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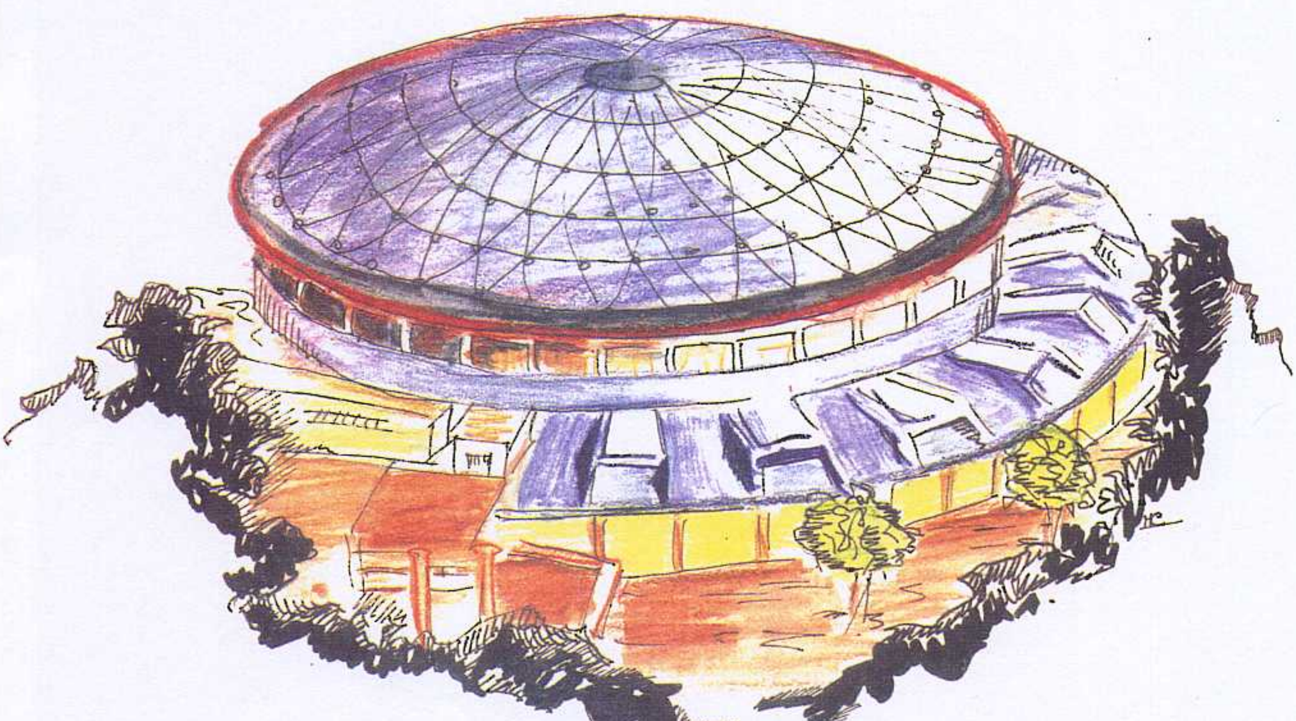

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N=3 SUPERSYMMETRIC EXTENSION OF KdV EQUATION[1]

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ABSTRACT

We construct a one-parameter family of N=3 supersymmetric and $SO(3)$ symmetric extensions of the KdV equation as a Hamiltonian flow on N=3 superconformal algebra and argue that it is non-integrable for any choice of the parameter. Then we propose a modified N=3 super KdV equation which possesses the higher order conserved quantities and so is a candidate for an integrable system. Upon reduction to N=2, it yields the recently discussed "would-be integrable" version of the N=2 super KdV equation. In the bosonic core it contains a coupled system of the KdV type equation and a three-component generalization of the mKdV equation. We give a Hamiltonian formulation of the new N=3 super KdV equation as a flow on some contraction of the direct sum of two N=3 superconformal algebras.

[1]Invited talk at "8th Workshop on Nonlinear Evolution Equations and Dynamical Systems", JINR, Dubna, Russia, July 6-17,1992. To be published in"Proceedings..."

1 Introduction

In recent years there has been an incredible growth of interest in studying integrable KdV-type hierarchies and their supersymmetric extensions, mainly due to the distinguished role these systems play in $2D$ (super)gravities and the related matrix models [1-8].

A remarkable feature of the KdV hierarchy is its relation, via the second Hamiltonian structure, to the Virasoro algebra [2]. This provides a link between the KdV hierarchy and $2D$ conformal field theories (and $2D$ gravity). The mKdV hierarchy is related in the same way to the $U(1)$ Kac-Moody algebra, the famous Miura map being recognized as the Sugawara-Feigin-Fuchs representation for the Virasoro algebra. Analogously, nonlinear W-algebras and their various generalizations define the second Hamiltonian structures for generalized KdV hierarchies which thus turn out to be relevant to W-gravities and proper generalizations of the latter. For instance, Zamolodchikov's W_3-algebra amounts to the second Hamiltonian structure for the Boussinesq hierarchy [3]. An important implication of these relationships is the possibility to construct new integrable systems of the KdV type and their superextensions in a regular way, starting with the structure relations of one or another infinite-dimensional algebra or superalgebra.

