Laboratori Nazionali di Frascati

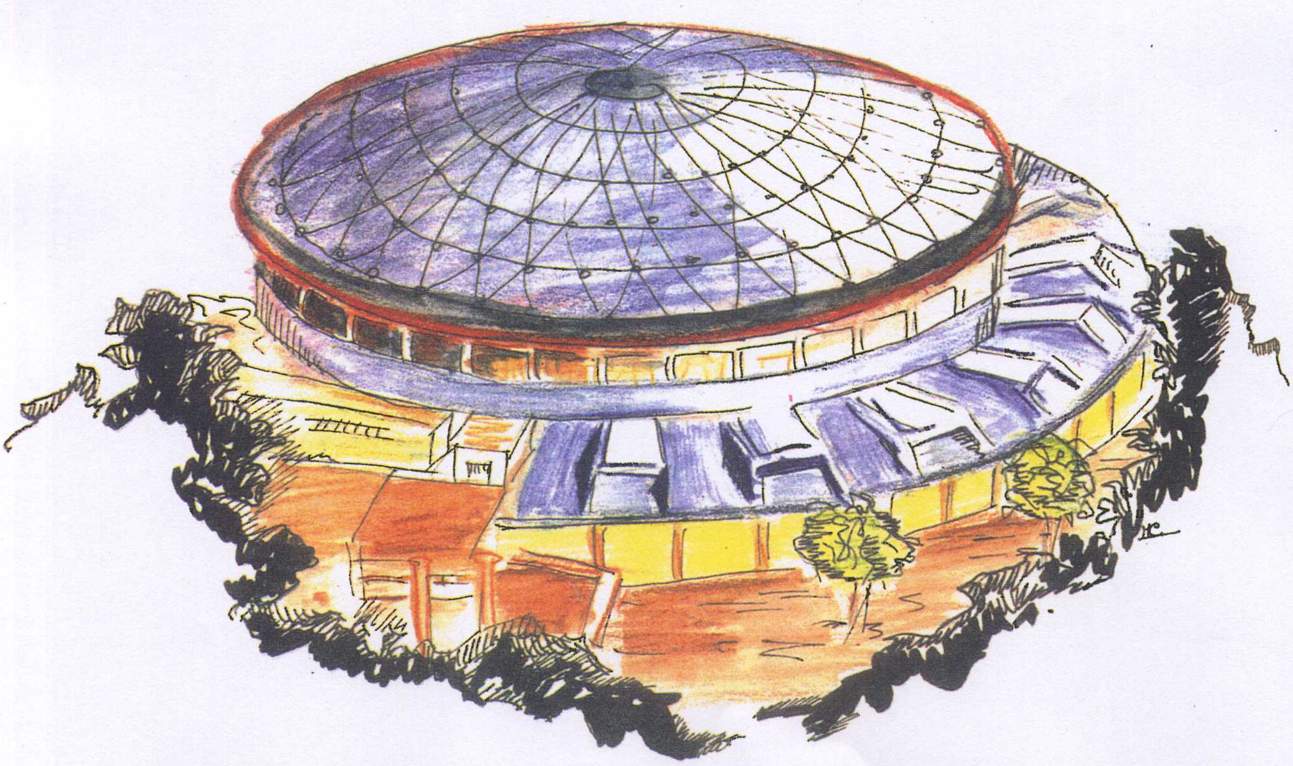
Submitted to Classical and Quantum Gravity

LNF-92/099(P)
23 Novembre 1992

ENSLAPP-L-409/92

S. Bellucci, E. Ivanov, S. Krivonos:

MULTI-FIELD COSET SPACE REALIZATIONS OF $w_{1+\infty}$



Servizio Documentazione
dei Laboratori Nazionali di Frascati
P.O. Box, 13 - 00044 Frascati (Italy)

INFN - Laboratori Nazionali di Frascati
Servizio Documentazione

LNF-92/099 (P)
23 Novembre 1992

ENSLAPP-L-409/92

MULTI-FIELD COSET SPACE REALIZATIONS OF $w_{1+\infty}$

Stefano Bellucci*
INFN-Laboratori Nazionali di Frascati, P.O. Box 13, 00044 Frascati (Italy)

Evgenyi Ivanov**
Laboratoire de Physique Theorique ENSLAPP, ENS Lyon, 46 Allee d'Italie, F–69364 Lyon Cedex 07, France.
and JINR–Laboratory of Theoretical Physics Dubna, Head Post Office, P.O. Box 79, 101 000 Moscow, Russia

Sergey Krivonos***
JINR–Laboratory of Theoretical Physics Dubna, Head Post Office, P.O. Box 79, 101 000 Moscow, Russia

ABSTRACT

We extend the coset space formulation of the one-field realization of $w_{1+\infty}$ to include more fields as the coset parameters. This can be done either by choosing a smaller stability subalgebra in the nonlinear realization of $w_{1+\infty}$ symmetry, or by considering a nonlinear realization of some extended symmetry, or by combining both options. We show that all these possibilities give rise to the multi-field realizations of $w_{1+\infty}$. We deduce the two-field realization of $w_{1+\infty}$ proceeding from a coset space of the symmetry group $\tilde{G}$ which is an extension of $w_{1+\infty}$ by the second self-commuting set of higher spin currents. Next, starting with the unextended $w_{1+\infty}$ but placing all its spin 2 generators into the coset, we obtain a new two-field realization of $w_{1+\infty}$ which essentially involves a $2D$ dilaton. In order to construct the invariant action for this system we add one more field and so get a new three-field realization of $w_{1+\infty}$. We re-derive it within the coset space approach, by applying the latter to an extended symmetry group $\hat{G}$ which is a nonlinear deformation of $\tilde{G}$. Finally we present some multi-field generalizations of our three-field realization and discuss several intriguing parallels with $N = 2$ strings and conformal affine Toda theories.

PACS number: 11.30.–j

* BITNET: BELLUCCI@IRMLNF
** BITNET: IVANOV@ENSLAPP.ENS-LYON.FR
*** BITNET: KRIVONOS@LTP.JINR.DUBNA.SU

1 Introduction

A universal geometric description of field systems respecting the invariance under a symmetry group G is provided by the method of nonlinear realizations (or the coset space realizations). One considers G as a group of transformations acting in some coset space G/H with an appropriately chosen stability subgroup H and identifies the coset parameters with fields. The authors of ref. [1] have constructed a nonlinear realization of $w_{1+\infty}$ and have shown that, after imposing an infinite number of the inverse Higgs-type constraints on the relevant Cartan forms, one is left with the well-known realization of $w_{1+\infty}$ on one scalar $2D$ field [2].

In the present paper we extend the results of ref. [1] to obtain some multi-field realizations of $w_{1+\infty}$. The main idea of our approach is to enlarge the initial coset, in order to find the appropriate place for additional scalar fields. This can be done in two ways, either by considering a larger group $\tilde{G}$ which includes $w_{1+\infty}$ as a subgroup[1], or by choosing a smaller stability subgroup $\tilde{H}$ in $w_{1+\infty}$. We demonstrate that both options (and their combination) give rise to the multi-field realizations of $w_{1+\infty}$, allowing us to obtain the two-field realization of [3], as well as essentially new realizations of $w_{1+\infty}$ on the set of scalar fields including dilaton-like ones. Besides providing new insights into the geometric origin of the $w_{1+\infty}$ transformations, this could shed more light on the geometry of the associated w gravity.

The paper is organized as follows.

In Sect.2 we briefly recall the main results of ref. [1] concerning the one-field realization of $w_{1+\infty}$.

In Sect.3 we recapitulate the basic facts about the two scalar field realization of $w_{1+\infty}$ and list some important symmetries of the corresponding action. In particular, we present an infinite number of conserved currents extending the standard $w_{1+\infty}$ transformations.

Our main results are collected in Sects.4 and 5.

In Sect.4 we utilize the symmetries presented in the previous section, in order to recover the two scalar field realization of $w_{1+\infty}$ within the coset space approach. It turns out that in this case one should start with an extension of the algebra $w_{1+\infty}$ by some infinite-dimensional ideal. We also construct a new coset two-field realization of $w_{1+\infty}$ which essentially involves a $2D$ dilaton.

