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Contribution to the DAΦNE Theory Study Group
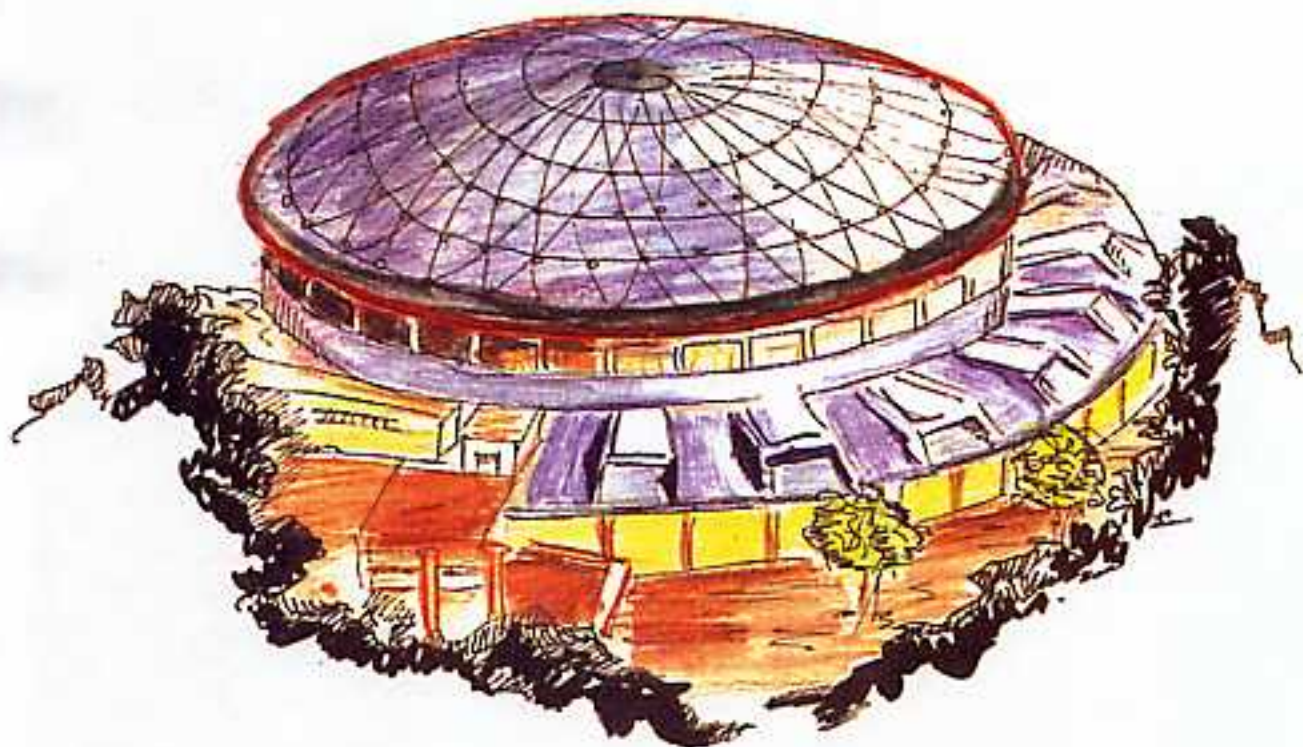
CORRECTIONS TO $K \to \pi\gamma\gamma$ FROM $K \to 3\pi$

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Abstract

In the framework of chiral perturbation theory we compute the $\Delta I = 3/2$ contribution to $K_L \to \pi^0\gamma\gamma$. As for the $\Delta I = 1/2$ transition, the result turns out to be unambiguously predicted and finite. We discuss also the effects on this decay coming from $O(p^4)$ corrections to $K \to 3\pi$. As a by-product we have also a prediction for the CP conserving amplitude $A(K_L \to \pi^0 e^+ e^-)$.

1. Introduction.

Rare meson decays are very useful for our understanding of particle physics [1]. Chiral perturbation theory (CHPT) [2,3] is very suitable for this research; it fulfills this task, satisfying symmetry requirements (the interactions have to be invariant under $SU(3)_L \otimes SU(3)_R$) and having the mesons as the Goldstone bosons associated with spontaneous breaking of the symmetry. Already at tree level all the low energy theorems, PCAC and soft pion properties are recovered. Furthermore, though the theory is non renormalizable, we do require unitarity, which is obtained perturbatively by considering also pion loops. Divergences in the loops are absorbed by corresponding counterterms, which then depend on the renormalization scale μ of the loops. Due to non renormalizability of the theory, new counterterms, which can be determined from experiment [3,4], have to be added order by order to the theory. Actually, the $O(p^4)$ correction to the lowest order strong Lagrangian can be predicted reasonably well by vector meson exchange [5], which also gives an $O(p^4)$ contribution to the $A(K \to 3\pi)$ amplitude, improving the lowest order weak Lagrangian result [6].

$K_S \to \gamma\gamma$ [7] and $K_L \to \pi^0\gamma\gamma$ [8] play an important role in CHPT; they have no $O(p^2)$ tree level contribution, since the external particles are neutral; for the same reason there are no $O(p^4)$ counterterms. This fact has two implications: 1) the chiral meson loops are finite and so free of the ambiguity of the cut-off; 2) these are the only $O(p^4)$ contributions; no dependence on counterterms or other unknown constants.

Thus, CHPT predicts unambiguously at $O(p^4)$ these two decays. While there is a substantial agreement between theory [7,9] and experiments [10] for $K_S \to \gamma\gamma$

$$\Gamma(K_S \to 2\gamma)_{Th} = 1.52 \cdot 10^{-11} eV, \tag{1.1}$$

$$\Gamma(K_S \to 2\gamma) = (1.8 \pm 0.8) \cdot 10^{-11} eV. \tag{1.2}$$

for $K_L \to \pi^0\gamma\gamma$ the situation is still controversial. The theoretical prediction of CHPT at $O(p^4)$ for the $\Delta I = 1/2$ amplitude has been calculated [8], giving the branching ratio

$$\mathrm{Br}(K_L \to \pi^0\gamma\gamma)_{Th} = 0.68 \cdot 10^{-6}, \tag{1.3}$$

to compare with the experimental values

$$Br(K_L \to \pi^0\gamma\gamma) = \begin{cases} (1.7 \pm 0.3) \cdot 10^{-6} & \text{NA31} \\ (1.86 \pm 0.6 \pm 0.6) \cdot 10^{-6} & (\sqrt{q^2} > 280\ MeV)\ \text{E731} \end{cases} \tag{1.4}$$

($\sqrt{q^2}$ is the two photon invariant mass) at CERN [11] and FNAL [12] respectively, which are somewhat bigger than the prediction. Nevertheless the predicted spectrum, which is dominated at this order by the absorptive part $K_L \to \pi^0 \pi^+ \pi^- \to \pi^0 \gamma \gamma$, looks in good agreement with experiment. Indeed, NA31 experiment seems to exclude a big dispersive contribution at low q^2.