With making use of this approach, in refs. [3-8] $N = 1$ and $N = 2$ supersymmetric KdV equations with $N = 1$ and $N = 2$ superconformal algebras as the second Hamiltonian structure have been found and their integrability properties have been studied. It is of interest to treat in the same context higher N superextensions of KdV, by relating them to the higher N superconformal algebras. Some preliminary steps in this direction for the $N = 3$ and $N = 4$ cases (however, without any discussion of the integrability issues) have been made in [9,10]. In the present paper we report on the results of a more thorough study of the $N = 3$ case.

Before displaying the main content of our paper let us briefly recall the precise meaning of the aforementioned interrelation between the KdV and super KdV systems on the one hand and Virasoro and super Virasoro algebras on the other.

As was shown in [2], the KdV equation

$$u_t = -u_{xxx} + 6uu_x \tag{1.1}$$

can be treated as a Hamiltonian system,

$$u_t = \{u, \mathcal{H}\} \quad ,$$

with the Hamiltonian and the Poisson brackets defined by

$$\mathcal{H} = \frac{1}{2}\int dx\, u^2(x) \quad , \quad \{u(x), u(y)\} = \left[-\partial^3 + 4u\partial + 2u_x\right]\delta(x-y) \quad . \tag{1.2}$$

Just this property is referred to as the existence of the second Hamiltonian structure for the KdV equation. For the Fourier modes of $u(x)$,

$$u(x) = \frac{6}{c}\sum_n \exp(-inx)L_n - \frac{1}{4} \quad , \tag{1.3}$$

the Poisson brackets in (1.2) imply the structure relations of the Virasoro algebra

$$i\{L_n, L_m\} = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \, . \tag{1.4}$$

So, from the formal point of view, the definition (1.2) means that the density of the KdV Hamiltonian $\mathcal{H}$ is the square of a conformal stress-tensor $u(x)$ obeying the Virasoro algebra (1.2), (1.4). Note that the Hamiltonian in eq. (1.2) has dimension 3 and is unique, i.e. it is the only Hamiltonian of such a dimension that can be built out of the dimension 2 field $u(x)$. The higher order conserved quantities of the KdV equation can be regarded as the Hamiltonians which generate, through the Poisson brackets (1.2), next equations from the KdV hierarchy.

The same idea was applied for constructing $N = 1$ and $N = 2$ superextensions of the KdV equation [3-8]. They were related in an analogous way, via the second Hamiltonian structure, to $N = 1$ and $N = 2$ superconformal algebras. In the latter case, starting from $N = 2$ superconformal algebra in the form

$$\{\Phi(X), \Phi(X')\} = [\mathcal{D}_1\mathcal{D}_2\partial + 2\Phi\partial - (\mathcal{D}_1\Phi)\mathcal{D}_1 - (\mathcal{D}_2\Phi)\mathcal{D}_2 + 2\Phi_x]\,\Delta(X - X') \tag{1.5}$$

with

$$X \equiv \{x, \theta_1, \theta_2\} \quad , \quad \mathcal{D}_i = \theta_i\partial + \frac{\partial}{\partial\theta_i} \quad ,$$

$$\Delta(X - X') = (\theta_2 - \theta_2')(\theta_1 - \theta_1')\delta(x - x')$$

and choosing the most general $N = 2$ supersymmetric Hamiltonian of dimension 3

$$\mathcal{H} = \frac{1}{2}\int dx d^2\theta \left(\Phi\mathcal{D}_1\mathcal{D}_2\Phi + \frac{a}{3}\Phi^3\right) \, , \tag{1.6}$$

where a is an arbitrary constant, one finds the following one-parameter family of supersymmetric evolution equations:

$$\Phi_t = -\Phi_{xxx} + 3\left(\Phi\mathcal{D}_1\mathcal{D}_2\Phi\right)_x + \frac{a-1}{2}\left(\mathcal{D}_1\mathcal{D}_2\Phi^2\right)_x + 3a\Phi^2\Phi_x \quad . \tag{1.7}$$

The $N = 1$ super KdV equation can be obtained as a proper reduction of this $N = 2$ one.

It was shown that the equation (1.7) is completely integrable, i.e. possesses a Lax pair representation and admits an infinite number of the conserved quantities, only for $a = -2,\ 4$ [7]. For $a = 1$ there still exist higher-order conservation laws [8], however, no standard Lax representation is known. Hence, the proof of complete integrability of the $N = 2$ super KdV equation for $a = 1$ is an open problem.