In Sect.5 we show that this $2D$ field system can be given a Lagrangian formulation at cost of adding one more field. The resulting three-field realization of $w_{1+\infty}$ is in a sense interpolating between the two-field ones constructed in Sect.4. As in the previous cases, it can be deduced in the framework of the coset space method and the inverse Higgs procedure, now applied to a nonlinear deformation of extended symmetry explored in Sect.4. We explicitly give the relevant conserved currents and their OPE's and discuss a possible relation to the conformal affine Toda theory [4, 5, 6]. We also mention some interesting multi-field generalizations of the $w_{1+\infty}$ realization constructed.

[1]We denote the algebra $w_{1+\infty}$ and the associate group of transformations by the same character, hoping that this will not give rise to a confusion.

2 A sketch of the one-field realization of $w_{1+\infty}$

For the reader's convenience we begin with recapitulating the basic steps of the construction proposed in ref.[1].

Its starting point is the truncated $w_{1+\infty}$ formed by the generators V_n^s with the following commutation relations

$$\left[V_n^s, V_m^k\right] = ((k+1)n - (s+1)m)\, V_{n+m}^{s+k}; \quad s,k \geq -1; \quad n \geq -s-1, m \geq -k-1 \,. \tag{2.1}$$

In what follows we will basically mean by $w_{1+\infty}$ just this algebra.

As observed in ref. [1], the standard one-field realization of $w_{1+\infty}$ [2] can be easily re-derived within the framework of a coset realization. It is induced by a left action of the group associated with the algebra (2.1) on the infinite-dimensional coset over the subgroup generated by

$$V_n^0 \; (n \geq 0) \quad , \quad V_m^s \; (s \geq 1, m \geq -s-1)\,. \tag{2.2}$$

An element of this coset space can be parametrized as follows:

$$g = e^{zV_{-1}^0}\, e^{v_0 V_0^{-1}}\, e^{\sum_{n\geq 1} v_n V_n^{-1}}\,. \tag{2.3}$$

Here all coset parameters are assumed to be $2D$ fields depending on z and the second $2D$ Minkowski space light-cone coordinate which is regarded as an extra parameter. In what follows we will never explicitly indicate the dependence on this extra coordinate and, where it is necessary, will identify z with the light-cone coordinate x^+.

As usual in nonlinear realizations, the group G (associated with $w_{1+\infty}$ in the present case) acts as left multiplications of the coset element:

$$g_0 \cdot g(z, v_0, v_n) = g(z', v_0', v_n') \cdot h(z, v_0, v_n; g_0) \quad , \quad g_0 = exp(\sum_{s,n} a_n^s V_n^s)\,, \tag{2.4}$$

where h is some induced transformation of the stability subgroup. This generates a group motion on the coset: the coordinate z together with the infinite tower of coset fields $v_0(z), v_n(z)$ constitute a closed set under the group action. For instance,

$$\begin{aligned} \delta^s z &= -a^s(z)(s+1)(v_1)^s \\ \delta^s v_0 &= -a^s(z) s (v_1)^{s+1} \quad \text{etc.} \quad , \end{aligned} \tag{2.5}$$

where

$$a^s(z) = -\sum_{n\geq -s-1} a_n^s z^{n+s+1}\,.$$

Thus one obtains the realization of $w_{1+\infty}$ on the coordinate z and the infinite number of coset fields v_0, v_n. At the next step of this game the inverse Higgs procedure [7] becomes involved, in order to find the kinematic equations for expressing the higher-order coset fields in terms of $v_0(z)$. This can be done by putting some covariant constraints on the Cartan forms.

The Cartan forms are introduced in the usual way

$$g^{-1}dg = \sum_{s,n} \omega_n^s V_n^s \tag{2.6}$$

and are invariant by construction under the left action of $w_{1+\infty}$ symmetry. They can be easily evaluated using the commutation relations (2.1). The first few ones are as follows:

$$\begin{aligned} \omega_{-1}^{0} &= dz \\ \omega_{0}^{-1} &= dv_0 - v_1 dz \\ \omega_{1}^{-1} &= dv_1 - 2v_2 dz, \quad \text{etc.} \end{aligned} \tag{2.7}$$

Note that the higher order forms, like ω_0^{-1} and ω_1^{-1}, contain the pieces linear in the relevant coset fields. Now, keeping in mind the invariance of these forms, one may impose the manifestly covariant inverse Higgs type constraints

$$\omega_n^{-1} = 0 \quad , \quad (n \geq 0) \tag{2.8}$$

which can be looked upon as algebraic equations for expressing the parameters-fields v_n $(n \geq 1)$ in terms of $v_0(z)$ and its derivatives; e.g., using eqs. (2.7) one finds the coset fields v_1 and v_2 to be expressed by

$$v_1 = \partial v_0 \quad , \quad v_2 = \frac{1}{2}\partial v_1 = \frac{1}{2}\partial^2 v_0 \quad . \tag{2.9}$$

Finally, substituting the expression for v_1 in the transformations laws for z and v_0 (2.5) one gets

$$\begin{aligned} \delta^s z &= -a^s(z)(s+1)(\partial v_0)^s \\ \delta^s v_0 &= -a^s(z)s(\partial v_0)^{s+1} \,. \end{aligned} \tag{2.10}$$

The active form of these transformations is just the standard one-field realization of $w_{1+\infty}$:

$$\tilde{\delta} v_0(z) \equiv v_0'(z) - v_0(z) = a^s(z)(\partial v_0)^{s+1} \quad . \tag{2.11}$$

Thus, the realization of $w_{1+\infty}$ on one scalar field can be deduced in a purely geometric way in the framework of the nonlinear realizations method. One of the most intriguing questions which arise when trying to advance this approach further is how to incorporate in it the multi-field realizations of $w_{1+\infty}$. It seems natural to extend the coset space by some new generators (either from the stability subgroup or by considering a larger group $\tilde{G}$). We will consider these possibilities in Sects.4 and 5, after briefly reviewing in the next section the well-known two scalar fields realization of $w_{1+\infty}$.

3 Two scalar fields realization of $w_{1+\infty}$

In this Section we briefly recall the two scalar fields realization of the $w_{1+\infty}$ algebra.

It is known that algebraically $W_{1+\infty}$ admits a contraction to $w_{1+\infty}$. So one may obtain the field realizations of $w_{1+\infty}$ as a contraction of those of $W_{1+\infty}$. This has been done in ref. [3]. The corresponding realization of $w_{1+\infty}$ looks as follows.

Let v and w be the scalar fields with the following OPE's:

$$v(z_1)v(z_2) = 0 \quad , \quad v(z_1)w(z_2) = w(z_1)w(z_2) = log(z_{12}) \; ; \quad z_{12} \equiv z_1 - z_2 \; . \tag{3.1}$$

Then, defining the current $V^{(s)}$

$$V^{(s)}(z) = (\partial v)^{s+1}\partial w - \frac{1}{s+2}(\partial v)^{s+2} \quad , \tag{3.2}$$

one can easily check that it satisfies the following OPE

$$V^{(s)}(z_1)V^{(k)}(z_2) = \frac{(s+k+2)V^{(s+k)}(z_2)}{z_{12}^2} + \frac{(s+1)\partial V^{(s+k)}(z_2)}{z_{12}} - \frac{\delta_{s+1,0}\delta_{k+1,0}}{z_{12}^2} \tag{3.3}$$

from which it follows that the Fourier components

$$V_n^s = \frac{1}{2\pi i}\oint dz z^{s+n+1}V^{(s)}(z) \tag{3.4}$$

obey the commutation relations of $w_{1+\infty}$, eq.(2.1)[2]. The associate transformations of v and w read

$$\begin{aligned} \delta^s v &= a^s(z)(\partial v)^{s+1} \\ \delta^s w &= a^s(z)(s+1)(\partial v)^s \partial w \quad . \end{aligned} \tag{3.5}$$

The free action for these fields

$$S = \int d^2z\,(-\partial_+ v\partial_- v + \partial_+ w\partial_- v + \partial_- w\partial_- w) \tag{3.6}$$

gives the simplest example of a two-field action invariant under $w_{1+\infty}$ transformations. In fact the first term in the action is invariant in its own right and can be removed by the redefinition of w

$$w \Rightarrow \tilde{w}\,, \quad \tilde{w} = w - \frac{1}{2}v \tag{3.7}$$

$$S = \int d^2z\,(\partial_+\tilde{w}\partial_- v + \partial_-\tilde{w}\partial_+ v) \tag{3.8}$$

$$v(z_1)\tilde{w}(z_2) = log(z_{12})\,, \quad \tilde{w}(z_1)\tilde{w}(z_2) = 0\,, \quad v(z_1)v(z_2) = 0\,. \tag{3.9}$$

Let us make a few comments concerning the realization (3.5) and action (3.6), (3.8).

