

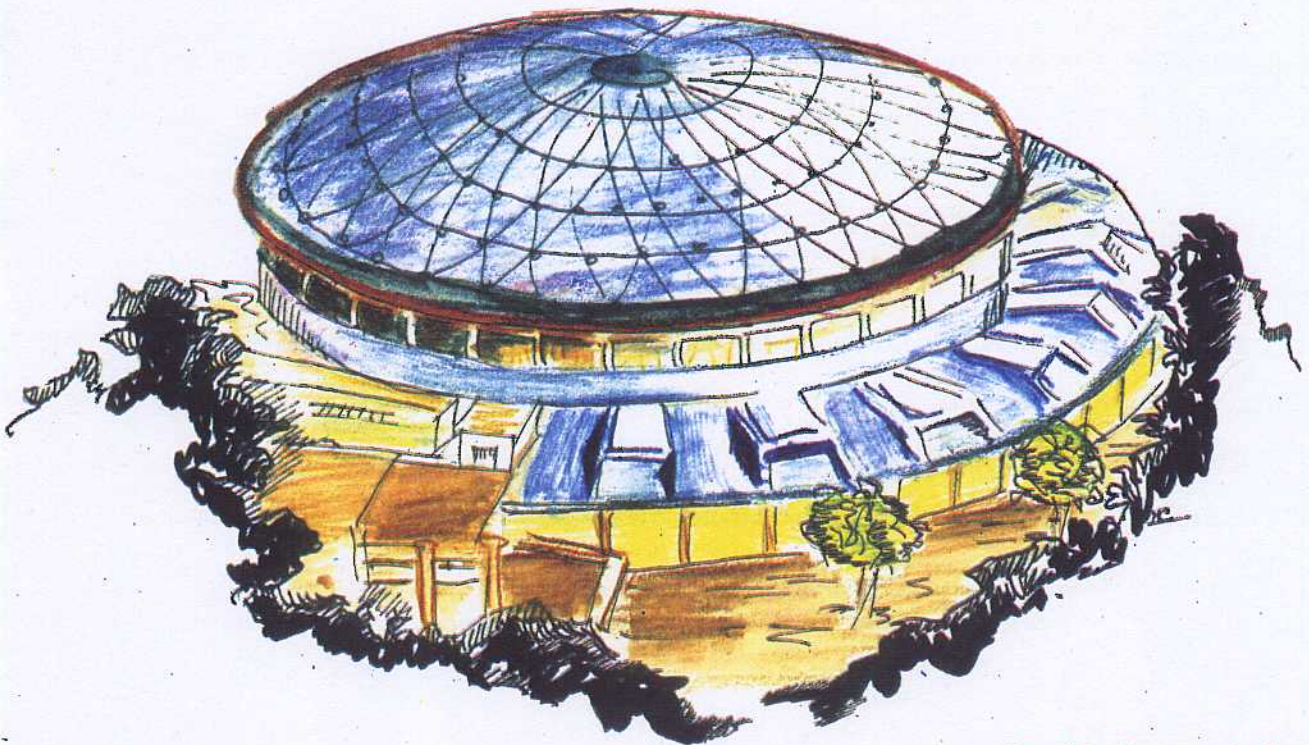


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ϕ^4 THEORY WITH EVEN ELEMENTS OF A GRASSMANN
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ϕ^4 theory with even elements of a Grassmann algebra

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Abstract Bilinear composites of anticommuting constituents are even elements of a Grassmann algebra, which are nilpotent commuting variables (NCV). We study a ϕ^4 theory where the Fourier components of the ϕ -field are NCV and suggest that it is asymptotically free.

1-Bilinear composites of fermionic constituents arise in many areas of physics. Well known examples are met in the theory of superconductivity, the Tomonaga model of the electron gas, the theory of spin waves in ferro-antiferromagnetic metals, the Interacting Boson Model in Nuclear Physics. In Particle Physics such composites appear for instance in the framework of dynamical symmetry breaking, as well as, of course, in models of composite fields.