A natural extension of the above scheme to the $N = 3$ case we are interested in is to start with the $N = 3$ supercurrent which is subject to the SOPE relations (or, equivalently, the Poisson brackets) generating $N = 3$ superconformal algebra [11], to construct the appropriate Hamiltonian out of this supercurrent and to define the $N = 3$ super KdV equation as the evolution equation with respect to this Hamiltonian structure. This is what we do in Sect.2 of the present paper. We show that the most general $N = 3$ super KdV Hamiltonian (respecting the automorphism $SO(3)$ symmetry along with $N = 3$ supersymmetry), like in the $N = 2$ case, involves one free parameter, thus generating a one-parameter family of the $N = 3$ super KdV equations [1]. Requiring the $N = 3$ KdV equation to yield, upon the reduction $N = 3 \rightarrow N = 2$, one of the integrable (or "would-be integrable") versions of the $N = 2$ KdV equation fixes the parameter at some non-zero values. Unfortunately, and this is the radical difference from the lower N cases, even for these special values of the parameter the $N = 3$ KdV equation turns out to be non-integrable: it does not admit the Lax representation (at least in the form employed earlier in the $N = 0$, $N = 1$ and $N = 2$ cases) and nontrivial local higher order conservation laws.

In Sect.3, in order to clear up the origin of this difficulty, we analyze the question of existence of the first non-trivial higher order conservation law for the most general $N = 3$ super KdV equation containing several free parameters. We find that requiring the existence of such a conservation law unambiguously fixes all the unknown coefficients in the $N = 3$ super KdV equation (as in the previous case we also require $SO(3)$ invariance and the existence of a proper reduction to $N = 2$ super KdV). The resulting equation is different from that constructed in Sect. 2. Upon the reduction to $N = 2$ it turns out to yield just the special would-be integrable case of the $N = 2$ super KdV equation with $a = 1$. It contains, as its bosonic core, the coupled system of the ordinary KdV equation for the dimension 2 scalar field $u(x)$ (conformal stress-tensor) and the special version of the matrix modified KdV one for the $SO(3)$ triplet of the dimension 1 fields $v^i(x)$ ($SO(3)$ Kac-Moody currents).

In Sect.4 we address the problem of the Hamiltonian description of our new $N = 3$ super KdV equation. We find that it can be obtained as a closed subsystem of an enlarged system of the superfield equations involving an extra $N = 3$ superfield $\tilde{J}$. The latter generates a centrally extended $N = 3$ superconformal algebra while the KdV superfield J itself is now treated as *quasi-primary* with respect to $\tilde{J}$, with an additional central charge. On its own right J generates a commutative superalgebra.

[1]In [10] a particular case of this general Hamiltonian has been considered, it corresponds to the zero value of the parameter.

2 $N = 3$ super KdV from $N = 3$ superconformal algebra

For deducing an $N = 3$ extension of the KdV equation we can try the same strategy as in the $N = 0$, $N = 1$ and $N = 2$ cases. Namely, we choose as the basic object a $N = 3$ conformal supercurrent

$$J(Z) = \psi(x) + \theta^i v^i(x) + \theta^{3-i}\xi^i(x) + \theta^3 u(x) \tag{2.1}$$

where $Z = (x, \theta^i), i = 1, 2, 3$ are the coordinates of $N = 3$, $1D$ superspace,

$$\theta^3 = \frac{1}{6}\epsilon^{kji}\theta^i\theta^j\theta^k \quad , \quad \theta^{3-i} = \frac{1}{2}\epsilon^{kji}\theta^j\theta^k \tag{2.2}$$

and the components $\psi(x)$, $v^i(x)$, $\xi^i(x)$, $u(x)$ form the supermultiplet of currents of $N = 3$ superconformal algebra [11] (respectively, the dimension $\frac{1}{2}$ singlet fermionic current, the triplet of the dimension 1 $SO(3)$ Kac-Moody currents, the triplet of the dimension $\frac{3}{2}$ fermionic currents and the conformal stress-tensor of dimension 2). The structure relations of the $N = 3$ superconformal algebra with an arbitrary central charge c can be summarized as the following Poisson brackets between the supercurrents $J(Z)$, $J(Z')$:

$$\{J(Z), J(Z')\}_+ = \left[\frac{c}{12}\mathcal{D}^3 - \frac{1}{2}J\partial + \frac{1}{2}\mathcal{D}^i J\mathcal{D}^i + \partial J\right]\Delta(Z - Z') \quad , \tag{2.3}$$

where we denoted

$$\Delta(Z - Z') = \frac{1}{6}\epsilon^{ijk}(\theta^i - \theta^{i'})(\theta^j - \theta^{j'})(\theta^k - \theta^{k'})\delta(x - x')$$

and defined the spinor covariant derivatives

$$\mathcal{D}^i = \frac{\partial}{\partial\theta^i} - \theta^i\frac{\partial}{\partial x} \quad , \quad \left\{\mathcal{D}^i, \mathcal{D}^j\right\} = -2\delta^{ij}\partial_x \quad , \tag{2.4}$$

$$\mathcal{D}^3 = \frac{1}{6}\epsilon^{ijk}\mathcal{D}^i\mathcal{D}^j\mathcal{D}^k \quad , \quad \mathcal{D}^{3-i} = \frac{1}{2}\epsilon^{ijk}\mathcal{D}^j\mathcal{D}^k \quad , \quad \mathcal{D}^{3-ij} = \epsilon^{ijk}\mathcal{D}^k \quad .$$