The decay $K_L \to \pi^0 e^+ e^-$ has three kinds of contributions [14,15,13]: direct CP violation, mass CP violation $K_L \to \pi^0 \gamma^* \to \pi^0 e^+ e^-$ and CP conserving $K_L \to \pi^0 \gamma^* \gamma^* \to \pi^0 e^+ e^-$. The CP violating contributions are expected to give a $Br(K_L \to \pi^0 e^+ e^-) \simeq 10^{-11}$; while, since $O(p^4)$ $K_L \to \pi^0 \gamma \gamma$ gives an helicity suppressed CP conserving amplitude, only the $O(p^6)$ contribution might give an appreciable rate to the branching ratio of the CP conserving process, which indeed naive power counting would predict of order 10^{-14} [16,13]; still in this framework $O(p^6)$ vector meson exchange diagrams [13,17,18] might enhance both $Br(K_L \to \pi^0 \gamma \gamma)$ and the CP conserving $Br(K_L \to \pi^0 e^+ e^-)_{CP}$, but it is unlikely to obtain values for these branching ratios of about $2 \cdot 10^{-6}$ and 10^{-11} respectively. Phenomenological models with large vector [19,20] or scalar [21] meson exchange diagrams can obtain these values, but would alter the spectrum $d\Gamma/dq^2$, particularly at low q^2, where they would predict a drastic increase.

In this paper we will first give the $O(p^4)$ $\Delta I = 3/2$ contribution to $K_L \to \pi^0 \gamma \gamma$; then we will discuss some implications of the $O(p^4)$ corrections to the $K \to 3\pi$ amplitude to $K_L \to \pi^0 \gamma \gamma$, $K_L \to \pi^0 e^+ e^-$ and to the CP violating charge asymmetry of $K^\pm \to \pi^\pm \gamma \gamma$.

2. $\Delta I = 3/2$

At order p^2, the chiral Lagrangian is

$$L = L_{\Delta S = 0} + L_{\Delta S = 1}. \tag{2.1}$$

One has

$$L_{\Delta S=0} = \frac{1}{4} f^2 Tr D_\mu U D^\mu U^\dagger + \frac{f^2}{2} Tr U^\dagger \mu M + \frac{f^2}{2} Tr U \mu M, \tag{2.2}$$

where

$$U = e^{\frac{2i}{f} \pi_a T_a}, \quad D_\mu U = \partial_\mu U + ie A_\mu [Q, U],$$

$$M = diag(m_u, m_d, m_s), \quad Q = diag(2/3, -1/3, -1/3), \tag{2.3}$$

$$T_a = \frac{\lambda_a}{2}, \quad Tr T_a T_b = \frac{1}{2} \delta_{ab}, \quad f \simeq F_\pi = 93.3 MeV;$$

the λ_a are the Gell-Mann matrices and μ is the correct factor to reproduce the observed meson masses.

The CP-conserving $\Delta S = 1$ weak Lagrangian consists of two pieces: the octet and 27-plet:

$$L_{\Delta S=1} = \frac{1}{4} f^2 h_8 Tr \lambda_6 D_\mu U D^\mu U^\dagger + \frac{h_{27} f^2}{4} [T_{ij}^{kl} (U D_\mu U^\dagger)_k^i (U D^\mu U^\dagger)_l^j + h.c.], \qquad (2.4)$$

where the tensor T is the U=1, ΔS=1, ΔQ=0 element of the 27 with the non-zero components:

$$T_{13}^{12} = T_{13}^{21} = T_{31}^{21} = T_{31}^{12} = \frac{3}{5}, \quad T_{23}^{22} = T_{32}^{22} = T_{33}^{23} = T_{33}^{32} = -\frac{3}{10}. \qquad (2.5)$$

From $K \to \pi\pi$ decays we have at order p^2

$$h_8 = 3.2 \cdot 10^{-7}, \quad h_{27} = -1 \cdot 10^{-8}. \qquad (2.6)$$

The general amplitude for $K_L \to \pi^0 \gamma\gamma$ is given by

$$M(K_L(p) \to \pi^0(p_3)\gamma(k_1,\varepsilon_1)\gamma(k_2,\varepsilon_2)) = \varepsilon_{1\mu}\varepsilon_{2\nu} M^{\mu\nu}(p_3,k_1,k_2), \qquad (2.7)$$

where ε_1, ε_2 are the photon polarizations and $M^{\mu\nu}$, if CP is conserved, is made of two invariant amplitudes:

$$M^{\mu\nu} = \frac{A(y,z)}{m_K^2} (k_2^\mu k_1^\nu - k_1 k_2 g^{\mu\nu})$$

$$+ \frac{2B(y,z)}{m_K^4} (-p k_1 p k_2 g^{\mu\nu} - k_1 k_2 p^\mu p^\nu + p k_1 k_2^\mu p^\nu + p k_2 p^\mu k_1^\nu), \qquad (2.8)$$

where

$$y = |p(k_1 - k_2)|/m_K^2, \qquad z = (k_1 + k_2)^2/m_K^2.$$

Note that $A(y,z)$ and $B(y,z)$ are symmetric for $k_1 \leftrightarrow k_2$, as required by Bose symmetry.

