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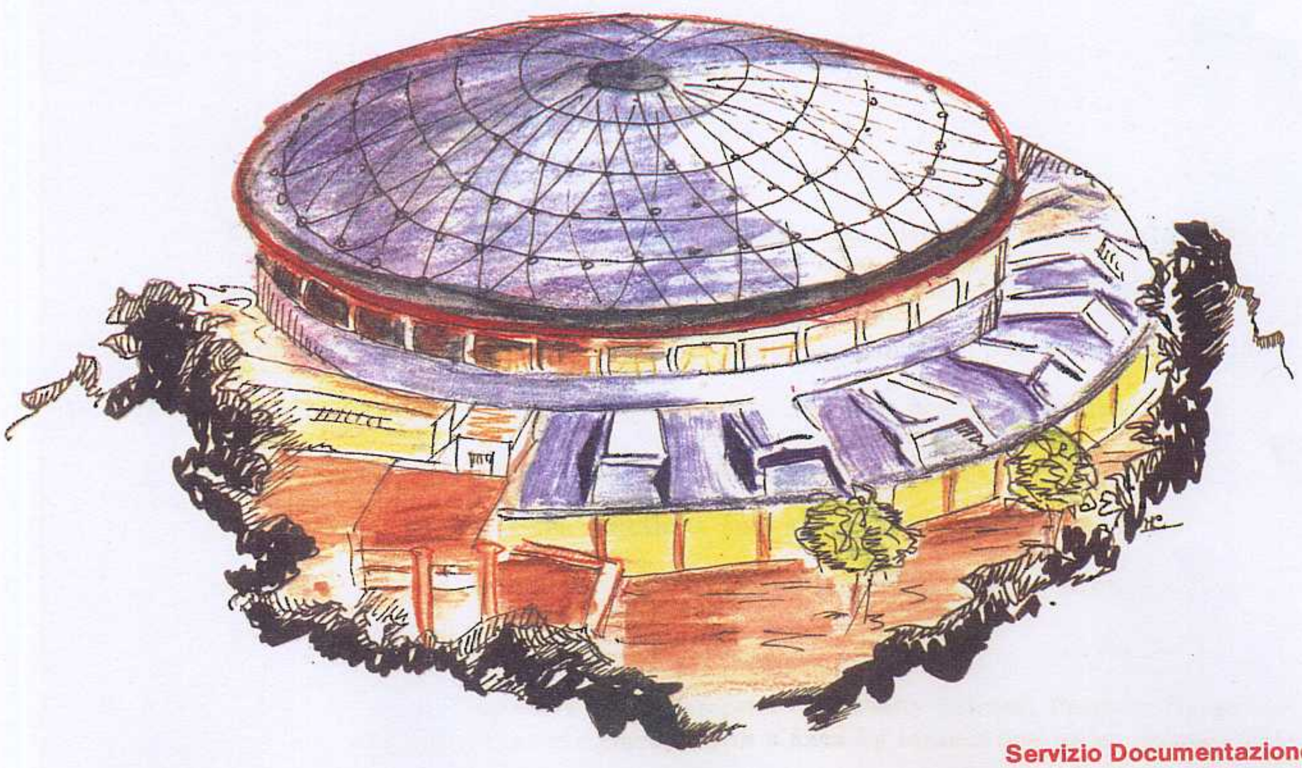

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Gauge transformations and boundary conditions in compact QED on the lattice

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Abstract We show that in the hamiltonian formalism of compact QED on the lattice gauge transformations appear as representations of translations on a circle or on a line according to the boundary conditions on the wave functions.When gauge invariance is enforced, one gets at strong coupling a linear potential in the first case and a Coulombic potential in the second.

1-It is generally accepted that the occurrence of a linear potential at strong coupling is an *intrinsic* feature of the compact formulation of abelian and nonabelian gauge theories on the lattice [1].Contrary to such a belief, some time ago it has been shown that the abelian theory can also be formulated in such a way that the potential is Coulombic at strong coupling [2].This can be achieved in the hamiltonian formalism by imposing appropriate quasiperiodic boundary conditions (b.c.) on the wave functions (not to be confused with the b.c. of the gauge fields as functions of the site, which we will always assume periodic).A partition function incorporating such b.c. has also been constructed.The above results have been derived in the case of static charges.