The $N = 3$ supercurrent $J(Z)$ has the dimension $\frac{1}{2}$, so the most general Hamiltonian having the dimension 3 (needed for the correspondence with the bosonic KdV) and respecting both $N = 3$ supersymmetry and the automorphism $SO(3)$ symmetry is given by the expression

$$\mathcal{H} = \int dx d^3\theta \left(J\mathcal{D}^3 J + \frac{\alpha}{3}J\mathcal{D}^i J\mathcal{D}^i J\right) \quad , \tag{2.5}$$

where α is an arbitrary parameter and the specific normalization has been chosen for further convenience (c.f. eq. (1.6)). Using the Poisson structure (2.3) it is then easy to verify that the Hamilton equation

$$J_t = \{J, \mathcal{H}\} \tag{2.6}$$

yields the following two-parameter family of the evolution equations for the supercurrent $J(Z)$

$$J_t = -\frac{c}{6}J_{xxx}+3\left(J\mathcal{D}^3 J\right)_x+\frac{6-c\alpha}{12}\mathcal{D}^3\left(\mathcal{D}^i J\mathcal{D}^i J\right)+\frac{12-c\alpha}{6}\mathcal{D}^3\left(J\partial J\right)+\alpha\left(J\mathcal{D}^i J\mathcal{D}^i J\right)_x \quad . \tag{2.7}$$

Note that we are at freedom to fix the central charge c at any non-zero value by rescaling the variables in eq. (2.7) as

$$t\rightarrow \frac{1}{b}t \quad , \quad J\rightarrow bJ \quad , \quad \alpha\rightarrow\frac{1}{b}\alpha \quad ,$$

b being an arbitrary parameter. So we actually deal with the one-parameter family. It is convenient to choose $c=6$. Eventually, the $N=3$ super KdV equation we will discuss in this Section is as follows

$$J_t = -J_{xxx}+3\left(J\mathcal{D}^3 J\right)_x+\frac{1-\alpha}{2}\mathcal{D}^3\left(\mathcal{D}^i J\mathcal{D}^i J\right)+(2-\alpha)\mathcal{D}^3\left(J\partial J\right)+\alpha\left(J\mathcal{D}^i J\mathcal{D}^i J\right)_x \quad . \tag{2.8}$$

It remains to find out whether the parameter α can be chosen so that the associated equation is completely integrable as in the $N=2$ case, i.e. admits a Lax pair representation and exhibits infinitely many conservation laws.

To start with, we note that eq. (2.8) has a proper reduction to the $N=2$ case.

The reduction $N=3\rightarrow N=2$ goes as follows. The $N=3$ supercurrent $J(x,\theta^i)$ contains the $N=2$ supercurrent $\Phi(x,\theta^1,\theta^2)$ as a coefficient before θ^3 in its θ^3 expansion while all the additional currents are contained in the θ^3 independent part of J. Thus one should put $J=\theta^3\Phi$ and substitute this ansatz into eq. (2.8).

Under this choice we immediately obtain the $N=2$ super KdV equation (1.7) with

$$a=\alpha \quad .$$

It is clear that the integrable version of the $N=3$ super KdV equation (if it exists) should yield the integrable $N=2$ super KdV upon the reduction. So it is natural to limit our study to the following values of α:

$$\alpha_1=-2\ ,\ \ \alpha_2=4\ ,\ \ \alpha_3=1\ ,$$

which correspond, respectively, to the two integrable and one would-be integrable $N=2$ super KdV equations.

Unfortunately, our equation (2.8) admits no standard Lax representation in the form [8]

$$L_t=[-4\ L_+^{3/2},L]$$

for any value of α. We have checked this by a tedious but straightforward computation. One might think that, like in the $N=2$ case, eq. (2.8) could have higher order conservation laws despite the non-existence of Lax representation. However, our attempts to find non-trivial higher order conservation laws reducible to those of the $N = 2$ super KdV upon the reduction $N = 3 \to N = 2$ have also failed for any value of α. Thus a straightforward application of the approach used previously for constructing integrable KdV equations in the $N = 0$, $N = 1$ and $N = 2$ cases leads to a non-integrable system in the $N = 3$ case. In the next Section we propose another way to obtain an integrable $N = 3$ super KdV equation by considering the most general $N = 3$ superfield extension of the KdV equation and finding the conditions under which it possesses non-trivial higher order conservation laws.

3 N=3 super KdV and conservation laws.

Now we turn to an explicit construction of $N = 3$ supersymmetric KdV equation possessing non-trivial conservation laws. We postpone to Sect.4 a discussion of how it can be given a Hamiltonian interpretation.