The physical region in the adimensional variables y and z is given by

$$0 \le y \le \frac{1}{2}\lambda^{\frac{1}{2}}(1,r_\pi^2,z), \quad 0 \le z \le (1 - r_\pi)^2, \qquad (2.9)$$

where

$$\lambda(1,r_\pi^2,z) = 1 + z^2 + r_\pi^4 - 2(z + z r_\pi^2 + r_\pi^2) \quad \text{and} \quad r_\pi = \frac{m_{\pi^0}}{m_{K^0}}. \qquad (2.10)$$

From (2.8), (2.9) and (2.10) we obtain the double differential rate for unpolarized photons:

$$\frac{d^2\Gamma}{dy\, dz} = \frac{m_K}{2^9 \pi^3} \left\{ z^2 |A + B|^2 + [y^2 - (\frac{(1 + r_\pi^2 - z)^2}{4} - r_\pi^2)]^2 |B|^2 \right\}. \qquad (2.11)$$

Since K_L and π^0 are neutral there are no tree level $O(p^2)$ and no $O(p^4)$ counterterm contributions to the amplitude. At order p^4 the amplitude $B(y, z)$ is zero since there are not enough powers of momenta. The result for the octet amplitude $A^{(8)}(z)$, which at this order depends only on z, is [8]

$$A^{(8)}(z) = \frac{h_8 \alpha m_K^2}{4\pi f^2} \left[(1 - \frac{r_\pi^2}{z})f(\frac{z}{r_\pi^2}) - (1 - \frac{r_\pi^2}{z} - \frac{1}{z})f(z) \right], \tag{2.12}$$

$$f(x) = 1 + \frac{1}{x}\ln^2\frac{\beta(x) - 1}{\beta(x) + 1}, \qquad \beta(x) = \sqrt{1 - \frac{4}{x}}. \tag{2.13}$$

The function $f(x)$ is real for $x \le 4$ and complex for $x \ge 4$. More explicitly it is written

$$f(x) = \begin{cases} 1 - \frac{4}{x}\arcsin^2 \frac{\sqrt{x}}{2} & x \le 4 \\ 1 + \frac{1}{x}[\ln^2\frac{1-\beta(x)}{1+\beta(x)} - \pi^2 + 2i\pi\ln\frac{1-\beta(x)}{1+\beta(x)}] & x \ge 4 \end{cases} . \tag{2.14}$$

In (2.12) the contribution proportional to $f(z)$ comes from the kaon loops and so does not have absorptive part, while the one proportional to $f(z/r_\pi^2)$ comes from pion loops and it has absorptive part, since the pions can be on shell. Correspondingly, the kaon contribution is much less than the one of pions. As for the $A(K_S \to \gamma\gamma)$ this is an excellent test for chiral perturbation theory. In particular the spectrum and the width depend upon the hypothesis of pion loop, and can be considered as a test of CHPT as a quantum field theory. Indeed, the peak in the z spectrum (Fig. 1) is due to the absorptive part.

It might be interesting to calculate the contribution of the 27, which due to the lacking of the corresponding counterterms, turns out to be finite and unambiguously predicted at $O(p^4)$:

$$A_{1/2}^{(27)}(z) = -\frac{h_{27}\alpha m_K^2}{20\pi f^2}(1 - \frac{r_\pi^2}{z})f(\frac{z}{r_\pi^2}); \tag{2.15a}$$

$$A_{3/2}^{(27)}(z) = \frac{h_{27}\alpha m_K^2}{8\pi f^2}\left[\frac{3 - r_\pi^2 - 14r_\pi^4 - (5 - 14r_\pi^2)z}{(1 - r_\pi^2)z}\right]f(\frac{z}{r_\pi^2}). \tag{2.15b}$$

The z spectrum for the y independent amplitudes (2.12, 2.15) is given by

$$\frac{d\Gamma}{dz} = \frac{m_K}{2^{10}\pi^3}[(1 + r_\pi)^2 - z]^{\frac{1}{2}}[(1 - r_\pi)^2 - z]^{\frac{1}{2}}z^2|A|^2. \tag{2.16}$$

Due to the finiteness of the $\Delta I = 3/2$ contribution to $K_L \to \pi^0\gamma\gamma$, CHPT predicts, at $O(p^4)$, the $\Delta I = 1/2$ rule for this decay. Indeed, a small negative interference is predicted for the branching ratio:

$$Br(K_L \to \pi^0\gamma\gamma)^{(8)} = 0.68 \cdot 10^{-6}, \quad Br(K_L \to \pi^0\gamma\gamma)^{(8+27)} = 0.61 \cdot 10^{-6}. \tag{2.17}$$

These are dominated by the absorptive contributions:

$$Br(K_L \to \pi^0\gamma\gamma)^{(8)}_{abs} = 0.46 \cdot 10^{-6}, \quad Br(K_L \to \pi^0\gamma\gamma)^{(8+27)}_{abs} = 0.43 \cdot 10^{-6}. \qquad (2.18)$$

In Fig.1 the spectrum $d\Gamma/dz$ for the $O(p^4)$ amplitudes (2.12) plus (2.15) is reported with the relative absorptive and dispersive contributions.

3. $O(p^6)$ from $K \to 3\pi$.

The amplitude for the process $K(p) \to \pi(p_3)\pi(p_1)\pi(p_2)$ is generally expanded in powers of the Dalitz plot variables

$$X = \frac{s_2 - s_1}{m_\pi^2}, \quad Y = \frac{s_3 - s_0}{m_\pi^2}, \qquad (3.1)$$

where $s_i = (p - p_i)^2$ and $s_0 = (s_1 + s_2 + s_3)/3$ and 3 indicates the "odd" pion. For the decays $K_L(p) \to \pi^0(p_3)\pi^+(p_1)\pi^-(p_2)$ and $K^+(p) \to \pi^-(p_3)\pi^+(p_1)\pi^+(p_2)$ the isospin decomposition, neglecting the phase shifts, up to quadratic terms, is written as [22,23,4],

$$A(K_L \to \pi^0\pi^+\pi^-) = (\alpha_1 + \alpha_3) - (\beta_1 + \beta_3)Y + (\zeta_1 - 2\zeta_3)(Y^2 + \frac{X^2}{3}) + (\xi_1 - 2\xi_3)(Y^2 - \frac{X^2}{3}) \qquad (3.2)$$

$$A(K^+ \to \pi^+\pi^+\pi^-) = (2\alpha_1 - \alpha_3) + (\beta_1 - \frac{1}{2}\beta_3 + \sqrt{3}\gamma_3)Y -$$

$$-2(\zeta_1 + \zeta_3)(Y^2 + \frac{X^2}{3}) - (\xi_1 + \xi_3 - \xi_3')(Y^2 - \frac{X^2}{3}), \qquad (3.3)$$

where the subscripts 1 and 3 refer to the $\Delta I = 1/2, 3/2$ transitions. Among the experimental values of α_i and β_i, obtained by a general fit of all $K \to 3\pi$ amplitudes [23,4], and the $O(p^2)$ CHPT theoretical predictions [24,22,4] from (2.4) and (2.6), there is a 20-30% disagreement. The inclusion of $O(p^4)$ contributions [4] overcomes this discrepancy, fixing also the quadratic terms, which are vanishing at the lowest order.