In a path integral formulation bilinear composites of anticommuting constituents are even elements of a Grassmann algebra, which are nilpotent commuting variables (NCV). When one is interested in correlation functions which do not involve the constituent fields but only those combinations which define the composites, one would like to be able to treat the composites themselves as independent variables. This would be particularly useful in the presence of confinement, because one could get rid of the constituents altogether, but it would be in general helpful in deriving an effective action from the fundamental one.

We considered a method of treating NCV as independent variables in the framework of a model of composite gauge fields with fermionic constituents [1]. This method is based on a

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definition of integral on even elements of a Grassmann algebra which, as far as composites correlation functions are concerned, gives the same result as the Berezin integral on the odd elements. We apply here this method to a ϕ^4 model where the Fourier components of the ϕ -field are NCV [2]. Of course nilpotent commuting ϕ -fields are nonlocal, in agreement with their nature of composites, so that a theory with these variables might turn out to be meaningful only as an effective theory.

In this paper we are not interested in how the ϕ -field might be related to fermionic constituents, but only in the properties of the model. We will see that it has perturbative UV properties similar to those of the ordinary theory, but the nonperturbative features are quite different. The ordinary ϕ^4 theory with negative coupling constant is perturbatively asymptotically free [3], but it has a euclidean action unbounded from below, so that its partition function is undefined. With NCV the partition function is instead well defined *irrespective* of the sign of the coupling. Of course this is not enough to ensure the existence of the theory at a nonperturbative level, but it opens the way to new possibilities.

A specific feature of NCV should be mentioned at the outset: The order of nilpotency (the smallest integer n^* for which $(NCV)^n=0$ for $n > n^*$) is changed by a linear change of variables. Such a feature might appear a too severe limitation for the notion of NCV of a given order to be of practical use. It would in fact seem that it is incompatible with most symmetries. This however is not the case, as it is shown by their relevance in the model of composite gauge fields with local gauge invariance [1] , where the ϕ -field is a NCV of order 1, and by the ϕ^4 theory we are going to investigate where Lorentz invariance can be imposed with fields whose Fourier components are NCV of order 1 .

Let us start by reporting the definition of integral. For a single complex NCV of order 1

$$a : \quad a^2 = 0, \quad a a^* = a^* a \quad (1)$$

the integral is defined according to

$$\int da^* da a^* a = 1 \quad (2)$$

all other integrals vanishing. If

$$a = c_1 c_2, \quad a^* = c_2^* c_1^* \quad (3)$$

the c_i 's being odd Grassmann variables, the definition (1) gives the same result as the Berezin integral on the c_i 's

$$\int dc_1^* dc_1 dc_2^* dc_2 c_2^* c_1^* c_1 c_2 = 1. \quad (4)$$

Notice that according to such a definition

$$\int da^* da \exp(a^* a) = 1 \quad (5)$$

with a plus sign in the exponent.

The generalization to more degrees of freedom

$$a_h : \quad a_h^2 = 0, \quad a_h a_k = a_k a_h, \quad a_h^* a_k = a_k a_h^* \quad (6)$$

is obvious and the integral is defined according to

$$\int \prod_h da_h^* da_h a_h^* a_h = 1 \quad (7)$$

all other integrals vanishing. It is then easy to see that

$$\int [da^* da] \exp \sum_{h,k} a_h^* A_{h,k} a_k = \text{per}(A) \quad (8)$$

where

$$[da^* da] = \prod_h da_h^* da_h \quad (9)$$

and $\text{per}(A)$ is the permanent of the matrix A .

Note that the integration measure is invariant under phase transformations

$$a_h \rightarrow e^{i\theta_h} a_h \quad (10)$$

but not under orthogonal transformations. We cannot therefore evaluate the permanent of A by diagonalizing A .

To take into account nilpotency it is convenient that the arguments of the nilpotent variables be discrete. Therefore we consider our system in a box, which is sufficient for nilpotent variables in momentum space. We will work in euclidean space.