First of all we note that the transformation law for the field w in (3.5) can be rewritten as

$$\delta^s w = -(\delta^s z)\,\partial w\,, \tag{3.10}$$

where $\delta^s z$ is given in (2.10). So the $w_{1+\infty}$ transformations of the field w are induced by the $w_{1+\infty}$ shift of its argument z. Thus this field behaves as a scalar under $w_{1+\infty}$ symmetry, while v supports the standard one-field realization of $w_{1+\infty}$ discussed in the previous section.

Secondly, the free action (3.6), (3.8) possesses a larger symmetry than $w_{1+\infty}$. Here we quote the infinite number of conserved currents which generate the symmetries we will

[2]The central term appearing in the spin 1 sector of the OPE (3.3) does not contribute to the commutation relations of the truncated $w_{1+\infty}$ algebra (2.1) due to the restrictions on the indices n, m.

discuss in the next Sections. These currents read as follows:

$$\begin{aligned} W_1^{(s)} &= \frac{1}{s+2}(\partial v)^{s+2} \\ W_2^{(s)} &= (\partial v)^{s-1}\partial^3 v \\ W_3^{(s)} &= (\partial v)^{s-2}\partial^4 v \\ &\dots \\ W_N^{(s)} &= (\partial v)^{s-N+1}\partial^{N+1} v\,, \quad \text{etc.} \end{aligned} \tag{3.11, 3.12}$$

For each N there exists a value s_0 such that the currents with the spins exceeding $s_0 + 2$ are independent in the sense that they cannot be reduced to the derivatives of the lower N currents. Using the OPE's (3.1) one can check that all these currents mutually commute and give rise to the transformations of w only ($\delta v = 0$):

$$\begin{aligned} \delta_1^s w &= b^s(z)(\partial v)^{s+1} \\ \delta_2^s w &= (s-1)b^s(z)(\partial v)^{s-2}\partial^3 v + \partial^2\left(b^s(z)(\partial v)^{s-1}\right) \\ \delta_3^s w &= (s-2)b^s(z)(\partial v)^{s-3}\partial^4 v - \partial^3\left(b^s(z)(\partial v)^{s-2}\right)\,, \quad \text{etc.} \end{aligned} \tag{3.13}$$

It is worth mentioning that the currents $W^{(s)}$ (3.11), (3.12) together with the $w_{1+\infty}$ currents $V^{(s)}$ (3.2) form a closed algebra. Moreover, they span an ideal $\hat{\mathcal{H}}$ in this extended algebra, so that the factor-algebra of the latter by $\hat{\mathcal{H}}$ coincides with $w_{1+\infty}$. Let us give the OPE's between $V^{(s)}$ and the first two currents $W_1^{(s)}$, $W_2^{(s)}$

$$\begin{aligned} V^{(s)}(z_1)W_1^{(k)}(z_2) &= \frac{(s+k+2)W_1^{(s+k)}(z_2)}{z_{12}^2} + \frac{(s+1)\partial W_1^{(s+k)}(z_2)}{z_{12}} + \frac{\delta_{s+1,0}\delta_{k+1,0}}{z_{12}^2} \\ V^{(s)}(z_1)W_2^{(k)}(z_2) &= (k-1)(s+1)\left[\frac{W_1^{(s-1)}(z_1)\,W_2^{(k-1)}(z_2)}{z_{12}^2} + 6\frac{W_1^{(s-1)}(z_1)\,W_1^{(k-3)}(z_2)}{z_{12}^4}\right] \\ &= A_1(s,k)\frac{W_1^{(s+k-2)}(z_2)}{z_{12}^4} + \ldots + A_2(s,k)\frac{W_2^{(s+k)}(z_2)}{z_{12}^2} + \ldots \\ &\quad + A_3(s,k)\frac{W_3^{(s+k+1)}(z_2)}{z_{12}}\,, \end{aligned} \tag{3.14}$$

where in the last two lines we have written down only the leading terms without specifying the numerical coefficients A_1, A_2, A_3 (actually in what follows we will never need their explicit form). The OPE's between $V^{(s)}$ and the next currents display a similar structure. Their most characteristic feature is that the currents $W_N^{(s)}$ form a not completely reducible set with respect to $w_{1+\infty}$: $W_1^{(s)}$ are transformed through themselves while the remaining currents are transformed through themselves and the currents $W_1^{(s)}$. So the minimal extended set of currents forming a closed algebra includes $V^{(s)}$ and $W_1^{(s)}$. In what follows this minimal extension of $w_{1+\infty}$ will be referred to as $\tilde{\mathcal{G}}$. Actually one can show that adding any other current $W_N^{(s)}$, $N \geq 2$, to $\tilde{\mathcal{G}}$ would produce, via commuting with the $w_{1+\infty}$ generators, the whole ideal $\hat{\mathcal{H}}$. In the next Section we will show that the two-field realization of $w_{1+\infty}$ which we are discussing here can be deduced from a coset realization of the group associated with $\tilde{\mathcal{G}}$. Note that the ideal $\hat{\mathcal{H}}$ is none other than the universal enveloping algebra for the centreless $U(1)$ Kac-Moody algebra generated by the spin 1 current $W_1^{(-1)}$. Indeed, this enveloping algebra is spanned by all possible products of ∂v and its derivatives of any order. All such products can be represented as linear combinations of the currents $W_N^{(s)}$ from the set (3.11), (3.12) and the derivatives of $W_N^{(s)}$, so these currents form a basis in the enveloping algebra in question regarded as a linear algebra.

The existence of extra symmetries in the above system entails an interesting consequence. One can modify the currents $V^{(s)}$ by adding $W_1^{(s)}$ with the proper coefficient so as the modified currents still close on $w_{1+\infty}$. This modification reads as follows:

$$V_{(\gamma)}^{(s)} = V^{(s)} + \gamma s W_1^{(s)} , \tag{3.15}$$

with γ being an arbitrary parameter. Thus, in the case at hand one actually deals with a one-parameter family of $w_{1+\infty}$ algebras. This fact has been noticed in ref. [8] where it has been also observed that, by adjusting the parameter value, one can cancel the central charge in the spin 1 sector of $w_{1+\infty}$ and so make the modified currents obey the centreless $w_{1+\infty}$. As it is seen from the OPE's (3.3) and (3.14), in our notation this cancellation occurs at $\gamma = -\frac{1}{2}$. It is easy to show that

$$V_{(-1/2)}^{(-1)} = \partial \tilde{w} ,$$

and the spin 1 current becomes self-commuting[3] as a consequence of the self-commutativity of the field $\tilde{w}$.

For any value of γ one can pick up the combination of the fields w, v, namely $w + \gamma v$, which transforms under the appropriate $w_{1+\infty}$ from the above family according to the transformation law (3.10) (the transformations of v do not depend on γ and are always given by eq.(3.5)). In particular, for the centreless $w_{1+\infty}$ (for $\gamma = -\frac{1}{2}$) this combination coincides with the self-commuting field $\tilde{w}$

$$V_{(-1/2)}^{(s)} : \quad \delta \tilde{w} = -(\delta^s z)\partial \tilde{w} . \tag{3.16}$$

Finally, we note that the action (3.6), (3.8) actually respects many more symmetries than those listed above. In particular, in view of invariance of (3.8) under the permutation $\tilde{w} \Leftrightarrow v$, the currents obtained by this permutation from (3.2), (3.11), (3.12), (3.15) also define symmetries of (3.6), (3.8). We discussed only those symmetries which are relevant to the subsequent coset space constructions.