Under natural conditions of $N = 3$ supersymmetry and $SO(3)$ symmetry the most general $N = 3$ super KdV equation is of the form

$$J_t = \mathcal{A}(J) \quad , \tag{3.1}$$

where $\mathcal{A}$ is a linear combination of all possible terms with proper dimension $(7/2)$ which can be constructed from the $N = 3$ superfield $J(Z)$ and covariant spinor derivatives. Explicitly, it is the six-parameter family of equations

$$\begin{aligned} J_t &= -J_{xxx} + a_1 \left(J\mathcal{D}^3 J\right)_x + a_2 \mathcal{D}^3 \left(J J_x\right) + a_3 \mathcal{D}^3 \left(\mathcal{D}^i J \mathcal{D}^i J\right) \\ &\quad + a_4 \left(\mathcal{D}^i J\right)_x \mathcal{D}^{3-i} J + a_5 J \left(\mathcal{D}^i J \mathcal{D}^i J\right)_x + a_6 J_x \left(\mathcal{D}^i J \mathcal{D}^i J\right) \quad . \end{aligned} \tag{3.2}$$

In order to reduce the number of parameters and thereby to simplify computations we impose the requirement that upon the reduction to the $N = 2$ case eq. (3.2) goes over to the known $N = 2$ KdV family (1.7).

The reduction requirement gives rise to the following relations between the parameters $a_1, \ldots, a_6$:

$$a_1 = 3 \quad , \quad a_3 = \frac{1-a}{2} \quad , \quad a_4 = 0 \quad , \quad 2a_5 + a_6 = 3a \quad , \tag{3.3}$$

where a is the parameter which enters the $N = 2$ super KdV equation. So, the $N = 3$ super KdV equation we will consider contains three undetermined parameters

$$\begin{aligned} J_t &= -J_{xxx} + 3\left(J\mathcal{D}^3 J\right)_x + a_2 \mathcal{D}^3 \left(J J_x\right) + \frac{1-a}{2} \mathcal{D}^3 \left(\mathcal{D}^i J \mathcal{D}^i J\right) \\ &\quad + \frac{1}{2}(3a - a_6)\, J \left(\mathcal{D}^i J \mathcal{D}^i J\right)_x + a_6 J_x \left(\mathcal{D}^i J \mathcal{D}^i J\right) \quad . \end{aligned} \tag{3.4}$$

The previously considered equation (2.8) is the particular case of (3.4) corresponding to the choice

$$a_2 = 2 - a \ , \quad a_6 = a \ .$$

Now we wish to inquire whether this three-parameter family of equations yields integrable systems for some specific values of the parameters. Here we do not concern the question of the existence of the relevant Lax pairs. Instead we search for the first non-trivial higher order conservation law.

The simplest candidate for the higher order conserved quantity is an integral of degree 5 over $N = 3$ superspace with the integrand constructed from all possible independent densities of degree 9/2, each multiplied by an undetermined coefficient[1]

$$
\begin{aligned}
H_5 &= \int dx d^3\theta \{ A_1 J\mathcal{D}^3 J_{xx} + A_2 J\mathcal{D}^i J\mathcal{D}^i J_{xx} + A_3 J J_x J_{xx} + A_4 J\mathcal{D}^3 J\mathcal{D}^3 J \\
&\quad + A_5 J J_x \mathcal{D}^i J\mathcal{D}^{3-i} J + A_6 J\mathcal{D}^i J\mathcal{D}^i J\mathcal{D}^3 J + J\left(\mathcal{D}^i J\mathcal{D}^i J\right)^2 \} \quad . \qquad (3.5)
\end{aligned}
$$

The coefficients are then fixed by requiring the integral to be conserved (i.e. time-independent) on the equation of motion (3.4),

$$
(H_5)_t = 0 \; .
$$

This also must fix the values of parameters a, a_2, a_6 in (3.4).

After tedious calculations one finds that *all coefficients* in the integral (3.5) and in eq. (3.4) are fixed to the unique values

$$
A_1 = -5 \; , \;\; A_2 = -\frac{5}{2} \; , \;\; A_3 = \frac{5}{2} \; , \;\; A_4 = 10 \; , \;\; A_5 = \frac{5}{3} \; , \;\; A_6 = \frac{20}{3} \qquad (3.6)
$$

$$
a = 1 \; , \;\; a_2 = 0 \; , \;\; a_6 = 0 \quad . \qquad (3.7)
$$

Thus in the $N = 3$ supersymmetric case there exists only one superfield extension of the KdV equation which possesses a nontrivial higher order conservation law

$$
J_t = -J_{xxx} + 3\left(J\mathcal{D}^3 J\right)_x + \frac{3}{2} J\left(\mathcal{D}^i J\mathcal{D}^i J\right)_x \quad . \qquad (3.8)
$$

It is curious that after reduction to the $N = 2$ case this equation goes over to the exceptional $N = 2$ super KdV equation with parameter $a = 1$. For completeness we write also the first two lower order conserved quantities for eq. (3.8)

$$
\begin{aligned}
H_1 &= \int dx d^3\theta J \\
H_3 &= \int dx d^3\theta \left(J\mathcal{D}^3 J + \frac{1}{3} J\mathcal{D}^i J\mathcal{D}^i J\right) \qquad (3.9)
\end{aligned}
$$

A few comments are needed concerning the equation (3.8).