The interplay of b.c. of the wave functions and gauge transformations, however, has not been elucidated.In this paper we reconsider the problem and we show that gauge

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transformations appear as representations of translations on a circle whith periodic b.c. and translations on a line whith quasiperiodic b.c. of the wave functions.When gauge invariance is enforced we get at strong coupling a linear potential in the first case and a Coulombic potential in the second.

We will discuss the problem using as independent variables the compact gauge fields

$$A_k(x) \; : \quad -\pi \leq A_k(x) \leq \pi. \tag{1}$$

We will then show how it can be reformulated in terms of the Wilson link variables

$$U_k(x) = e^{iA_k(x)} \tag{2}$$

by using a unitary transfomation.

We will first consider the case of the coupling to static sources and then we will outline the generalization to dynamical charged fields.The construction of the partition function in the presence of dynamical charged fields can be done by a suitable modification of the method of ref.(2) for static charges, and it will not be reported here.

2-The gauge field hamiltonian in units $\hbar$=c=1 is

$$H_G = \frac{1}{a}\sum_x\{\frac{e^2}{2}\sum_k E_k^2(x) + \frac{1}{2e^2}\sum_{h,k}[1-\cos(\Delta_h A_k(x) - \Delta_k A_h(x))]\} \tag{3}$$

where e is the electric charge, a the lattice spacing , Δ_h the right derivative

$$\Delta_h\, A_k(x) = A_k(x+e_h) - A_k(x), \quad (e_h)_k = \delta_{h,k} \tag{4}$$

and $E_k(x)$ the electric strength

$$E_k(x) = -i\frac{\partial}{\partial A_k(x)}. \tag{5}$$

We assume the gauge fields to satisfy periodic b.c.

$$A_k(x + e_h L) = A_k(x), \tag{6}$$

where L is the side of the lattice.It then follows from the Gauss constraint that the total charge in the lattice must vanish.

The coupling of the gauge field to static charges takes place only via the Gauss constraint on the admissible wave functions Ψ

$$G(x)\Psi = 0 \tag{7}$$

where

$$G(x) = \sum_k \triangle_k^{(-)} E_k(x) + j_0(x) \tag{8}$$

$\triangle_k^{(-)}$ being the left derivative

$$\triangle_h^{(-)} A_k(x) = A_k(x) - A_k(x - e_h). \tag{9}$$

We start by considering the coupling to two opposite static charges localized at y, z whose charge density is

$$j_0(x) = \delta_{x,y} - \delta_{x,z}. \tag{10}$$

Let $\mathcal{E}_A$ be the Hilbert space of square integrable functions of $A_k(x)$ whith scalar product

$$< \Psi_1 \mid \Psi_2 >= \prod_{x,k} \frac{1}{2\pi} \int_{-\pi}^{\pi} dA_k(x)\ \Psi_1^*\ \Psi_2. \tag{11}$$

In the presence of compact variables there are several different dense domains in the Hilbert space $\mathcal{E}_A$ where the hamiltonian is selfadjoint.Some of these domains are carachterized , among other requirements specified below, by the socalled quasiperiodic b.c.

$$\Psi(A_k(x) = -\pi) = e^{-i2\pi\alpha_k(x)}\Psi(A_k(x) = \pi), \quad \forall\, A_k(x) \tag{12}$$

where the $\alpha_k(x)$ are arbitrary real parameters.We restrict ourselves to those domains, which we call $\mathcal{D}_\alpha$, which are completely characterized in the following way. Let us introduce the complete orthonormal system

$$e^{(\alpha)}_{\{n_k\}} = e^{i\sum_{x,k}(\alpha_k(x)+n_k(x))A_k(x)} \tag{13}$$

where $n_k(x)$ are arbitrary integers .Then the $\mathcal{D}_\alpha$ consist of all the wave functions

$$\sum_{\{n_k\}} c_{\{n_k\}} e^{(\alpha)}_{\{n_k\}} \tag{14}$$

satisfying the condition

$$\sum_{\{n_k\}} \mid c_{\{n_k\}} \mid^2 n_k^4 < \infty. \tag{15}$$

Introducing the functions χ which describe the gauge degrees of freedom of the static charges, the wave functions Ψ can be written

$$\Psi = \sum_{\{n_k\}} c_{\{n_k\}} e^{(\alpha)}_{\{n_k\}} \chi^*(y)\chi(x). \tag{16}$$