The absorptive part of $A(K_L \to \pi^0\gamma\gamma)$ is due to $K_L \to \pi^0\pi^+\pi^- \to \pi^0\gamma\gamma$; thus we can improve the $O(p^4)$ result, using for the absorptive part the physical amplitude $A(K_L \to 3\pi)$, obtaining

$$Br(K_L \to \pi^0\gamma\gamma)^{physical}_{abs} = 0.61 \cdot 10^{-6} \qquad (3.4)$$

to compare with the second branching ratio in (2.18).

Dispersive contributions are in general divergent, and one needs a (model dependent) prescription for the subtraction. Very interestingly, the authors of ref. [25] have shown that the $\Delta I = 1/2$, $O(p^4)$ corrections to the $K \to 3\pi$ amplitude could be effectively reproduced (neglecting loop contributions) by two $O(p^4)$ chiral operators . For the $K_L \to \pi^+\pi^-\pi^0$, for instance, the corrections to the $O(p^2)$ amplitude

$$A^{(2)}(K_L \to \pi^+\pi^-\pi^0) = \frac{h_8 m_K^2}{12f^2}(1 + \frac{3m_\pi^2}{m_K^2}Y) \tag{3.5}$$

are written as (p_3 is the neutral pion four momentum):

$$A^{(4)}(K_L \to \pi^+\pi^-\pi^0) = a_1 p \cdot p_3 p_+ \cdot p_- + a_2(p \cdot p_+ p_3 \cdot p_- + p \cdot p_- p_3 \cdot p_+). \tag{3.6}$$

a_1 and a_2 are the coefficients of the operators [a]. Writing (3.6) in terms of the kinematical variables X and Y in (3.1), one obtains

$$A^{(4)}(K_L \to \pi^+\pi^-\pi^0) = a_1 m_K^4 [\frac{1}{18}(1 - 3r_\pi^2) + \frac{1}{12}r_\pi^2(1 + 3r_\pi^2)Y - \frac{1}{4}r_\pi^4 Y^2] +$$

$$a_2 m_K^4 [\frac{1}{9}(1 - 3r_\pi^2) - \frac{1}{12}r_\pi^2(1 + 3r_\pi^2)Y - \frac{1}{8}r_\pi^4(Y^2 + X^2)]. \tag{3.7}$$

Thus, comparing the sum of the expressions (3.5) and (3.7) to the experimental amplitudes (3.2) [23,4], a_1 and a_2 can be fixed by fitting the coefficients of either the quadratic or the linear terms. Their values are:

$$a_1^{lin} = 1.0 \cdot 10^{-5}/m_K^4, \quad a_1^{quad} = -8(\zeta_1 + 2\xi_1)/3m_\pi^4 = 1.6 \cdot 10^{-5}/m_K^4,$$
$$a_2^{lin} = -2.6 \cdot 10^{-6}/m_K^4, \quad a_2^{quad} = -8(\zeta_1 - \xi_1)/3m_\pi^4 = -4.8 \cdot 10^{-6}/m_K^4. \tag{3.8}$$

The slight discrepancy gives somehow the size of the model dependence. The inclusion of loops [4] of course changes the size of the coefficients, but confirms the relevance of the (3.6) corrections.

We have used the effective operators (3.6) as effective vertices of the $O(p^4)$ $K \to 3\pi$ amplitude and calculated the corresponding $O(p^6)$ correction to the $K_L \to \pi^0\gamma\gamma$ amplitude. There are no pole diagrams, but just a charged pion loop, with photons attached to it or directly emitted at the $K \to 3\pi$ vertex (as required by gauge invariance for derivative vertices).

Since one does not integrate on the π^0 momentum, we can write the Y variable in terms of z defined in (2.8):

$$Y = \frac{2}{3}\frac{m_K^2}{m_\pi^2}(1 - \frac{3p \cdot p_3}{m_K^2}) = \frac{z}{r_\pi^2} - 1 - \frac{1}{3r_\pi^2} \tag{3.9}$$

[a] Our definition of a_1 and a_2 differs by a factor four from the one in ref.[25].

and so all the terms in (3.7), but the one proportional to X^2, are constant over the charged pion loop and, very similarly to the $O(p^4)$ result, they give a finite $A(z)$ amplitude proportional to $f(z/r_\pi^2)$, where the function f is defined in (2.13).

With this in mind, to solve the ambiguous choice of a_1 and a_2 shown in (3.8), we can directly use the parametrization given in (3.2), and get for the contribution of all terms not proportional to X^2:

$$A^{(Y)}(z) = \frac{\alpha}{\pi}[\alpha_1 + \alpha_3 - (\beta_1 + \beta_3)Y + (\zeta_1 - 2\zeta_3 + \xi_1 - 2\xi_3)Y^2]\frac{1}{z}f(\frac{z}{r_\pi^2}). \qquad (3.10)$$

The X^2 term gives a more sophisticated amplitude, both terms B and A in (2.8) will be present, and the A amplitude depends also on the y variable defined in (2.8). Both contributions are divergent and we have performed a dimensional regularization, absorbing the divergence with a suitable renormalization of the corresponding counterterm coupling constants, providing a consistent prescription for the dispersive contribution. The loop amplitude will then depend on a renormalization scale μ. The finite part of the counterterm coupling constant has been discussed in the VMD framework [13,15], but it will not be considered here. The B amplitude is given by:

$$B(z) = \frac{4}{3\pi}\frac{\alpha}{r_\pi^4}(\zeta_1 - 2\zeta_3 - \xi_1 + 2\xi_3)\times$$

$$\left[\frac{1}{6}\ln\frac{m_\pi^2}{\mu^2} + \frac{5}{9} - \frac{13}{12}\beta^2 + \frac{1}{4}(\frac{5}{3}\beta^2 - 1)\beta\,\ln(\frac{\beta+1}{\beta-1}) + \frac{1}{16}(1 - \beta^2)^2\ln^2(\frac{\beta+1}{\beta-1})\right], \qquad (3.11)$$

where $\beta = \sqrt{1 - 4r_\pi^2/z}$.