First of all, we have started with the most general $N = 3$ superfield equation (3.2). The only extra demands we have employed from the beginning were rigid $SO(3)$

[1] Recall that the $N = 3$ superspace integration measure ($dx d\theta^3$) has the dimension 1/2, so the integral (3.5) has the dimension 5.

symmetry and the existence of a proper reduction to the $N = 2$ case. It seems very intriguing that under such general assumptions we were eventually left with the unique candidate for the integrable $N = 3$ KdV equation.

Secondly, recall that even for the $N = 2$ super KdV equation the integrability at $a = 1$ is an open problem due to lacking of the standard Lax representation in this case. The problem of proving integrability remains, of course, in our case too. Up to now we know only the first non-trivial conservation law for the equation (3.8). Let us stress, however, that the set of equations that must be satisfied by the coefficients a, a_i, A_i is highly overdetermined. There are about five times as many equations compared to the unknowns. So the very existence of this first nontrivial conservation law is a strong indication for the complete integrability of the corresponding equation.

Finally, we briefly discuss the bosonic core of our $N = 3$ super KdV equation (3.8).

It is straightforward to find the set of bosonic equations to which eq. (3.8) is reduced after putting all fermions equal to zero

$$
\begin{aligned}
u_t &= -u_{xxx} + 3\left(u^2 - v^i v^i_{xx} + u v^i v^i\right)_x \\
v^i_t &= -v^i_{xxx} + 3\left(u v^i\right)_x + 3 v^i v^j v^j_x \ ,
\end{aligned}
\tag{3.10}
$$

where

$$
v^i = \mathcal{D}^i J| \quad , \quad u = \mathcal{D}^3 J| \quad .
$$

Here we indicate by $|$ the superfield projection to $\theta = 0$. It is a crucial novel feature of the $N = 3$ KdV equation compared to the $N = 2$ one that in its bosonic sector, besides the dimension 2 KdV field $u(x)$ which is identified with a conformal stress-tensor and generates a Virasoro subalgebra in the $N = 3$ superconformal algebra (2.3), there is also a triplet of the dimension 1 fields $v^i(x)$ which generate an $SO(3)$ Kac-Moody subalgebra of (2.3). In the $N = 2$ case only one such a field is present and it generates a $U(1)$ Kac-Moody algebra.

So we see that the bosonic subsector of our $N = 3$ super KdV equation contains the two coupled equations – the KdV equation for the scalar field u and a three-component generalization of the mKdV equation, both with the extra mixed terms in the r.h.s. These equations cannot be decoupled by a redefinition of u. While the first equation is a kind of the perturbed KdV equation, the second one can be viewed as a perturbation of the equation

$$
v^i_t = -v^i_{xxx} + 3 v^i (v^2)_x \ , \tag{3.11}
$$

which is a particular case of the general $SO(3)$ matrix mKdV equation

$$
v_t = -v_{xxx} + A\frac{i}{2}[v, v_{xx}] + B v_x \,(v^2) + C v \,(v^2)_x \quad , \quad v \equiv v^i \tau^i \ , \tag{3.12}
$$

τ^i being Pauli matrices and A, B, C arbitrary numerical coefficients. Eq. (3.11) arises under the choice

$$
A = B = 0 \ , \quad C = \frac{3}{2} \ . \tag{3.13}
$$

Note that in ref. [12] the integrability has been shown for another particular case of eq.(3.12) corresponding to the option

$$A = 1\ ,\ \ B = -C = \frac{1}{6}\ .$$

Our consideration suggests that, being extended to a coupled system including a KdV-type equation, this matrix mKdV equation can be as well integrable for the choice of parameters as in eq.(3.13). Anyway, it is clear that the complete analysis of the integrability properties of the new $N = 3$ super KdV equation (3.8) should essentially rely upon the study of such properties of the bosonic subsystem (3.10) and the matrix mKdV equation (3.12). We hope to return to these issues in the future.

4 The Hamiltonian structure of new N=3 super KdV equation

In the previous Section we have found the unique $N = 3$ super KdV equation (3.8) which possesses a nontrivial higher-order conserved quantity. This equation cannot be obtained within the standard Hamiltonian approach of Section 2 as a Hamiltonian flow on $N = 3$ superconformal algebra. Indeed, the only conserved quantity having the dimension of the Hamiltonian for eq.(3.8) is H_3 defined in eq. (3.9). It is easy to see that it coincides with the Hamiltonian (2.5) at $\alpha = 1$. So the equation produced for J by this Hamiltonian via the Poisson structure (2.3) is a particular case of eq.(2.7). But this is just the non-integrable case we started with.

Thus in order to give a Hamiltonian interpretation to $N = 3$ super KdV equation (3.8) we need to examine the question of existence of another Hamiltonian structure for this system.

The only way to construct a Hamiltonian formalism for eq.(3.8) we have succeeded to invent is to introduce one more spinor $N = 3$ superfield $\tilde{J}$ and to re-obtain (3.8) as a closed subsystem of some Hamiltonian system of equations for this extended set of superfields.