4 Two-field realizations of $w_{1+\infty}$ from the coset space approach

In this Section we explain how the two-field realizations of $w_{1+\infty}$ reviewed in the previous Section can be reproduced in the framework of the coset space approach and also present a new kind of two-field realization.

[3]The central term is still retained in the commutator (or OPE) of $V_{(-1/2)}^{(-1)}$ with $W_1^{(-1)}$.

As we noted previously, in order to find the appropriate place for the additional fields we need to enlarge the coset space we start with. The first possibility is to enlarge the symmetry group G (as the simplest example we may consider the nonlinear realization of the extended symmetry $\tilde{G}$ defined in the previous Section). We do this in Subsect. 4.1 and demonstrate that the two-field realizations of ref. [3, 8] can be derived in this way. Secondly, one may still deal with the same $w_{1+\infty}$ as in ref. [1], but restrict the stability subgroup, transferring some of its generators in the coset. We do this in Subsect. 4.2 and deduce a new example of the two-field realization of $w_{1+\infty}$.

4.1 Nonlinear realization of the extended symmetry $\tilde{G}$

Let us consider the algebra $\tilde{\mathcal{G}}$ with the following commutation relations:[4]

$$\begin{aligned} \left[V_n^s, V_m^k\right] &= (n(k+1) - m(s+1))\, V_{n+m}^{s+k} \\ \left[W_n^s, V_m^k\right] &= (n(k+1) - m(s+1))\, W_{n+m}^{s+k} \\ \left[W_n^s, W_m^k\right] &= 0 \end{aligned} \tag{4.1}$$

The generators V_n^s give rise to the standard $w_{1+\infty}$ algebra and the mutually commuting generators W_n^s are in the adjoint representation of $w_{1+\infty}$. They are none other than the Fourier modes of the currents $V^{(s)}$ and $W_1^{(s)}$ introduced in the previous Section (eqs. (3.2) and (3.11)). From eqs.(4.1) it is evident that W_n^s constitute an infinite-dimensional ideal in $\tilde{\mathcal{G}}$.

As discussed in Section 2, for constructing a nonlinear realization of the associated symmetry group $\tilde{G}$ one needs to define the appropriate infinite-dimensional coset space with a suitably chosen stability subgroup H. In the present case there are many possible choices for the stability subgroup, due to the commutativity of the generators W_n^s. Here we consider the simplest possibility, with the stability subalgebra formed by the following generators:

$$V_n^0\ (n \geq 0) \quad , \quad V_m^s\ (s \geq 1)\,;\ W_n^k\ (k \geq 0)\,. \tag{4.2}$$

An element of the associate coset space can be parametrized as follows:

$$g = e^{zV_{-1}^0}\, e^{\sum_{n\geq 0} v_n V_n^{-1}}\, e^{\sum_{m\geq 0} w_m W_m^{-1}}\,. \tag{4.3}$$

As usual, the group $\tilde{G}$ acts as the left multiplications on the coset element (we will first consider the action of the $w_{1+\infty}$ subgroup of $\tilde{G}$ with the generators V_n^s):

$$g_0 \cdot g(z, v_n, w_m) = g(z', v_n', w_m') \cdot h(z, v_n, w_m; g_0) \quad , \quad g_0 = exp\Big(\sum_{n \geq -s-1} a_n^s V_n^s\Big) \tag{4.4}$$

Now the coordinate z constitutes a closed set under the group action together with the infinite tower of coset fields $v_n(z), w_m(z)$:

$$\begin{aligned} \delta^s z &= -a^s(z)(s+1)(v_1)^s \\ \delta^s v_0 &= -a^s(z)s(v_1)^{s+1} \\ \delta^s w_0 &= 0\,, \quad \text{etc.}\,, \end{aligned} \tag{4.5}$$

[4] This algebra can be regarded as a contraction of the sum of two independent $w_{1+\infty}$ algebras.

where

$$a^s(z) = -\sum_{n\geq -s-1} a^s_n z^{n+s+1} .$$

Thus, we obtained the realization of $w_{1+\infty}$ on the coordinate z and an infinite number of coset fields v_n, w_m. Now we may impose the inverse Higgs constraints in order to find the kinematic equations for expressing the higher-order coset fields in terms of $v_0(z), w_0(z)$. The appropriate set of constraints reads as follows:

$$\omega_n^{-1} = 0 \quad , \quad \tilde{\omega}_n^{-1} = 0 \quad , \quad (n \geq 0) \tag{4.6}$$

where $\omega^s_n, \tilde{\omega}^k_m$ are Cartan forms:

$$g^{-1}dg = \sum_{s,n} \omega^s_n V^s_n + \sum_{k,m} \tilde{\omega}^k_m W^k_m \quad . \tag{4.7}$$

After straightforward calculations one finds that the first higher-order coset fields v_1, w_1 are expressed by

$$v_1 = \partial v_0 \quad , \quad w_1 = \partial w_0 \quad . \tag{4.8}$$

Finally, substituting the expression for v_1 in the transformations laws for z and v_0 (4.5), we obtain

$$\begin{aligned} \delta^s z &= -a^s(z)(s+1)(\partial v_0)^s \\ \delta^s v_0 &= -a^s(z)s(\partial v_0)^{s+1} \\ \delta^s w_0 &= 0 \quad . \end{aligned} \tag{4.9}$$

These transformations, being rewritten in the active form, are recognized as the standard two-field realization of $w_{1+\infty}$ (cf. eqs.(3.5))

$$\begin{aligned} \tilde{\delta} v_0(z) &\equiv v_0'(z) - v_0(z) = a^s(z)(\partial v_0)^{s+1} \\ \tilde{\delta} w_0(z) &\equiv w_0'(z) - w_0(z) = a^s(z)(s+1)(\partial v_0)^s \partial w_0 \quad . \end{aligned} \tag{4.10}$$

As for the transformations with the generators W^s_n, they do not touch the coordinate and the parameter-fields associated with the generators V_n^{-1} and act only on $w_m(z)$. In particular,

$$\delta^s w_0 = b^s(z)(v_1)^{s+1} = b^s(z)(\partial v_0)^{s+1} \quad . \tag{4.11}$$

In accordance with the remark in the end of Sect.3, there exists a one-parameter family of embeddings of $w_{1+\infty}$ in $\tilde{G}$. Introducing the new basis in $\tilde{G}$

$$V^s_{(\gamma)n} = V^s_n + \gamma s W^s_n \; , \quad W^s_n \; , \tag{4.12}$$

one easily checks that the newly defined generators satisfy the same commutation relations as the old ones. In order to study the action of the $w_{1+\infty}$ transformations with a given γ on the coset fields, it is convenient to pass in eq. (4.3) to the new set of generators (4.12). This entails an appropriate redefinition of the coset fields, on the new set of fields the $w^{(\gamma)}_{1+\infty}$ transformations being realized just as the original $w_{1+\infty}$ transformations (corresponding to $\gamma = 0$) on v_m and w_n. It is easy to check that the field v_0 is not redefined while the generator W_0^{-1} now enters with the following combination of original fields:

$$w_0 + \gamma v_0 .$$

The relevance of such a combination has been already mentioned in the end of Sect.3. Here we see how it appears within the coset space approach.