Due to the different tensor structure in (2.8), the A and B parts of the amplitude give rise to contributions to the differential decay rate which have different dependence on the two photon invariant mass. In particular, the second term in (2.8) gives a non vanishing contribution to $d\Gamma(K_L \to \pi^0\gamma\gamma)/dz$ in the $z \to 0$ limit; thus the $O(p^6)$ spectrum starts with a non zero value, contrary to the $O(p^4)$ case. This would allow the emission of collinear photons.

We report in the Appendix the analytic expressions for the $O(p^6)$ corrections to A'. Note that both the A and B amplitudes remain finite in the limit $m_\pi \to 0$ ($\beta \to 1$), consistently with CHPT.

As for the amplitude in (3.10), we have evaluated the contribution of the terms proportional to X^2 using the parameters in (3.2).

We observe that separately $A(y, z)$ and $B(z)$ give both a significant dispersive contribution to the amplitude; while the real part of $f(\frac{z}{r_\pi^2})$ decreases rapidly with $q^2 \to \infty$, as required by the finiteness of the amplitude, the dispersive contributions in (3.11) and (A.1) are not finite and remain separately large.

The $O(p^6)$ A amplitude interferes destructively with the $O(p^4)$ amplitude. The B amplitude has instead a constructive interference with (2.12). As a result, there is an almost complete (model independent) cancellation between these two contributions, independently on the size of the coefficient of the X^2 term. The contribution of the dominant term $|A+B|$ to the total branching ratio remains very much unaltered by the $O(p^6)$ corrections. To show the model independence of the final result, we report in the following table the values of the total branching ratio (according to the renormalization scale μ) using respectively either the experimental values for α, β, ζ, ξ [4] in (A.1) and (3.11) or the parametrization of the $O(p^6)$ amplitude (A.12), with the two sets of values given in (3.8) for a_1 and a_2:

μ	$Br(K_L \to \pi^0 \gamma\gamma)$ $exp.\alpha, \beta, \zeta, \xi$	$Br(K_L \to \pi^0 \gamma\gamma)$ a_1^{lin}, a_2^{lin}	$Br(K_L \to \pi^0 \gamma\gamma)$ a_1^{quad}, a_2^{quad}	
$770\ MeV$	$8.4 \cdot 10^{-7}(2.3)$	$9.6 \cdot 10^{-7}(2.4)$	$11.2 \cdot 10^{-7}(2.8)$	(3.12)
$1000\ MeV$	$8.6 \cdot 10^{-7}(2.5)$	$9.7 \cdot 10^{-7}(2.5)$	$11.3 \cdot 10^{-7}(2.9)$	

The values in parenthesis refer to the dispersive part (in units 10^{-7}). Of course, one of the targets of this paper is to give an estimate of the size of the dispersive contribution. Indeed we can see that all three parametrizations lead to a comparable small result. Obviously, only the data in the first column reproduce the absorptive contribution in (3.4), since the others are obtained from a parametrization which gives only an approximate description of the $K \to 3\pi$ amplitude. In Fig. 2 and in Fig. 3, we plot respectively the z-spectrum and the y-spectrum for the $K_L \to \pi^0 \gamma\gamma$ amplitude given by (A.1) and (3.11).

Thus we find the interesting result that the dispersive analytic $O(p^6)$ contribution to $K_L \to \pi^0 \gamma\gamma$ is almost negligible, showing that, at least in this case, CHPT is indeed perturbative.

4. The CP conserving decay $K_L \to \pi^0 e^+ e^-$ and $K^+ \to \pi^+ \gamma\gamma$.

The CP conserving two-photon exchange decay $K_L \to \pi^0 e^+ e^-$ has the absorptive part coming from the two photon discontinuity. Due to the tensor structure shown in (2.8) the part of the $K_L \to \pi^0 \gamma\gamma$ amplitude containing $A(y,z)$ gives a contribution to the two photon discontinuity in $A(K_L \to \pi^0 e^+ e^-)_{CP}$ which is suppressed by a factor m_e/m_K. This suppression is not present in the contribution coming from the term containing $B(y,z)$ in (2.8) [16]. Dispersion relations for $A(K_L \to \pi^0 e^+ e^-)_{CP}$ using the $O(p^4)$ amplitude (2.12), give rise to an m_e/m_K suppression also for the dispersive part of $A(K_L \to \pi^0 e^+ e^-)_{CP}$ [16]. Thus the main contribution to this process in CHPT is expected to come from the B-type $O(p^6)$ contribution of $K_L \to \pi^0 \gamma\gamma$ to the two photon discontinuity amplitude. The dispersive part should not change the size of $\Gamma(K_L \to \pi^0 e^+ e^-)_{CP}$.

While the $O(p^6)$ contributions coming from VMD have been studied by other authors [15,19,20,13] (with different conclusions), here we will be concerned with the value of $B(y,z)$ in (3.11) obtained by including the $O(p^4)$ corrections to the $A(K \to 3\pi)$ amplitudes.

The absorptive part of $A(K_L \to \pi^0 \gamma\gamma)$ from (3.2) will give rise to a model independent $B(y,z)_{abs}$ in (2.8) coming from the terms proportional to X^2 (actually depending only on z),

$$B(z)_{abs} = \frac{\alpha}{3r_\pi^4}(\zeta_1 - 2\zeta_3 - \xi_1 + 2\xi_3)\left[\beta - \frac{5}{3}\beta^3 - \frac{1}{2}(1-\beta^2)^2 \ln\frac{1+\beta}{1-\beta}\right], \qquad (4.1)$$

which gives

$$Br(K_L \to \pi^0 e^+ e^-)_{CP}^{B_{abs}} \simeq 10^{-14}. \qquad (4.2)$$

We have computed the $K_L \to \pi^0 \gamma\gamma$ two photon discontinuity contribution when the A amplitude in (A.1) and the B amplitude in (3.11) are included. We find indeed that the B-term gives a substantial branching ratio, which changes very little with the value of μ:

μ	$Br(K_L \to \pi^0 e^+ e^-)_{CP}$	
770 MeV	$0.8 \cdot 10^{-12}$	(4.3)
1000 MeV	$1.0 \cdot 10^{-12}$	

As final issue, we discuss some implications of the $O(p^4)$ corrections to the $K \to 3\pi$ for the decay $K^+ \to \pi^+ \gamma\gamma$.