We define the Fourier transform of a function f

$$f(x) = \frac{1}{L^2} \sum_p \tilde{f}(p) e^{ipx} \quad (11)$$

where L is the side of the box and the sum is over the discrete momenta

$$p_\mu(n) = \frac{2\pi}{L} n_\mu, \quad n_\mu \text{ integer.} \quad (12)$$

For a scalar field

$$\tilde{\phi}(p) = \frac{1}{\omega(p)} [a^*(p) + a(-p)], \quad (13)$$

where

$$\frac{1}{\omega(p)} = (p^2 + m^2)^{-\frac{1}{2}} \exp \frac{-p^2}{\Lambda^2} \quad (14)$$

is the regulated propagator. We are going to investigate the case where $a^*(p), a(p)$ are NCV of order 1. Note that, with the regularization (14), the ϕ -field is of infinite order of nilpotency, but it can be made of finite order by, for instance, a lattice regularization.

We assume the free action to be

$$S_0 = \int d^4x \phi(x) [-\square + m^2]_\Lambda \phi(x), \quad (15)$$

the wave operator being regulated in such a way as to have the propagator (14).

Because of the invariance of the integration measure under the transformations (10), S_0 is invariant under Lorentz transformations (in the formal limit of continuous momenta). Note that S_0 differs from the action of ordinary scalars by a factor $\frac{1}{2}$.

Free propagators are defined by

$$\langle \tilde{\phi}(p_1) \tilde{\phi}(p_2) \rangle_0 = \frac{1}{Z_0} \int [da^* da] \tilde{\phi}(p_1) \tilde{\phi}(p_2) \exp S_0 \quad (16)$$

where

$$Z_0 = \int [da^* da] \exp S_0. \quad (17)$$

This definition is analogous to that of the propagators of ordinary scalars in terms of holomorphic variables, apart from the plus sign in the exponent. As we will see this sign is necessary, as a consequence of eq.(5), to get the right propagator.

It is convenient to introduce the variables

$$A(p) = a^*(p) + a(-p); \quad B(p) = A(p)A(-p) \quad (18)$$

which simplify the expression of

$$S_0 = \sum_p B(p) \quad (19)$$

and

$$\exp S_0 = [1 + B(0)] \prod_p^* [1 + 2B(p) + 2B^2(p)]. \quad (20)$$

The star means that the product must be taken over all directions and absolute values of $p \neq 0$, but not over its orientations.

In order to evaluate correlation functions we must arrange the products of $\tilde{\phi}(p)$ into products of $B(p)$'s and use the relations

$$\int [da^* da] B(0) \exp S_0 = Z_0 \quad (21)$$

$$\int [da^* da] B(p) \exp S_0 = Z_0; \quad \int [da^* da] B^2(p) \exp S_0 = \frac{1}{2} Z_0, \quad p \neq 0. \quad (22)$$

In such a way we get for the free propagator

$$\langle \tilde{\phi}(p_1) \tilde{\phi}(p_2) \rangle_0 = \delta(p_1 + p_2) \frac{1}{\omega^2(p_1)}, \quad (23)$$

namely the expression valid for ordinary scalars. Many-point correlation functions, however, are different from the corresponding ones of ordinary scalars, due to the exclusion principle obeyed by $\tilde{\phi}(p)$.

Let us now introduce the interaction

$$S_I = \frac{1}{4!} g \int d^4 x \phi^4(x) = \frac{1}{4!} g \frac{1}{L^4} \sum_{p_1, p_2, p_3, p_4} \delta(p_1 + p_2 + p_3 + p_4) \prod_{i=1}^4 \frac{A(p_i)}{\omega(p_i)}, \quad (24)$$

and consider the correlation functions

$$\langle \tilde{\phi}(p_1) \dots \tilde{\phi}(p_n) \rangle = \frac{1}{Z} \int [da^* da] \tilde{\phi}(p_1) \dots \tilde{\phi}(p_n) \exp(S_0 + S_I), \quad (25)$$

where

$$Z = \int [da^* da] \exp(S_0 + S_I). \quad (26)$$

We will perform the usual loop expansion, but it must be observed that at finite L it cannot be arranged exactly into an expansion of connected terms. Consider in fact a disconnected term consisting of two pieces, one involving some $B(p)$ and the other one $B^2(p)$ with the same p . One such term is present in the expansion of Z in the denominator. But to be present also in the expansion of the numerator, it should have been originated by a contraction of a factor $B^3(p)$, which is instead vanishing. This is an example of a disconnected term which is not cancelled. The contribution of such terms to n -point correlation functions, however, is negligible in the thermodynamic limit, unless n is of the order of $(\frac{L\Lambda}{2\pi})^4$. Therefore, unless we are studying nonperturbative properties, we can disregard disconnected terms.