Thus, we have succeeded in deducing the standard realizations of $w_{1+\infty}$ on two scalar fields in the framework of the nonlinear realizations method. One of these fields, v, as before is the coset field associated with the generator V_0^{-1} from the spin 1 sector of $w_{1+\infty}$, while the other, w (or $\tilde{w}$), is associated with the generator W_0^{-1} from the spin 1 sector of the infinite-dimensional ideal $\tilde{\mathcal{H}}$ extending $w_{1+\infty}$ to $\tilde{\mathcal{G}}$. It is worth mentioning that the same fields can be alternatively interpreted as the coset parameters corresponding to a nonlinear realization of the symmetry isomorphic to $\tilde{\mathcal{G}}$ and generated by the currents following from (3.2), (3.11) via the change $\tilde{w} \Leftrightarrow v$. With respect to this symmetry, the roles of the fields $\tilde{w}$, v are inversed: $\tilde{w}$ comes out as the coset parameter related to the subalgebra $w_{1+\infty}$ while v is associated with the ideal.

4.2 Nonlinear realization of $w_{1+\infty}$ with two essential fields

Now we consider another possibility to obtain a two-field realization of $w_{1+\infty}$. This realization turns out to be of an essentially new kind compared to the previously known ones.

The starting point of our construction will be the realization of $w_{1+\infty}$ in the coset space $w_{1+\infty}/H$, where the stability subgroup H is now generated by

$$V_m^s \; (s \geq 1) \,. \tag{4.13}$$

In other words, we put in the coset space along with the spin 1 generators all the generators from the (truncated) Virasoro subalgebra in the spirit of ref. [10, 11, 12].

An element of this coset space can be parametrized as follows:

$$g = e^{zV_{-1}^0} \, e^{\sum_{n\geq 0} v_n V_n^{-1}} \, e^{\sum_{m\geq 0} u_m V_m^0} \,. \tag{4.14}$$

Thus we have now two infinite series of the coset parameters-fields associated with the generators V_n^{-1} and V_m^0.

As in the previous cases we may easily find the transformation properties of the fields v_n, u_m under the $w_{1+\infty}$ symmetry realized by left multiplications of the coset (4.14)

$$\begin{aligned} \delta^s z &= -(s+1)\, a^s \,(v_1)^s \\ \delta^s v_0 &= -s\, a^s \,(v_1)^{s+1} \\ \delta^s u_0 &= -(s+1)\, \partial a^s \,(v_1)^s - 2s\, a^s \,(v_1)^{s-1} v_2 \;, \quad \text{etc.} \end{aligned} \tag{4.15}$$

Using the inverse Higgs-type constraints on the Cartan forms

$$g^{-1}dg = \sum \omega_n^s V_n^s$$

$$\omega_n^{-1} = \omega_n^0 = 0 \quad , \quad (n \geq 0) \tag{4.16}$$

one finds that all the coset fields are covariantly expressed through the two independent ones, v_0 and u_0. In particular, the coset fields v_1 and v_2 are represented by

$$v_1 = \partial v_0 \quad , \quad v_2 = \frac{1}{2}\partial v_1 = \frac{1}{2}\partial^2 v_0 \quad . \tag{4.17}$$

Finally, substituting the expressions for v_1, v_2 in the transformations laws of z, v and u (4.15), we find the new two-field realization of $w_{1+\infty}$

$$\begin{aligned} \delta^s z &= -(s+1)\, a^s\, (\partial v_0)^s \\ \delta^s v_0 &= -s\, a^s\, (\partial v_0)^{s+1} \\ \delta^s u_0 &= -(s+1)\, \partial\left(a^s\, (\partial v_0)^s\right) \end{aligned} \tag{4.18}$$

or, in the active form

$$\begin{aligned} \tilde{\delta}^s v_0 &= a^s\, (\partial v_0)^{s+1} \\ \tilde{\delta}^s u_0 &= -(s+1)\, \partial\left(a^s\, (\partial v_0)^s\right) + (s+1)\, a^s\, (\partial v_0)^s \partial u_0 \quad . \end{aligned} \tag{4.19}$$

In this new realization the field v_0 and the coordinate z form as before a representation of $w_{1+\infty}$ in their own, while u_0 transforms like the dilaton in the standard conformal theories. To see this, it is instructive to rewrite the transformation law of u_0 in the form

$$\tilde{\delta}^s u_0 = \partial(\delta^s z) - \delta^s z\, \partial u_0 \, . \tag{4.20}$$

Under the conformal group ($s = 0$) u_0 transforms as a $2D$ dilaton, with the purely chiral parameter $\delta^0 z = -a^0(z)$. However, for the higher spin transformations $\delta^s z$ ceases to be chiral because of the explicit presence of ∂v_0, so the $w_{1+\infty}$ transformations of u_0 can be reduced to the conformal ones only provided the field v_0 is on shell.

Several comments are in order here.

First, it should be emphasized that the second realization can be straightforwardly extended by adding more generators (with spins 3, 4, etc.) to the coset. In this case we will obtain some multi-field realizations of $w_{1+\infty}$. Moreover, we are at freedom to combine the approaches used in the Subsections 4.1 and 4.2, including more generators in the coset as well as adding more currents, in order to get new realizations of $w_{1+\infty}$. This possibility will be exploited in the next Section.

As a second remark closely related to the first one, we note that including in this game the generators with higher spins ($s \geq 3$) immediately leads to the appearance of the pure shifts of the corresponding fields under higher spin transformations (due to the goldstone nature of these fields). Classically, if these systems admit any Lagrangian formulation, such shifts may appear only through Feigin-Fuchs-like terms in the relevant currents. At the same time, these terms would inevitably yield central charges in the commutators of the higher spin currents. On the other hand, the algebra $w_{1+\infty}$ is known to admit central extensions in the spin 1 and spin 2 sectors only, otherwise it should be deformed into $W_{1+\infty}$. So the nonvanishing central charges in the commutators of the higher spin currents could signal an existence of a hidden $W_{1+\infty}$ structure in these systems. Another, more prosaic solution to this controversy could consist in that these systems admit no Lagrangian formulations at all (see the next comment) and the only correct way to include the higher spin generators in the cosets is to start from the beginning from nonlinear realizations of $W_{1+\infty}$ or to extend $w_{1+\infty}$ in a proper way. We postpone the complete analysis of this interesting question to the future.

Finally, we wish to emphasize that any new field realization of $w_{1+\infty}$ (or of $W_{1+\infty}$) seems to make sense only if it follows from some Lagrangian system (at least from the system of free fields). But just in the second, dilaton-like realization we did not succeed in finding any meaningful Lagrangian possessing invariance under the transformations (4.19). Respectively, it seems impossible to define currents which would generate (4.19) via Poisson brackets[5]. However, in a funny way it turns out that such a Lagrangian can be constructed if one adds to the system $\{v_0, u_0\}$ one more field, with a specific transformation law under $w_{1+\infty}$. We discuss this possibility in the next Section.

5 New three-field coset space realization of $w_{1+\infty}$

In this Section we consider a more complicated coset realization of $w_{1+\infty}$. It is based on a combination of the two approaches used in the previous Section. The main idea is to start with some extended group $\hat{G}$ which contains $w_{1+\infty}$ as a factor group. We first present the invariant Lagrangian and the transformation laws for the relevant field system and then explain how these laws can be derived from the coset space formalism.