The amplitude (3.3) can be used to extract the absorptive part of the loop amplitude of $K^+ \to \pi^+ \gamma\gamma$. For the dispersive part, CHPT is less predictive than for the case of $K_L \to \pi^0 \gamma\gamma$, since counterterms are present in the chiral Lagrangian at $O(p^4)$. Nevertheless, the $O(p^4)$ loop amplitude turns out to be finite [16].

Due to CP violation, the counterterm amplitude A_{CT} is in general complex; actually its imaginary part has been estimated for $m_t/m_c \sim 100$ [16]:

$$A_{CT} = \frac{h_8 \alpha m_K^2}{8\pi f^2} \hat{c}, \qquad |Im\hat{c}| \sim 3 \cdot 10^{-3}, \qquad (4.4)$$

where $\hat{c}$ is a scale independent constant. The interference between the absorptive part of the loop amplitude and the counterterm contribution might give a non vanishing value to the charge asymmetry of the process, defined by

$$\frac{\Gamma(K^+ \to \pi^+ \gamma\gamma) - \Gamma(K^- \to \pi^- \gamma\gamma)}{\Gamma(K^+ \to \pi^+ \gamma\gamma) + \Gamma(K^- \to \pi^- \gamma\gamma).} \qquad (4.5)$$

Disregarding Wess-Zumino type contributions (irrelevant to the present discussion), the amplitude for $K^+ \to \pi^+ \gamma\gamma$ has the form given in eq.(2.8). One has:

$$|\Gamma(K^+ \to \pi^+ \gamma\gamma) - \Gamma(K^- \to \pi^- \gamma\gamma)| = \frac{m_K^3}{2^7 \pi^3} \frac{h_8 \alpha}{8\pi f^2} \times$$

$$|Im\hat{c}| \int_{4r_\pi^2}^{(1-r_\pi)^2} z^2 dz \int_0^{\frac{1}{2}\lambda^{1/2}(1, r_\pi^2, z)} dy \, |(A_{abs}(y, z) + B_{abs}(z))|, \qquad (4.6)$$

where $r_\pi = m_{\pi^+}/m_{K^+}$ and $\lambda(1, r_\pi^2, z)$ is defined in (2.10).

The use of the amplitude (3.3) with the constants fitted to the experimental data in ref. [4] does not alter significantly the value of the charge asymmetry (4.6) which results equal to $6 \cdot 10^{-23}$ MeV, i.e. of the same order of magnitude of the value obtained at $O(p^4)$ in CHPT by the authors of ref.[16]: $4 \cdot 10^{-23}$ MeV.

5. Conclusions.

We have shown that the 27 contribution of $O(p^4)$ to $K_L \to \pi^0 \gamma\gamma$, is suppressed by $\Delta I = \frac{1}{2}$ rule, while absorptive contribution from $O(p^4)$ $K \to 3\pi$ increase by about 30% the width keeping unaltered the normalized spectrum $\Gamma^{-1} d\Gamma/dz$. We have also analyzed the corresponding dispersive contributions, finding that, due to a cancellation of the A and B term corrections, the total branching ratio remains essentially unchanged, while the shape of the spectrum is modified. Thus, if the experimental value of the branching ratio is confirmed, we think that our analysis shows that other effects, like for instance VMD, have to be relevant.

The fact that the contributions of the A and B terms of the amplitude $K_L \to \pi^0 \gamma\gamma$ are differently weighted in the two photon discontinuity contribution to $K_L \to \pi^0 e^+ e^-$, makes these corrections very relevant for this process, giving a branching ratio of 10^{-12}.

We have also considered the effects of the $O(p^4)$ $K \to 3\pi$ corrections to $K^{\pm} \to \pi^{\pm} \gamma\gamma$ asymmetry.

Appendix.

We report here the full expression of $O(p^4) + O(p^6)$ amplitude $A(y,z)$:

$$A(y,z) = \frac{\alpha}{\pi}[a \frac{1}{z} f(\frac{z}{r_\pi^2}) + c(A_3(z) + A_4(z) + A_6(y,z) + \frac{1}{2}A_2(z))]; \qquad (A.1)$$

where

$$a = \alpha_1 + \alpha_3 - (\beta_1 + \beta_3)Y + (\zeta_1 - 2\zeta_3 + \xi_1 - 2\xi_3)Y^2;$$
$$c = \frac{8}{3}\frac{m_K^4}{m_\pi^4}(\zeta_1 - 2\zeta_3 - \xi_1 + 2\xi_3). \qquad (A.2)$$

The α, β, ζ and ξ parameters are defined in (3.2); Y is defined in (3.1) and can be expressed in terms of z through (3.9).

$$A_2(z) = \frac{1}{4}z[1 - (\frac{1}{z} - \frac{r_\pi^2}{z})^2]f(\frac{z}{r_\pi^2}); \qquad (A.3)$$

$$A_3(z) = \frac{1}{4}z(1 + \frac{1}{z} - \frac{r_\pi^2}{z})^2(f_4(z) + f_6(z) - 4f_5(z)); \qquad (A.4)$$

$$A_4(z) = -[(r_\pi^2 + 1 - z)f_1(z) + \frac{1}{2}(1 - r_\pi^2 + z)(2f_5(z) - 2f_8(z) - f_6(z))]; \qquad (A.5)$$

$$A_6(y,z) = -(\frac{y^2}{z})(f_4(z) + f_6(z) - 4f_5(z)). \qquad (A.6)$$

The function $f(\frac{z}{r_\pi^2})$ is given by (2.13), whilst the $f_i(z)$ are defined as follows:

$$f_4(z) = \frac{1}{3}log(\frac{m_\pi^2}{\mu^2}) - \frac{1}{8}(1 - \beta^2)log^2(\frac{\beta+1}{\beta-1}) - \frac{1}{6}(\beta^2 - 3)\beta log(\frac{\beta+1}{\beta-1}) + \frac{1}{3}\beta^2 - \frac{25}{18}; \qquad (A.7)$$

