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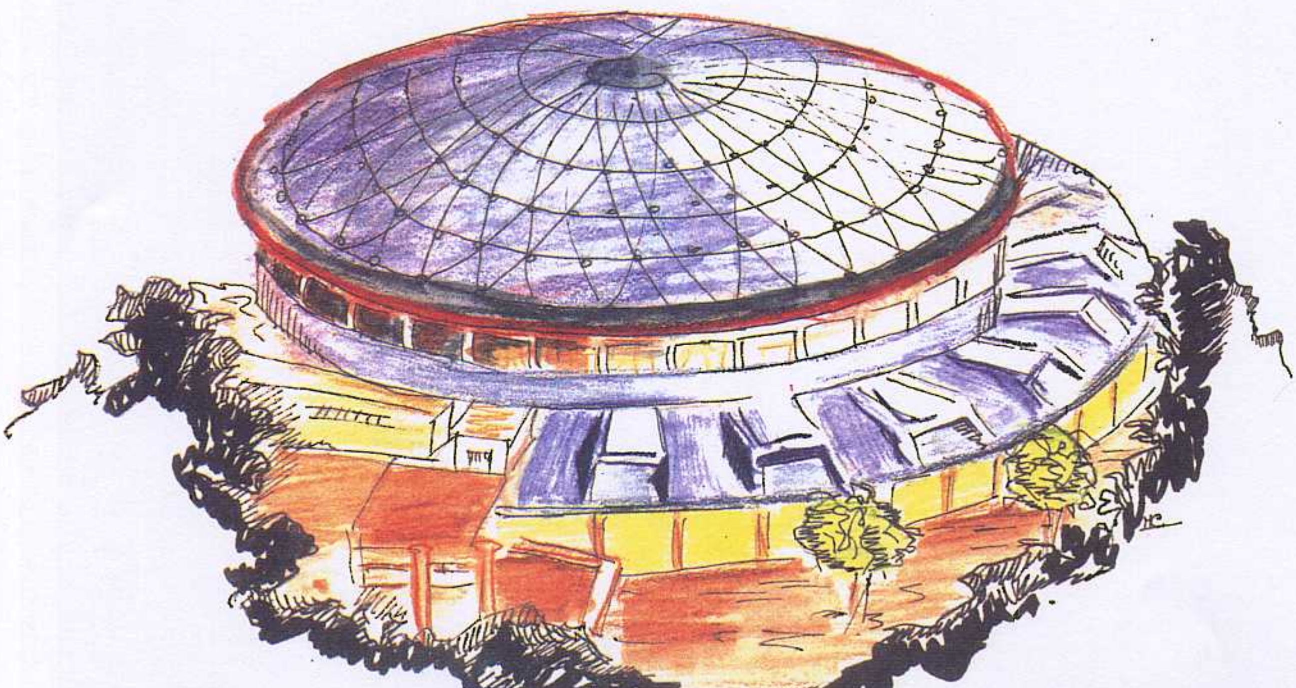

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TOWARDS AN INTEGRABLE N=3 SUPER KdV EQUATION

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Abstract

Starting with a most general $N = 3$ superfield extension of KdV equation and requiring the existence of both a higher order conservation law for it and a proper reduction to $N = 2$ super KdV equation we deduce a new $N = 3$ super KdV equation which is a unique candidate for being integrable. Upon reduction to the $N = 2$ case it yields the recently discussed "would-be" integrable version of $N = 2$ super KdV equation. It can be interpreted as a Hamiltonian flow on some contraction of the direct sum of two $N = 3$ superconformal algebras.

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1. Integrable supersymmetric extensions of the KdV hierarchy are expected to have important applications in $2D$ supergravities and the related matrix models (see, e.g. [1]). They can also bear an intimate relation to superconformal (super Virasoro) algebras via their second Hamiltonian structure, like the ordinary KdV equation is related to the Virasoro algebra [2].

In refs. [3-8] $N = 1$ and $N = 2$ supersymmetric KdV equations have been constructed as Hamiltonian flows on $N = 1$ and $N = 2$ superconformal algebras, and their integrability properties have been studied. In the $N = 2$ case the Hamiltonian approach gives rise to a one-parameter family of equations, however only for two selected values of the parameter the relevant equations have been found to be integrable [7], that is possessing a Lax pair representation and an infinite set of the conserved quantities. For one more value of the parameter the $N = 2$ KdV equation still admits higher-order conserved quantities, but no Lax pair in the standard form [8]. Despite the latter circumstance, this case was also conjectured to be integrable.

It is interesting to construct and analyze along similar lines higher N superextensions of the KdV equation. Some preliminary results for the $N = 3$ and $N = 4$ cases (however, without any discussion of the integrability issues) have been given in [9, 10].

In the present letter we address the question of existence of the first non-trivial higher order conservation law for the most general $N = 3$ super KdV equation. We show that requiring the existence of such a conservation law together with assuming a reducibility to the $N = 2$ super KdV family upon the reduction $N = 3 \rightarrow N = 2$ unambiguously fix all the unknown coefficients in this equation. The resulting equation, in contradistinction to the lower N cases, *cannot* be obtained as a Hamiltonian flow on the relevant ($N = 3$) superconformal algebra. After the reduction to $N = 2$ it yields just the special would-be integrable case of the $N = 2$ super KdV equation discussed in [8]. It contains, as its bosonic core, the coupled system of the ordinary KdV equation for the dimension 2 scalar field $u(x)$ and a matrix modified KdV equation for the $SO(3)$ triplet of the dimension 1 fields $v^i(x)$. We show that the $N = 3$ super KdV equation constructed can be given a Hamiltonian interpretation with a contraction of a direct sum of two $N = 3$ superconformal algebras as the second Hamiltonian structure.