Let us consider the following three-field modification of the action (3.6), (3.8)

$$\begin{aligned} S &= \int d^2z\,(-\partial_+ v\partial_- v + \partial_+ w\partial_- v + \partial_- w\partial_+ v + \partial_+ u\partial_- u) \\ &= \int d^2z\,(\partial_+ \tilde{w}\partial_- v + \partial_- \tilde{w}\partial_+ v + \partial_+ u\partial_- u) \quad . \end{aligned} \tag{5.1}$$

The latter is invariant under the transformations

$$\begin{aligned} \tilde{\delta}^s v &= a^s(z)(\partial v)^{s+1} \\ \tilde{\delta}^s u &= -\alpha\partial\left(a^s(z)(s+1)(\partial v)^s\right) + a^s(z)(s+1)(\partial v)^s\partial u \\ \tilde{\delta}^s w &= \frac{1}{2}s(s+1)a^s(z)(\partial v)^{s-1}\left((\partial u)^2 + 2\alpha\partial^2 u\right) \\ &+ a^s(z)(s+1)(\partial v)^s\partial w \quad , \end{aligned} \tag{5.2}$$

with α being an arbitrary parameter. One sees that the transformations of v and u coincide with (4.19) (the parameter α can be introduced in (4.19) via a rescaling of u)

[5]The sum of the free actions for the fields u_0 and v_0 is invariant under the following modified transformations: u_0 is still transformed according to (4.19) while v_0 according to

$$\tilde{\delta} v_0 = a^s(z)(\partial v_0)^{s+1} + s(s+1)a^s(z)(\partial v_0)^{s-1}(\partial^2 u_0 + \frac{1}{2}(\partial u_0)^2\,) \,.$$

However, the Lie bracket structure of these transformations is not that of $w_{1+\infty}$. Rather this is a kind of $W_{1+\infty}$ symmetry analogous to the one found by two of us (S.B.& E.I.) in the system of two fields, one of which enters with a Liouville potential term [9]. Note that the Liouville term for u_0, $exp(-u_0)$, is in itself invariant under the above transformations (modulo a total derivative).

and so constitute $w_{1+\infty}$ with the classical central charge $\sim \alpha^2$ in the Virasoro sector. Thus, for these fields we still have the dilaton-like realization of $w_{1+\infty}$ constructed in Subsect. 4.2 (for the moment we discard the subscript 0 of these fields). However, the above action is invariant only due to the presence of the third field w (or $\tilde{w}$). Its transformation law for $s \geq 1$ essentially differs from (3.5) and can by no means be reduced to the latter, since putting u or the stress-tensor of u, $T(u) = (\partial u)^2 + 2\alpha\partial^2 u$, equal to zero would clearly break the $w_{1+\infty}$ symmetry.

To understand why this reduction does not work, one needs to consider a commutator of two $w_{1+\infty}$ transformations on w. One finds that the commutator of two such transformations, with the spins $s+2$ and $k+2$, besides the original $w_{1+\infty}$ transformation of w with the spin $s+k+2$ and the bracket parameter

$$a_{br}^{s+k} = (s+1)a_1^s \partial a_2^k - (1 \leftrightarrow 2,\ s \leftrightarrow k)\ ,$$

necessarily contains some extra terms proportional to α^2. They are none other than the transformations from the set (3.13), not only those corresponding to the variation δ_1^s, but also those associated with higher order variations. Their contribution appears already in the commutator of the spin 2 and spin 3 $w_{1+\infty}$ variations (a pure chiral shift of w generated by the spin 1 current $W_1^{(-1)}$), so the spin 3 current in the present case turns out to be not primary. Commuting these extra variations of w with the original ones (5.2) we actually produce all the transformations from the set (3.13) which, as mentioned in Sect. 3 (see discussion below eq. (3.14)), form an infinite-dimensional ideal $\hat{\mathcal{H}}$ coinciding with the enveloping algebra of the spin 1 current $W_1^{(-1)}$. Thus, on the field w the $w_{1+\infty}$ symmetry is defined only modulo the ideal $\hat{\mathcal{H}}$ formed by the mutually commuting transformations (3.13). In other words, $w_{1+\infty}$ now appears as a factor algebra of an extended algebra $\hat{\mathcal{G}}$ by the ideal $\hat{\mathcal{H}}$. The transformations from this ideal are realized only on the field w, which explains why on the fields v and u the extended symmetry is reduced to $w_{1+\infty}$. For the time being, we do not completely understand why we need the additional field w and the factor-algebra interpretation of $w_{1+\infty}$ in order to construct the invariant action for the system v, u.

For a better insight into the structure of $\hat{\mathcal{G}}$ and in order to learn how to regain the above three-field realization from the coset space approach, we should construct the relevant currents and compute their OPE's, proceeding from the canonical formalism corresponding to the action (5.1).

The $w_{1+\infty}$ currents are easily found to be

$$\hat{V}^{(s)} \doteq (\partial v)^{s+1}\partial w - \frac{1}{s+2}(\partial v)^{s+2} + (s+1)(\partial v)^s \left(\frac{1}{2}(\partial u)^2 + \alpha\partial^2 u\right) \quad . \tag{5.3}$$

They generate the transformations (5.2) via the canonical OPE's

$$w(z_1)v(z_2) = w(z_1)w(z_2) = u(z_1)u(z_2) = log(z_{12})\ , \quad v(z_1)v(z_2) = 0 \tag{5.4}$$

following from the action (5.1) (or, equivalently, via the canonical Poisson brackets). The currents $\hat{W}_1^{(s)}$ and the higher-order currents generating the $\hat{\mathcal{H}}$ transformations (3.13) are still given by the expressions (3.11) and (3.12).

Now it is easy to write the whole set of OPE's defining the algebra $\hat{\mathcal{G}}$ [6]

$$
\begin{aligned}
\hat{V}^{(s)}(z_1)\hat{V}^{(k)}(z_2) &= \frac{(s+k+2)\hat{V}^{(s+k)}(z_2)}{z_{12}^2} + \frac{(s+1)\partial\hat{V}^{(s+k)}(z_2)}{z_{12}} - 6\alpha^2\frac{\delta_{s,0}\delta_{k,0}}{z_{12}^4} \\
&- 6\alpha^2(s^2+s)(k^2+k)\frac{\hat{W}_1^{(s-2)}(z_1)\hat{W}_1^{(k-2)}(z_2)}{z_{12}^4} - \frac{\delta_{s+1,0}\delta_{k+1,0}}{z_{12}^2} \\
\hat{V}^{(s)}(z_1)\hat{W}_1^{(k)}(z_2) &= \frac{(s+k+2)\hat{W}_1^{(s+k)}(z_2)}{z_{12}^2} + \frac{(s+1)\partial\hat{W}_1^{(s+k)}(z_2)}{z_{12}} + \frac{\delta_{s+1,0}\delta_{k+1,0}}{z_{12}^2} \\
\hat{W}_1^{(s)}(z_1)\hat{W}_1^{(k)}(z_2) &= 0 \ . \qquad (5.5)
\end{aligned}
$$

The main difference between $\hat{\mathcal{G}}$ and $\tilde{\mathcal{G}}$ considered in Sect.4 (recall that $\tilde{\mathcal{G}}$ is defined by the OPE (3.3) and by the first OPE in (3.14)) is the appearance of the bilinears of the currents $\hat{W}_1^{(s)}$ (along with the currents $\hat{W}_1^{(s)}$ themselves) in the OPE of $\hat{V}^{(s)}(z_1)\hat{V}^{(s)}(z_2)$. Hence $\hat{\mathcal{G}}$ is a kind of *nonlinear* deformation of $\tilde{\mathcal{G}}$. From the above OPE's it is clear (after re-expanding the bilinears in the r.h.s. of eq. (5.5) over the local products given at the point z_2) that actually not only the currents $\hat{W}_1^{(s)}$ and their products, but also the products involving derivatives of $\hat{W}_1^{(s)}$ appear. Recalling that all such products can be represented as linear combinations of the currents (3.11), (3.12) and the derivatives of the latter and that these combinations constitute the ideal $\hat{\mathcal{H}}$ which coincides with the enveloping algebra of $\hat{W}_1^{(-1)}$, one concludes that $\hat{\mathcal{G}}$ admits a twofold interpretation. Namely, it can be viewed either as a nonlinear deformation of $\tilde{\mathcal{G}}$ with the same two sets of defining currents $\hat{V}^{(s)}$, $\hat{W}_1^{(s)}$ or as a linear algebra generated by $\hat{V}^{(s)}$ and the infinite sequence of currents $\hat{W}_1^{(s)}$, $\hat{W}_2^{(s)}$, ... $\hat{W}_N^{(s)}$, ...