$$f_5(z) = \frac{1}{8}log(\frac{m_\pi^2}{\mu^2}) + \frac{1}{64}(1-\beta^2)^2 log^2(\frac{\beta+1}{\beta-1}) - \frac{1}{16}(1-3\beta^2)\beta log(\frac{\beta+1}{\beta-1}) - \frac{7}{16}\beta^2 + \frac{1}{12}; \qquad (A.8)$$

$$f_6(z) = \frac{1}{6}log(\frac{m_\pi^2}{\mu^2}) + \frac{1}{6}\beta^3 log(\frac{\beta+1}{\beta-1}) - \frac{1}{3}\beta^2 - \frac{1}{9}; \qquad (A.9)$$

$$f_8(z) = \frac{1}{192r_\pi^2}[3(1 + r_\pi^2)(1 - \beta^2)^2 log^2(\frac{\beta + 1}{\beta - 1})$$

$$+(6\beta^4(r_\pi^2 - 1) + 12\beta^2(3r_\pi^2 + 1) - 6(7r_\pi^2 + 1))\beta log(\frac{\beta + 1}{\beta - 1}) \qquad (A.10)$$

$$-12\beta^4(r_\pi^2 - 1) - 8\beta^2(11r_\pi^2 + 4) + 2(38r_\pi^2 + 10)].$$

Here $\beta = \sqrt{1 - 4r_\pi^2/z}$. By introducing a further expression:

$$A_1(z) = -\frac{1}{2}(1 - \frac{1}{z} - \frac{r_\pi^2}{z})(r_\pi^2 - \frac{1}{2}z)f(\frac{z}{r_\pi^2}), \qquad (A.11)$$

we can give an alternative parametrization of the $O(p^6)$ part of the $A(y, z)$ amplitude and of the $B(z)$ amplitude:

$$A^{(6)}(y, z) = -a_1 m_K^4 A_1(z) - a_2 m_K^4(A_2(z) + A_3(z) + A_4(z) + A_6(y, z)),$$
$$B(z) = a_2 \frac{\alpha}{\pi} m_K^4(f_6(z) - 2f_5(z)), \qquad (A.12)$$

corresponding to the separate contributions of the two effective vertices given in (3.6). Note that using a_2^{quad} given in (3.8) one obtains for $B(z)$ the $\Delta I = 1/2$ contribution to (3.11).

Acknowledgements

Note added. Some of the considerations of this paper have been also suggested by J.Donoghue in "Introduction to non linear effective field theory" presented at International Workshop on Effective Field Theory, Budapest, August 1991, World Scientific, 1992.