Gauge transformations are naturally defined by exponentiating the Gauss operator G(we use the spectral representation of G, which is selfadjoint on a domain containing $\mathcal{D}_\alpha$) .The resulting transformation of the basis functions is

$$e^{i\theta(x)G(x)} e^{\{\alpha_k\}}_{\{n_k\}} = e^{i\theta(x)[\Delta_k^{(-)}(\alpha_k(x)+n_k(x))+j_0(x)]} e^{\{\alpha_k\}}_{\{n_k\}}, \tag{17}$$

$$e^{i\theta(x)G(x)}\chi(x) = e^{i\theta(x)}\chi(x), \tag{18}$$

where $\theta(x)$ is the parameter of the transformation.

Therefore for the total wave function we have

$$(e^{i\sum_x \theta(x)G(x)}\Psi)(A_k) = \sum_{\{n_k\}} c_{\{n_k\}} e^{i\sum_{x,k}(\alpha_k(x)+n_k(x))(A_k(x)-\Delta_k\theta(x))}\chi^*(y)\chi(z). \tag{19}$$

Collecting the terms independent of n_k in the exponent we get

$$(e^{i\sum_x \theta(x)G(x)}\Psi)(A_k) = e^{-i2\pi\sum_{x,k}\alpha_k(x)m(A_k,\theta_k)}\Psi(A_k(x) - \Delta_k\theta(x) + 2\pi m(A_k,\theta)), \tag{20}$$

where the integers $m(A_k, \theta)$ are defined by the condition

$$-\pi < A_k(x) - \Delta_k\theta(x) + 2\pi m(A_k,\theta) \leq \pi. \tag{21}$$

We recognize a representation of the product of the translations group on the line, which, for periodic b.c., $\alpha_k(x) = 0$, can in a natural way be interpreted as a representation of the product of the translations group on the circle, i.e. the U(1) group.

The point is that there are solutions to the Gauss constraint in both cases.In the case of periodic b.c. there are solutions of the form

$$\Psi_L = e^{i\sum_\Gamma A_k(x)}\chi^*(y)\chi(z) \tag{22}$$

where Γ is an arbitrary simple line going from the negative to the positive charge.Such solutions, as it is well known, give rise at strong coupling to a linear potential.

In the case of quasiperiodic b.c. a solution is given by

$$\Psi_C = e^{i\sum_{x,k}\alpha_k(x)A_k(x)}\chi^*(y)\chi(z) \tag{23}$$

where

$$\alpha_k(x) = -\Delta_k\,[\Delta^{-1}(x-y) - \Delta^{-1}(x-z)]. \tag{24}$$

As shown in [2] at strong coupling it gives a Coulombic potential

$$\frac{e^2}{2a}\sum_x\sum_k E_k^2(x)\Psi_C = -\frac{e^2}{a}[\Delta^{-1}(y-z) - \Delta^{-1}(0)]\Psi_C. \tag{25}$$

In order to formulate the problem in terms of the unitary Wilson variables $U_k(x)$, we observe that there is a unitary transformation relating the Hilbert space $\mathcal{E}_A$ to the Hilbert space $\mathcal{E}_U$ of square integrable functions of $U_k(x)$ whith scalar product

$$< \Phi_1 \mid \Phi_2 >= \int \prod_{x,k} d\mu(U_k(x))\ \Phi_1^*\ \Phi_2 \tag{26}$$

where $d\mu(U_k(x))$ is the Haar measure on the group U(1) normalized to 1. Such transformation and its inverse are

$$(V\Psi)(U_k(x)) = \Psi(\frac{1}{i} ln U_k(x)) \tag{27}$$

$$(V^{-1}\Phi)(A_k) = \Phi(e^{iA_k(x)}) \tag{28}$$

By means of this transformation we can rewrite all the above in terms of the $U_k(x)$.

3-We now extend our analysis to the case of dynamical charged fields, confining ourselfs to the case of Dirac fermions described by field operators χ satisfying the anticommutation relations

$$\{\chi^*_{\alpha_1}(x), \chi_{\alpha_2}(y)\} = \delta_{\alpha_1,\alpha_2}\ \delta_{x,y}. \tag{29}$$

For the present discussion we do not need to specify the fermionic hamiltonian. It is sufficient that it be hermitian, bounded, and that it commute whith the Gauss operator.