It is easy to figure out from the above OPE's that the currents $\hat{W}$ indeed form an ideal $\hat{\mathcal{H}}$ in $\hat{\mathcal{G}}$, so that the factor-algebra of $\hat{\mathcal{G}}$ by $\hat{\mathcal{H}}$ is $w_{1+\infty}$. As before, there exists a one-parameter family of embeddings of this $w_{1+\infty}$ into $\hat{\mathcal{G}}$: one can check that the redefinition of $\hat{V}^{(s)}$ as in eq. (3.15) does not affect the OPE's (5.5) except for the central terms where there appears a dependence on the parameter γ, such that the central charge in the spin 1 sector of $w_{1+\infty}$ vanishes at $\gamma = -\frac{1}{2}$. Note that the closure of the Jacobi identities for $\hat{\mathcal{G}}$ could be straightforwardly checked starting with its defining OPE's. But there is no actual need to do this, since we have derived these OPE's by specializing to the field model (5.1) and making use of the canonical formalism for the involved fields.

The interpretation of $\hat{\mathcal{G}}$ as a nonlinear algebra makes a bit tricky the construction of the associated nonlinear realization. Nonetheless, this can be done using a proposal of ref. [11] which seems to work for any nonlinear algebras. Namely, one treats all the bilinears of the basic currents appearing in the defining OPE's (or the bilinears of the defining generators, if the standard commutation relations are used) as some new independent objects, thus formally replacing the original nonlinear algebra by some huge linear one which can already be handled by the standard methods of coset realizations. Normally one puts all these new currents (or generators) into the stability subalgebra so that only the original, basic generators turn out to be actually involved in the relevant coset constructions.

[6]We prefer to write here the OPE's instead of the commutation relations, as the latter look rather intricated for $\hat{\mathcal{G}}$.

We will make use of all these ideas to construct a nonlinear realization of the symmetry associated with $\hat{\mathcal{G}}$. In the case at hand, the trick proposed in [11] corresponds just to sticking to the interpretation of $\hat{\mathcal{G}}$ as a linear algebra involving as a subalgebra the whole enveloping algebra of the spin 1 current $\hat{W}_1^{(-1)}$. We combine the two approaches used in Sect.3, namely we put in the coset the generators with spin 1 and spin 2 from the set of $\hat{V}^{(s)}$, as well as the spin 1 generators coming from $\hat{W}_1^{(k)}$. All the generators coming from the composite currents (or, equivalently, from $\hat{W}_N^{(s)}$ for $N \geq 2$) are placed in the stability subalgebra.

Thus, the representative of our coset reads

$$\hat{g} = e^{z\hat{V}_{-1}^0}\, e^{\sum_{n\geq 0} v_n \hat{V}_n^{-1}}\, e^{\sum_{m\geq 0} w_m W_m^{-1}}\, e^{\sum_{m\geq 0} u_m \hat{V}_m^0} \, . \tag{5.6}$$

The machinery for invoking the inverse Higgs phenomenon is the same as in the previous Sections. After not very difficult labour we find that the whole symmetry $\hat{\mathcal{G}}$ can be realized on the three essential scalar fields v_0, u_0 and w_0, whose transformations under the action of the $w_{1+\infty}$ generators are just those given by the equations (5.2). Thus we have succeeded in deriving the above three-field realization of $w_{1+\infty}$ in a purely geometric way, proceeding from the coset space of the associated extended symmetry $\hat{G}$.

Before ending this Section, let us make a few remarks concerning possible generalizations of the above three-field realization and its relation to other models.

First of all, let us note that in our realization the field u enters in the currents and in the transformations of the field w only through its stress-tensor $T(u)$. Thus, we may easily extend our realization, in a close analogy with the W_3 realizations [13], to the multi-scalar case, replacing $T(u)$ by the corresponding stress-tensor

$$T(u) \to T(u, \phi, \ldots) = T(u) + (\partial\phi)^2 + \ldots \quad ,$$

where the fields $\phi, \ldots$ transform like u, but without inhomogeneous pieces. For the moment it is not quite clear how to reproduce such multi-field realizations in the framework of the coset space approach. One could interpret the extra fields ϕ, ... as scalars of $\hat{G}$ because these actually transform only due to the $w_{1+\infty}$ shift of one of their arguments (recall eq. (4.20)), but then it is unclear how to ensure the appearance of these fields in the transformation law of w (where they enter through the modified stress-tensor $T(u, \phi, \ldots)$) that is necessary for the invariance of the action. These reasonings prompt that the fields $\phi, \ldots$ should be somehow incorporated from the beginning in the coset space approach as the coset parameters.

The most intriguing possibility in what concerns these multi-scalar realizations seems to be as follows. Let us consider the realization with one extra field ϕ. One may add its kinetic term to the action (5.1) with the sign opposite to the sign of the kinetic term of u, assume that ϕ transforms under $w_{1+\infty}$ by an inhomogeneous law similar to the transformation law of u and, respectively, add the appropriate Feigin-Fuchs term for ϕ to the stress-tensor. Then, due to the wrong sign of the kinetic term for ϕ, the contributions of u and ϕ to the central charge of the classical stress-tensor can be cancelled (one may equivalently add ϕ to the action with the normal kinetic term, but add a purely imaginary Feigin-Fuchs term for this field to the stress-tensor). Recalling that the $\hat{W}_N^{(s)}$ transformations which appear in the commutators of the $w_{1+\infty}$ variations of w (and, respectively, the $\hat{W}$ terms in the first of the OPE's (5.5)) are proportional to the Virasoro central charge, the resulting transformations close on $w_{1+\infty}$ without any contributions from the $\hat{W}^{(s)}$ transformations and we end up with the original $w_{1+\infty}$ symmetry, now realized on four fields with the signature $(2+2)$ in the target space. More precisely, this realization is obtained by changing

$$T(u) \rightarrow T(u,\phi) = T(u) - (\partial\phi)^2 + 2\alpha\partial^2\phi$$

in the transformation laws (5.2) and ascribing the following transformation law to ϕ

$$\delta^s\phi = \alpha\partial(a^s(z)(s+1)(\partial v)^s) + a^s(z)(s+1)(\partial v)^s\partial\phi\ .$$

This $(2+2)$ system deserves a further study in view of its possible relation to the $N=2$ string of Ooguri and Vafa [14]. A closely related observation is that it naturally appears as the bosonic subsystem in the following $N=2$ supersymmetric extension of the $(1+1)$ action (3.6), (3.8): one can $N=2$ supersymmetrize this action by introducing two $N=2$ chiral superfields with the kinetic terms of the opposite sign. Then one may hope that the $(2+2)$ system in question could be given a coset space interpretation in the framework of the nonlinear realizations of $N=2$ super $w_{1+\infty}$ algebras [15, 8], along the lines of [1] and the present paper.

Finally, we wish to point out that our three-field system action (5.1) coincides with the free part of the action for the conformal affine Toda (CAT) system [4], e.g. under the identification

$$v = \mu\ , \quad \tilde{w} = \nu\ , \quad u = \phi\ ,$$

where in the r.h.s. the standard notation for the CAT fields is used. So one may expect that some relationships exist between our realization of $w_{1+\infty}$ and the infinite-dimensional symmetries of the CAT model found in [5, 6].

Let us remind that the basic objects in the CAT model are the stress-tensor $W^2(z)$ and the spin 1 current $J^c(z)$. If the current $J^c(z)$ has a non-zero central charge c_J with respect to $W_2(z)$ and commutes with itself, i.e. obeys the following OPE's

$$\begin{aligned} W_2(z_1)J^c(z_2) &= \frac{J^c(z_2)}{z_{12}^2} - \frac{\partial J^c(z_2)}{z_{12}} + \frac{c_J}{z_{12}^3} \\ J^c(z_1)J^c(z_2) &= 0\ , \end{aligned} \qquad (5.7)$$

then it is possible to construct the infinite number of currents generating the higher spin symmetries of this model [5, 6]. It has been shown in [5] that in a special limit this symmetry is reduced to $w_{1+\infty}$.