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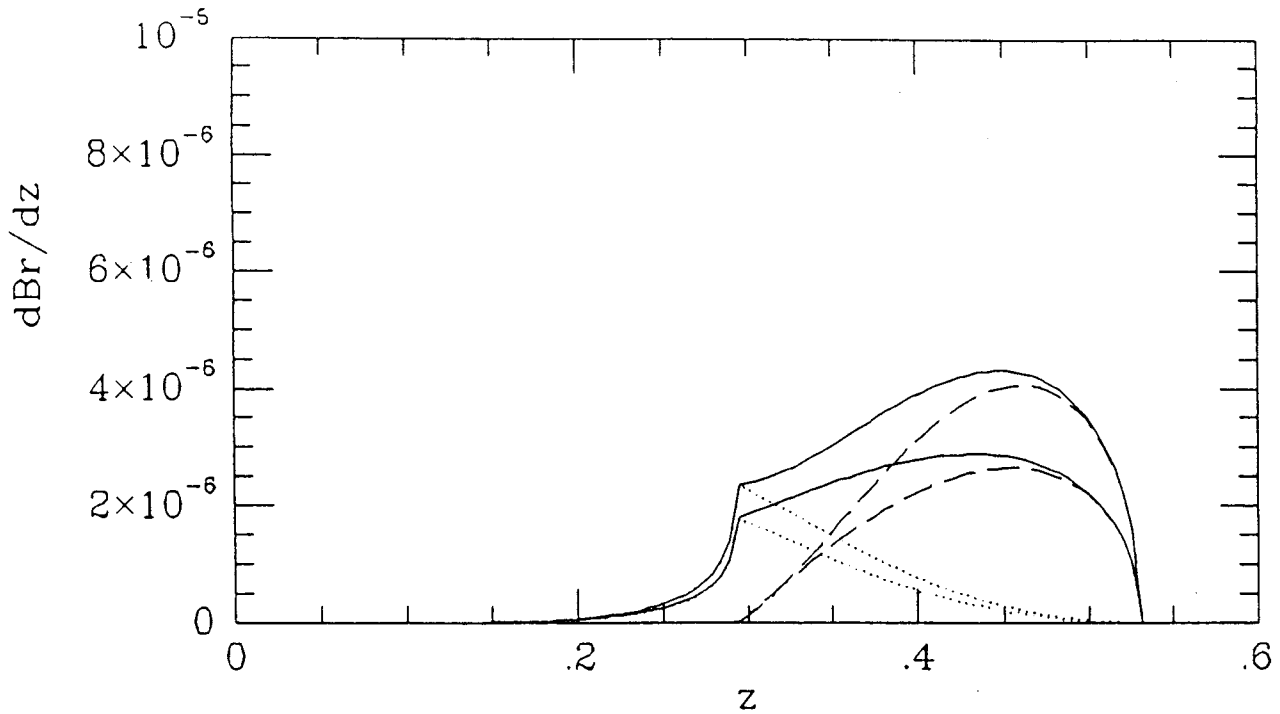Fig.1. The full lines are total contributions. The lower curve is the $O(p^4)$ 8 + 27 contribution, the upper curve is obtained by adding all $O(p^4)$ $K \rightarrow 3\pi$ contributions in the absorptive part and taking h_8 and h_{27} from α_1 and α_3 in the dispersive part. The dashed and dotted lines are the absorptive and dispersive contributions respectively.
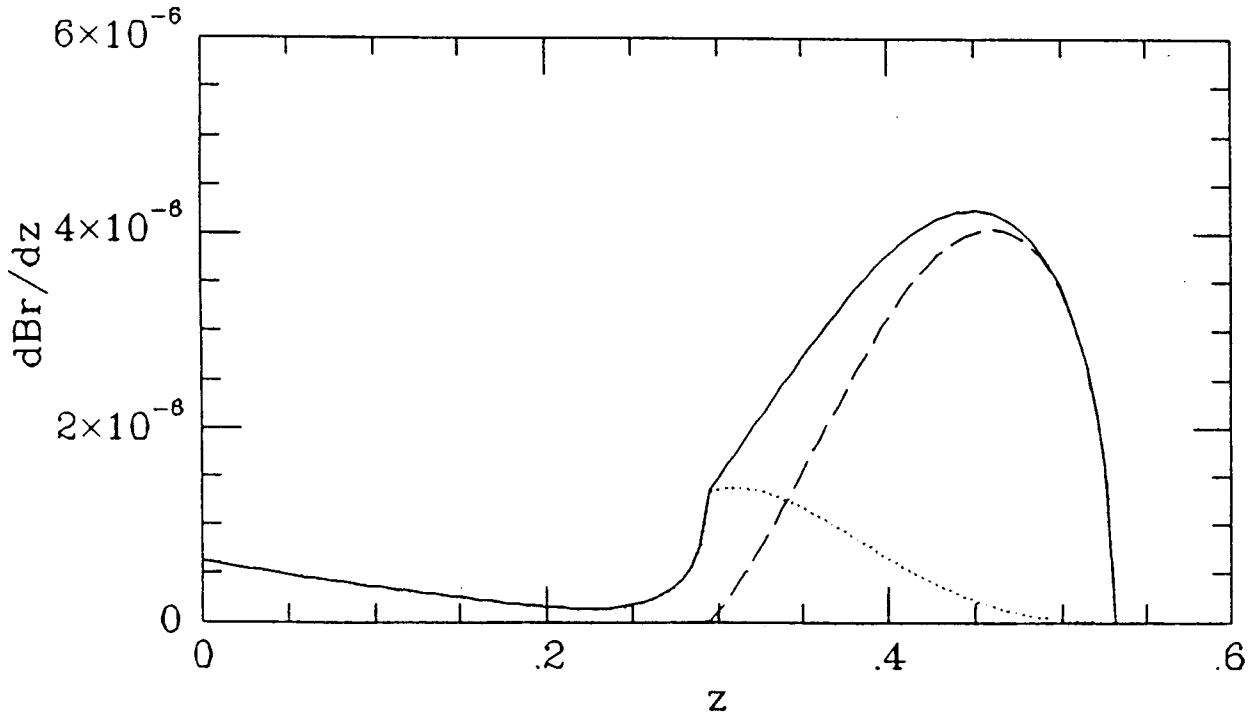
Fig.2. dBr($K_L \rightarrow \pi^0\gamma\gamma$)/dz obtained from the $O(p^4)+O(p^6)$ amplitude, using the experimental values of the parameters in (3.2). The full line is the total contribution, the dashed and dotted lines are the absorptive and dispersive contributions respectively.
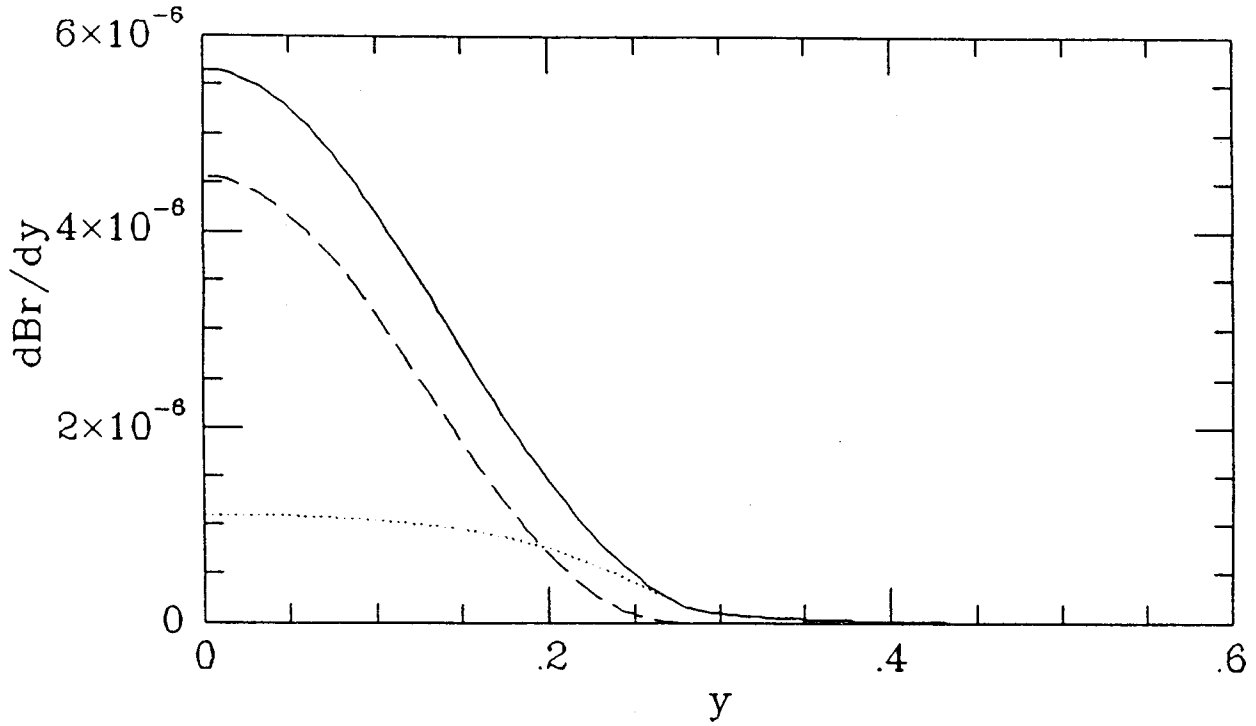
Fig.3. $dBr(K_L \to \pi^0\gamma\gamma)/dy$ obtained from the $O(p^4)+O(p^6)$ amplitude, using the experimental values of the parameters in (3.2). The full line is the total contribution, the dashed and dotted lines are the absorptive and dispersive contributions respectively.