The 2-point connected correlation function to one loop is

$$\begin{aligned} \langle \tilde{\phi}(p_1) \tilde{\phi}(p_2) \rangle_1 &= \frac{1}{Z_0} \frac{1}{4!} g \int [da^* da] \frac{A(p_1)}{\omega(p_1)} \frac{A(p_2)}{\omega(p_2)} \\ &\quad \frac{1}{L^4} \sum_{q_1, q_2, q_3, q_4} \delta(q_1 + q_2 + q_3 + q_4) \prod_{i=1}^4 \frac{A(q_i)}{\omega(q_i)} \exp S_0 |_C \end{aligned} \quad (27)$$

where the subscript C means that only connected contributions should be included. To get one of such contributions we must pair p_1 to one of the q_i and p_2 to one of the remaining q_i . There are 12 ways to do that. For each such way, after rearranging the A's into B's we get

$$\langle \tilde{\phi}(p_1)\tilde{\phi}(p_2) \rangle_1 = \delta(p_1 + p_2) \frac{1}{\omega^4(p_1)} \frac{1}{Z_0} \frac{1}{2} g \int [da^* da] 2B^2(p_1) \frac{1}{L^4} \sum_q \frac{B(q)}{\omega^2(q)} \exp S_0 \quad (28)$$

so that finally integrating over the $a^*(p), a(p)$ we find

$$\langle \tilde{\phi}(p_1)\tilde{\phi}(p_2) \rangle_1 = \delta(p_1 + p_2) \frac{1}{\omega^4(p_1)} \frac{1}{2} g \frac{1}{L^4} \sum_{q \neq \pm p_1} \frac{1}{\omega^2(q)}. \quad (29)$$

The restriction $q \neq \pm p$ has been kept to give an example of a consequence of nilpotency, but it is obviously irrelevant.

Let us now evaluate to one loop the 4-point connected correlation function

$$\begin{aligned} \langle \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) \rangle_1 &= \frac{1}{Z_0} \frac{1}{2} \frac{g^2}{(4!)^2} \frac{1}{L^8} \sum_{q_i, t_i} \int [da^* da] \delta(q_1 + q_2 + q_3 + q_4) \\ &\quad \delta(t_1 + t_2 + t_3 + t_4) \prod_{i=1}^4 \frac{A(p_i)A(q_i)A(t_i)}{\omega(p_i)\omega(q_i)\omega(t_i)} \exp S_0 |_C \quad (30) \end{aligned}$$

The above equation holds for momenta p_i different from one another. Proceeding as above we find that its divergent contribution is

$$\langle \tilde{\phi}(p_1)\tilde{\phi}(p_2)\tilde{\phi}(p_3)\tilde{\phi}(p_4) \rangle_{1div} = \delta(p_1 + p_2 + p_3 + p_4) \prod_{i=1}^4 \frac{1}{\omega^2(p_i)} \frac{1}{L^4} \frac{3}{16\pi^2} g^2 \ln \frac{\Lambda}{m}. \quad (31)$$

This result is the same as in the ordinary ϕ^4 theory. Due to the plus sign in the exponential in Z , however, the counterterm has opposite sign w.r. to that of the ordinary theory with the same sign of g , so that the β -function is

$$\beta(g) = -\frac{3}{16\pi^2} g^2. \quad (32)$$

It would be interesting to investigate the UV properties of a model where the ϕ -field itself, rather than its Fourier components, is a NCV of order 1. This is the case relevant to ref.[1].

To have a mass term, such a field must be complex, or there must be more flavours. More flavours are in any case necessary to have a quartic selfinteraction, which can for instance be written in terms of the field

$$\Phi(\mathbf{x}) = \phi_1(\mathbf{x}) + \phi_2(\mathbf{x}) + \phi_1^*(\mathbf{x}) + \phi_2^*(\mathbf{x}), \quad (33)$$

where $\phi_h(\mathbf{x})$ are complex NCV of order 1. The propagator of a complex nilpotent field of order 1 has already been evaluated and found equal to that of a selfavoiding random walk [2].

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