In our realization we have an analog of J^c, the current $\hat{W}_1^{(-1)} = \partial v$, but it has a vanishing central charge with respect to the stress-tensor $2V^{(0)}$. However, if we add to the set of our $\hat{\mathcal{G}}$ currents one more set of higher spin currents

$$A^{(s)} = (s+1)(\partial v)^s\, \partial^2\tilde{w}\ , \qquad (5.8)$$

which also generate symmetries of the action (5.1), we become able to construct the CAT stress-tensor as a linear combination of the currents $V^{(0)}$ and $A^{(0)}$ at $\alpha = -\frac{1}{2}$

$$W^2(z) = 2V^{(0)}(z) - 2A^{(0)}(z) = 2\partial v\partial\tilde{w} + (\partial u)^2 - \partial^2 u - 2\partial^2\tilde{w}\ . \qquad (5.9)$$

With respect to this stress-tensor, the current $\hat{W}_1^{(-1)}$ behaves as a quasi-primary field with nonvanishing c_J and so we recover the CAT current structure on the spin 1 and spin 2 levels. Nevertheless it remains not quite clear how our factor-algebra realization of $w_{1+\infty}$ could be incorporated into the CAT higher spin algebra. We have checked that the realization of $w_{1+\infty}$ following from the CAT algebra in the large J^C limit [5] is different from ours: it does not involve the field $u(=\phi)$ and is rather reduced to the two-field realization discussed in Sect. 3 and Subsect. 4.1 (with some $\gamma \neq -\frac{1}{2}$). Perhaps, in order to clarify these points one needs to consider a nonlinear realization of the extension of $\hat{G}$ by the currents $A^{(s)}$. One of the most interesting problems in this context is how to obtain the CAT field equations starting from some nonlinear realization and applying the covariant reduction procedure (i.e. implementing a dynamical version of the inverse Higgs constraints [10, 11, 12]).

6 Summary and discussion

In this paper we discussed how the nonlinear (coset space) realization approach of ref.[1] could be extended to produce multi-field realizations of $w_{1+\infty}$. We considered two possibilities, one of which is to narrow the stability subalgebra (respectively, to enlarge the coset space) still staying with the original $w_{1+\infty}$ and the other is to embed $w_{1+\infty}$ into a larger symmetry and to constuct a nonlinear realization of the latter. In both cases we found non-trivial examples of multi-field realizations of $w_{1+\infty}$. We reproduced the two-field realization of $w_{1+\infty}$ [3] as a coset space one corresponding to the extension of $w_{1+\infty}$ by one more set of self-commuting higher spin currents, i.e. to the algebra $\tilde{\mathcal{G}}$. Further we constructed a new dilaton-like two-field realization of $w_{1+\infty}$ in the coset space involving all the spin 2 (Virasoro) generators. In order to find a reasonable invariant action for this system it turned out necessary to add one more field with a specific transformation law under $w_{1+\infty}$. We have shown that this new three-field realization corresponds to embedding $w_{1+\infty}$ as a factor-algebra into some nonlinear deformation $\hat{\mathcal{G}}$ of the previously employed algebra $\tilde{\mathcal{G}}$. This new algebra can as well be regarded as an extension of the universal enveloping of some commuting centreless $U(1)$ Kac-Moody algebra, so that this enveloping forms an ideal while the factor-algebra by this ideal is $w_{1+\infty}$. We proposed some generalizations of the three-field realization to include more fields. One of them is the four-field realization with the signature $(2+2)$ in the target space, such that the extended algebra $\hat{\mathcal{G}}$ is reduced for it to the standard $w_{1+\infty}$. This system is expected to have an intimate relation to the $N=2$ string [14] and to a $N=2$ supersymmetrization of the two-field realization of ref. [3]. Finally, we discussed possible parallels between our three-field system and the affine conformal Toda systems [4, 5, 6].

There remain many unsolved problems in the coset space approach to $w_{1+\infty}$. First of all, it is as yet unclear how to reproduce in this framework the most interesting multi-field realizations of $w_{1+\infty}$ [2] which basically correspond to the replacement of the singlet field in the one-field realization by some matrix field taking values in the fundamental representation of some classical algebra or its Cartan subalgebra. In the light of the discussion in the present paper, it seems that the only way to do this is to embed $w_{1+\infty}$ into some extended algebra which would contain as many spin 1 generators as the fields in the given realization, in order to enable us to interpret these fields as coset parameters. A link with the classical Lie algebras could arise as the property that the spin 1 generators, being mutually commuting, transform according to the fundamental representations of these algebras, treated as some outer automorphism ones (as in extended supersymmetries). It is very intriguing to reveal what such huge higher spin algebras could be. If existing, they could have innumerable implications in the theory of W strings, W gravities, etc.

Regarding the realizations presented in this paper, there are also some points about them which need a further clarification. Some of these issues were already discussed in the main body of the paper. Here we wish to fix one more interesting problem, concerning the geometric interpretation of the extended symmetries explored in this paper. The algebra $w_{1+\infty}$ has a nice interpretation as the algebra of the volume-preserving diffeomorphisms of a cylinder [1, 2]. It is desirable to have analogous geometric images for the algebras $\tilde{\mathcal{G}}$ and $\hat{\mathcal{G}}$. While the first algebra can be considered as a contraction of a direct sum of two algebras $w_{1+\infty}$ and so can hopefuly be understood from the point of view of two independent diffeomorphism groups, it remains a mystery what is the geometric meaning of the algebra $\hat{\mathcal{G}}$. Perhaps it could be somehow related (e.g. through a contraction) to the recently discovered algebra $\hat{W}^{\infty}$ [16] which is claimed to contain all possible W type algebras, either via a contraction or a truncation.

Finally, we wish to point out that the obvious interesting problem ahead is to gauge the above realizations of $w_{1+\infty}$. This might clarify their relation to w_∞ strings and w_∞ gravity.

Acknowledgements

Two of us (E.I.& S.K.) are very grateful to K.Stelle for bringing to their attention the preprint [1] before publication and for illuminating discussions at the early stage of the present study. They also wish to thank the INFN for financial support and the Laboratori Nazionali di Frascati for hospitality extended to them during the course of this work. E.I. thanks ENSLAPP in Lyon for hospitality.

References

[1] K.S.Stelle and E.Sezgin, "Nonlinear realizations of $w_{1+\infty}$", Preprint IC/92/122, Trieste 1992.

[2] E.Bergshoeff, C.N.Pope, L.J.Romans, E.Sezgin,X.Shen and K.S.Stelle, Phys. Lett., B 243 (1990) 350.

[3] X.Shen and X.J.Wang, Phys. Lett., B 278 (1992) 63.

[4] O.Babelon and L.Bonora, Phys. Lett., B 244 (1990) 220.

[5] H.Aratyn, C.P.Constantinidis, L.A.Ferreira, J.F.Gomes and A.H.Zimerman, Phys. Lett., B 281 (1992) 245.

[6] H.Aratyn, L.A. Ferreira, J.F.Gomes and A.H.Zimerman, Phys. Lett., B 293 (1992) 67.

[7] E.A.Ivanov and V.I.Ogievetsky, Teor. Mat. Fiz., 25 (1975) 164.

[8] E.Bergshoeff and M. de Roo, "$N = 2$ W-supergravity", Preprint UG-64/91, Groningen 1991.

[9] S.Bellucci and E.Ivanov, Mod. Phys. Lett., A 6 (1991) 1269.

[10] E.A.Ivanov and S.O.Krivonos, Teor. Mat. Fiz., 58 (1984) 200; Lett.Math.Phys., 8 (1984) 39.

[11] E.Ivanov, S.Krivonos and A.Pichugin, Phys. Lett., B 284 (1992) 260.

[12] E.A.Ivanov, S.O.Krivonos and R.P.Malik, "Boussinesq-type equations from nonlinear realizations of W_3", Preprint JINR E2-92-301, Dubna 1992.

[13] H.Lu, C.N.Pope, S.Schrans and X.J.Wang, Mod. Phys. Lett., A 7 (1992) 1835.

[14] H.Ooguri and C.Vafa, Mod. Phys. Lett., A 5 (1990) 1389; Nucl. Phys., B 361 (1991) 469.

[15] C.N.Pope and X.Shen, Phys. Lett., 236B (1990) 21.

[16] F.Yu and Y.-S.Wu, Nucl. Phys., B 373 (1992) 713.