}$$

Moreover, since

$$< \lambda_i^*(x)\lambda_i(y) >= 0 \; for \; x \neq y \tag{26}$$

the constituent fermions do not propagate. In conclusion the action (12) for a nilpotent commuting scalar is equivalent to the action (25) of two nonpropagating fermions with the attractive quartic interaction S_H.

Let us now speculate on possible interactions of our nilpotent scalar field assuming its correlation functions to be the free ones in the continuum limit. Let us start with the coupling to an abelian gauge field. In scalar electrodynamics a quartic selfcoupling of the scalar field is necessary to make the theory renormalizable.It provides the counterterm for the 4-point ϕ-field correlation function which to one loop has a divergent part due to exchange of two photons. We cannot write such a counterterm but one might hope that, due to nilpotency, it is no longer necessary. Unfortunately, we will see that this is not the case.

We assume the usual interaction lagrangian density

$$\mathcal{L}_\phi = \phi^*(x)\frac{1}{a^2}[U_\mu(x)\phi(x+\mu) + U^*_\mu(x-\mu)\phi(x-\mu)] + \tilde{M}^2\phi^*(x)\phi(x) \tag{27}$$

where U_μ is Wilson link variable. In the Feynman gauge

$$< A_\mu(x)A_\nu(y) >_0 = \delta_{\mu,\nu}K(x-y) \tag{28}$$

where

$$K(y) = \frac{1}{L^4}\prod_\mu \sum_{|n_\mu|<\frac{N}{2}} K_n e^{i\frac{2\pi}{N}ny}. \tag{29}$$

The Fourier transform appearing in the above equation is

$$K_n = \frac{1}{2\sum_\mu[1-cos(\frac{2\pi}{N}n_\mu)]}. \tag{30}$$

We can now evaluate the 4-point ϕ-field correlation function to one loop

$$\begin{aligned} < \phi^*(x_1)\phi(x_2)\phi^*(x_3)\phi(x_4) > &= a^8\sum_{y,z}\mathcal{F}(x_1,x_2,x_3,x_4,y,z) \\ &\quad [K^2(y-z) - \delta_{y,z}K^2(0)] \end{aligned} \tag{31}$$

where

$$\mathcal{F} = [(1-\delta_{x_1,z})(1-\delta_{x_2,z})(1-\delta_{x_3,y})(1-\delta_{x_4,y}) + y \rightleftharpoons z]\mathcal{G}(x_1,x_2,x_3,x_4). \tag{32}$$

$\mathcal{G}$ is a product of free propagators G_0. The second term of Eq.(31) cannot cancel the divergence of the first one, because it remains finite in the limit a$\rightarrow\infty$

$$\delta_{y,z} K^2(0) \rightarrow \delta(y-z) \frac{1}{(2\pi)^4} \prod_\nu \int_{-\pi}^{\pi} d\eta_\nu \frac{1}{2\sum_\mu (1 - cos\ \eta_\mu)}. \tag{33}$$

In the above equation $\delta(y-z)$ is the Dirac function (while $\delta_{y,z}$ is the Kronecker function). The 4-point function of the ϕ-field has therefore a divergent contribution for which we cannot write a counterterm.

Let us finally consider 2 flavours, with the lagrangian density

$$\mathcal{L} = \sum_{i=1,2} \{\phi_i^*(x) \frac{1}{a^2} [\phi_i(x+\mu) + \phi_i(x-\mu)] + \tilde{M}^2 \phi_i^*(x) \phi_i(x)\} + g\phi_1^*(x)\phi_1(x)\phi_2^*(x)\phi_2(x). \tag{34}$$

To one loop we have a divergent contribution to the 4-point function $< \phi_1^*(x_1)\phi_1(x_2)\phi_1^*(x_3)\phi_1(x_4) >$ for wich we cannot write a counterterm.

In conclusion, to have renormalizable interactions we need nilpotent fields of higher order.

Let us summarize our results.We have studied the free theory of a complex nilpotent commuting scalar of order 1. We have found that it can be related to a random walk where paths with odd number of crossings give a negative contribution, in such a way that in a number of dimensions where the self-avoiding random walk is a free theory, a free nilpotent commuting scalar behaves as an ordinary scalar.

We also addressed the problem of renormalizability. We found that the criterion of power counting does not work with NCV, since some counterterms, which exist with ordinary variables, cannot be written because of nilpotency.

On the basis of our analysis, we do not see any a priori obstruction for renormalizability of the models we considered with NCV of higher order. Actually we regard this as an interesting possibility, also in connection with the quoted model of composite gauge fields, where nilpotent commuting scalars of higher order can be introduced as well. Let us finally emphasize the relevance of this possibility to an asymptotically free ϕ^4 model where the ϕ-field is a real NCV of order 4.

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