Let us start from two independent $N = 3$ supercurrents $J_1(Z)$ and $J_2(Z)$ and assume that the Poisson bracket structure for these superfields is given by a direct product of the two standard structures (2.3) with arbitrary central charges c_1 and c_2:

$$\begin{aligned}
\{J_1(Z),\ J_2(Z')\}_+ &= 0 \\
\{J_1(Z),\ J_1(Z')\}_+ &= \left[\frac{c_1}{12}\mathcal{D}^3 - \frac{1}{2}J_1\partial + \frac{1}{2}\mathcal{D}^i J_1 \mathcal{D}^i + \partial J_1\right]\Delta(Z - Z') \qquad (4.1) \\
\{J_2(Z),\ J_2(Z')\}_+ &= \left[\frac{c_2}{12}\mathcal{D}^3 - \frac{1}{2}J_2\partial + \frac{1}{2}\mathcal{D}^i J_2 \mathcal{D}^i + \partial J_2\right]\Delta(Z - Z')
\end{aligned}$$

In other words, at this step we deal with two independent $N = 3$ superconformal algebras, J_1 and J_2 being the relevant supercurrents.

Now we wish to show that the second Hamiltonian structure for eq. (3.8) can be obtained as a contraction of the product structure (4.1). To this end, let us pass to the new superfields J and $\tilde{J}$ defined as follows

$$J = J_1 - J_2 \quad , \quad \tilde{J} = J_1 + J_2 \quad . \tag{4.2}$$

These objects, respectively $\tilde{J}$ and J, can be identified with the supercurrents generating the diagonal $N = 3$ superconformal group in the above product and the coset over this subgroup. The Poisson bracket structure for these new superfields is simply another form of (4.1)

$$\begin{aligned}
\{J(Z),\, J(Z')\}_+ &= \left[\frac{c_1+c_2}{12}\mathcal{D}^3 - \frac{1}{2}\tilde{J}\partial + \frac{1}{2}\mathcal{D}^i\tilde{J}\mathcal{D}^i + \partial\tilde{J}\right]\Delta(Z-Z') \\
\left\{\tilde{J}(Z),\, J(Z')\right\}_+ &= \left[\frac{c_1-c_2}{12}\mathcal{D}^3 - \frac{1}{2}J\partial + \frac{1}{2}\mathcal{D}^iJ\mathcal{D}^i + \partial J\right]\Delta(Z-Z') \\
\left\{\tilde{J}(Z),\, \tilde{J}(Z')\right\}_+ &= \left[\frac{c_1+c_2}{12}\mathcal{D}^3 - \frac{1}{2}\tilde{J}\partial + \frac{1}{2}\mathcal{D}^i\tilde{J}\mathcal{D}^i + \partial\tilde{J}\right]\Delta(Z-Z') \quad .
\end{aligned} \tag{4.3}$$

Let us now deform this structure in the following self-consistent way:

$$\begin{aligned}
J &\longrightarrow \frac{1}{\kappa}J\ , \quad (c_1 - c_2) \equiv \frac{1}{\kappa}\, c\ , \quad (c_1 + c_2) \equiv \tilde{c} \\
\kappa &\longrightarrow 0\ .
\end{aligned} \tag{4.4}$$

In the contraction limit (4.4) goes over to

$$\begin{aligned}
\{J(Z),\, J(Z')\}_+ &= 0 \\
\left\{\tilde{J}(Z),\, J(Z')\right\}_+ &= \left[\frac{c}{12}\mathcal{D}^3 - \frac{1}{2}J\partial + \frac{1}{2}\mathcal{D}^iJ\mathcal{D}^i + \partial J\right]\Delta(Z-Z') \\
\left\{\tilde{J}(Z),\, \tilde{J}(Z')\right\}_+ &= \left[\frac{\tilde{c}}{12}\mathcal{D}^3 - \frac{1}{2}\tilde{J}\partial + \frac{1}{2}\mathcal{D}^i\tilde{J}\mathcal{D}^i + \partial\tilde{J}\right]\Delta(Z-Z') \quad .
\end{aligned} \tag{4.5}$$

Now we consider the most general $N = 3$ supersymmetric (and $SO(3)$ symmetric) Hamiltonian which is linear in $\tilde{J}$

$$H = \int dx d^3\theta \left(\gamma\tilde{J}\mathcal{D}^3J + \alpha\tilde{J}\mathcal{D}^iJ\mathcal{D}^iJ + \beta\tilde{J}JJ_x\right) \quad . \tag{4.6}$$

This Hamiltonian gives rise, as one of the associated Hamiltonian equations

$$J_t = \{H,\, J\}_- \ ,$$

to the following evolution equation for J:

$$\begin{aligned}
J_t \;=\; & -\frac{c\gamma}{12}J_{xxx} + \frac{3\gamma}{2}\left(J\mathcal{D}^3J\right)_x + \left(\frac{\gamma}{4} + \frac{c\alpha}{12}\right)\mathcal{D}^3\left(\mathcal{D}^iJ\mathcal{D}^iJ\right) + \left(\gamma + \frac{c\beta}{12}\right)\mathcal{D}^3\left(JJ_x\right) \\
& -\frac{2\alpha+\beta}{4}J\left(\mathcal{D}^iJ\mathcal{D}^iJ\right)_x - \frac{4\alpha-\beta}{4}J_x\mathcal{D}^iJ\mathcal{D}^iJ \quad .
\end{aligned} \tag{4.7}$$