It is convenient to define the basis states

$$\Lambda_{\beta_1...\beta_n,\gamma_1...\gamma_n}(y_1...y_n, z_1...z_n) = \chi^*_{\beta_1}(y_1)...\chi^*_{\beta_n}(y_n)...\chi_{\gamma_1}(z_1)...\chi_{\gamma_n}(z_n)\Omega \tag{30}$$

where the indices $\beta_i = 1,2$, $\gamma_i = 3,4$ and

$$\chi_\beta(x)\Omega = \chi_\gamma^*(x)\Omega = 0. \tag{31}$$

It should be noted that these states contain the same number of positive and negative charges as required by the Gauss constraint for periodic b.c. of the gauge fields.They are eigenstates of the charge density operator

$$j_0(x) = \frac{1}{2}[\chi^*(x), \chi(x)] = \sum_{\beta=1,2} \chi_\beta^*(x)\chi_\beta(x) - \sum_{\gamma=3,4} \chi_\gamma(x)\chi_\gamma^*(x). \tag{32}$$

The complete orthonormal system in the Hilbert space of the $A_k(x)$ will be denoted by

$$e^{(\alpha)}_{\{n_k\}}(y_1...y_n, z_1...z_n) = e^{i\sum_{x,k}[\alpha_k(x,y_1...y_n,z_1...z_n)+n_k(x)]A_k(x)} \tag{33}$$

where

$$\alpha_k(x, y_1...y_n, z_1...z_n) = -\Delta_k \sum_{i=1}^{n} \Delta^{-1}(x-y_i) - \Delta^{-1}(x-z_i). \tag{34}$$

The wave functions Ψ have the expansion

$$\begin{aligned}\Psi = &\sum_{\{n_k\},\{\beta_i\},\{\gamma_i\}} \sum_{y_1...y_n,z_1...z_n} c_{\{n_k\},\{\beta_i\},\{\gamma_i\}}(y_1...y_n, z_1...z_n) \\ &e^{(\alpha)}_{\{n_k\}}(y_1...y_n, z_1...z_n)\Lambda_{\beta_1...\beta_n,\gamma_1...\gamma_n}(y_1...y_n, z_1...z_n).\end{aligned} \tag{35}$$

It is now easy to see that the $e^{(\alpha)}_{\{n_k\}}(y, z)$ must satisfy the Gauss condition appropriate to static charges localised at $y_1...y_n, z_1...z_n$

$$[\sum_k \Delta_k^{(-)} E_k(x) + \sum_i \delta_{x,y_i} - \delta_{x,z_i}]e^{(\alpha)}_{\{n_k\}}(y_1...y_n, z_1...z_n) = 0. \tag{36}$$

The potential at strong coupling is given by

$$\frac{e^2}{2a}\sum_x\sum_k E_k^2(x)e^{(\alpha)}_{\{0\}}(y_1,..y_n,z_1,..z_n) = -\frac{e^2}{a}\sum_{i,j}[\Delta^{-1}(y_i-z_j)-\frac{1}{2}\Delta^{-1}(y_i-y_j) -\frac{1}{2}\Delta^{-1}(z_i-z_j)]e^{(\alpha)}_{\{0\}}(y_1,..y_n,z_1,..z_n). \quad (37)$$

The Hilbert space of the total system can be written as a direct sum of orthogonal subspaces, $\mathcal{E}_A = \sum_C^{\oplus}\mathcal{E}_{A,C}$, where $\mathcal{E}_{A,C}$ is the set of all the wave functions whith a given configuration C (positions and values) of the charges.It is then easy to see that, over a domain $\mathcal{D} = \sum_C^{\oplus}\mathcal{D}_{\alpha_C}$ where each of the $\mathcal{D}_{\alpha_C}$ is constructed in the previous way, the total hamiltonian is a selfadjoint operator.

In conclusion, we have shown that there are different definitions of compact QED on the lattice .The formulation whith periodic b.c.is simpler, but the one whith quasiperiodic b.c. is more natural from the physical point of view.

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