Making in eq.(4.7) arbitrary rescalings of x, t, θ and J, and observing that only two of these rescalings are actually independent, we are at liberty to fix two parameters. We choose the following option

$$\gamma = 2 \quad , \quad c = 6 \quad . \tag{4.8}$$

As a result our equation takes the form

$$\begin{aligned} J_t &= -J_{xxx} + 3\left(J\mathcal{D}^3 J\right)_x + \frac{\alpha+1}{2}\mathcal{D}^3\left(\mathcal{D}^i J \mathcal{D}^i J\right) + \frac{\beta+4}{2}\mathcal{D}^3\left(JJ_x\right) \\ & -\frac{2\alpha+\beta}{4}J\left(\mathcal{D}^i J\mathcal{D}^i J\right)_x - \frac{4\alpha-\beta}{2}J_x\mathcal{D}^i J\mathcal{D}^i J \quad . \end{aligned} \tag{4.9}$$

If we now compare this equation with our previous equation (3.8), we immediately find that they coincide for the following values of parameters α and β:

$$\alpha = -1 \quad , \quad \beta = -4 \quad . \tag{4.10}$$

For $\tilde{J}$ one also obtains some evolution equation whose precise form is of no interest for us here.

So we have succeeded in interpreting our $N = 3$ super KdV equation as a Hamiltonian equation in the framework of an extended system which includes the additional superfield $\tilde{J}$. It is worthwhile to emphasize that in this approach the KdV superfield J generates a commutative translation superalgebra instead of the $N = 3$ superconformal algebra; the crucial point in deducing eq.(3.8) from the Hamiltonian (4.6) is that J behaves as a "quasi-primary superfield" with respect to an extra $N = 3$ superconformal algebra generated by $\tilde{J}$. This latter property manifests itself as the presence of a nonvanishing central charge c in the second relation (4.5).

It is worth mentioning that the scalar field KdV equation (1.1) can also be obtained starting from the system of two scalar fields $u(x), \tilde{u}(x)$ with the Poisson bracket structure given by

$$\begin{aligned} \{u(x),\, u(y)\} &= 0 \\ \{\tilde{u}(x),\, u(y)\} &= \left[-\partial^3 + 4u\partial + 2u_x\right]\delta(x-y) \\ \{\tilde{u}(x),\, \tilde{u}(y)\} &= \left[-\partial^3 + 4\tilde{u}\partial + 2\tilde{u}_x\right]\delta(x-y) \end{aligned} \tag{4.11}$$

and the Hamiltonian

$$H = \frac{1}{2}\int dx\tilde{u}u \quad . \tag{4.12}$$

This doubling of fields looks rather artificial for the scalar KdV equation, owing to the existence of the standard Hamiltonian (1.2), but the lacking of such a Hamiltonian for the $N = 3$ super KdV equation (3.8) immediately leads us to make use of this possibility (it is the only one known to us at present).

Let us note, at the end of this Section, that almost all known systems with $N = 3$ supersymmetry respect as well $N = 4$ supersymmetry. Thus, the above doubling of fields could perhaps be interpreted as an extension of our $N = 3$ multiplet of currents to the $N = 4$ one or at least as coming from a contraction of the second Hamiltonian structure for $N = 4$ super KdV equation. This question certainly warrants further investigation. We postpone its discussion to the future.

5 Conclusion

In this paper we have demonstrated that in the case of the $N = 3$ super KdV equation the standard second Hamiltonian structure based on $N = 3$ superconformal algebra and respecting both global $N = 3$ supersymmetry and the automorphism $SO(3)$ symmetry gives rise to a non-integrable system. We have deduced a new $N = 3$ super KdV equation by considering the most general $N = 3$ supersymmetric extension of the KdV equation with the abovementioned symmetries and checking the existence of the higher order non-trivial superfield conservation laws for it. It is interesting that there exists a unique $N = 3$ supersymmetric extension of the KdV equation which possesses non-trivial conservation laws. After reduction to the $N = 2$ case this equation turns into the exceptional $N = 2$ super KdV equation (with parameter $a = 1$) whose integrability is under investigation [8]. The bosonic core of our modified $N = 3$ super KdV equation contains the new system of coupled KdV and matrix mKdV equations which has a great chance to be integrable.

We have also proposed the Hamiltonian structure for our $N = 3$ super KdV equation. It appears as some contraction of the direct sum of two $N = 3$ superconformal algebras. It is an open question whether this structure can be somehow related to $N = 4$ superconformal algebras. So it seems very interesting to consider possible integrable $N = 4$ superextensions of the KdV equation.

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