2. As the basic object of $N = 3$ super KdV equation we choose a dimension 2 spinor $N = 3$ superfield $J(Z)$

$$J(Z) = \psi(z) + \theta^i v^i(z) + \theta^{3-i}\xi^i(z) + \theta^3 u(z) \tag{1}$$

where $Z = (x, \theta^i), i = 1, 2, 3$ are the coordinates of $N = 3$, $1D$ superspace,

$$\theta^3 = \frac{1}{6}\epsilon^{kji}\theta^i\theta^j\theta^k \quad , \quad \theta^{3-i} = \frac{1}{2}\epsilon^{kji}\theta^j\theta^k \tag{2}$$

and the components $\psi(x)$, $v^i(x)$, $\xi^i(x)$, $u(x)$ constitute a minimal $N = 3$ supermultiplet containing the KdV field $u(x)$.

Under natural assumptions of $N = 3$ supersymmetry and $SO(3)$ symmetry the most general $N = 3$ super KdV equation is of the form

$$J_t = \mathcal{A}(J) \quad , \tag{3}$$

where $\mathcal{A}$ is a linear combination of all possible terms with proper dimension (7/2) which can be constructed from the superfield $J(Z)$ and covariant spinor derivatives of the latter. Explicitly, it is the six-parameter family of equations

$$\begin{aligned} J_t &= -J_{xxx} + a_1 \left(J\mathcal{D}^3 J\right)_x + a_2 \mathcal{D}^3 \left(J J_x\right) + a_3 \mathcal{D}^3 \left(\mathcal{D}^i J \mathcal{D}^i J\right) \\ &\quad + a_4 \left(\mathcal{D}^i J\right)_x \mathcal{D}^{3-i} J + a_5 J \left(\mathcal{D}^i J \mathcal{D}^i J\right)_x + a_6 J_x \left(\mathcal{D}^i J \mathcal{D}^i J\right) \quad , \end{aligned} \tag{4}$$

where

$$\mathcal{D}^i = \frac{\partial}{\partial \theta^i} - \theta^i \frac{\partial}{\partial x} \quad , \quad \left\{\mathcal{D}^i, \mathcal{D}^j\right\} = -2\delta^{ij}\partial_x \quad , \tag{5}$$

$$\mathcal{D}^3 = \frac{1}{6}\epsilon^{ijk}\mathcal{D}^i\mathcal{D}^j\mathcal{D}^k \quad , \quad \mathcal{D}^{3-i} = \frac{1}{2}\epsilon^{ijk}\mathcal{D}^j\mathcal{D}^k \quad , \quad \mathcal{D}^{3-ij} = \epsilon^{ijk}\mathcal{D}^k \quad .$$

In order to reduce the number of parameters we from the beginning impose the requirement that upon the reduction to the $N = 2$ case eq. (4) goes over to the known $N = 2$ KdV family

$$\Phi_t = -\Phi_{xxx} + 3\left(\Phi \mathcal{D}_1 \mathcal{D}_2 \Phi\right)_x + \frac{a-1}{2}\left(\mathcal{D}_1 \mathcal{D}_2 \Phi^2\right)_x + 3a\Phi^2\Phi_x \quad , \tag{6}$$

which is integrable for $a = -2, 4$ and "would-be" integrable for $a = 1$ [8].

This requirement amounts to the following relations between the parameters $a_1, \ldots, a_6$:

$$a_1 = 3 \quad , \quad a_3 = \frac{1-a}{2} \quad , \quad a_4 = 0 \quad , \quad 2a_5 + a_6 = 3a \quad , \tag{7}$$

Thus the $N = 3$ super KdV equation we will consider contains three undetermined parameters

$$\begin{aligned} J_t &= -J_{xxx} + 3\left(J\mathcal{D}^3 J\right)_x + a_2 \mathcal{D}^3 \left(J J_x\right) + \frac{1-a}{2}\mathcal{D}^3 \left(\mathcal{D}^i J \mathcal{D}^i J\right) \\ &\quad + \frac{1}{2}(3a - a_6)\, J \left(\mathcal{D}^i J \mathcal{D}^i J\right)_x + a_6 J_x \left(\mathcal{D}^i J \mathcal{D}^i J\right) \quad . \end{aligned} \tag{8}$$

Now we wish to inquire whether this three-parameter family of equations yields integrable systems for some specific values of the parameters. Here we do not concern the question of the existence of the relevant Lax pair. Instead we search for the first non-trivial higher order conservation law.

The simplest candidate for the higher order conserved quantity is an integral of dimension 5 over $N = 3$ superspace with the integrand constructed from all possible independent densities of dimension 9/2, each multiplied by an undetermined coefficient

$$\begin{aligned} H_5 &= \int dx d^3\theta \{A_1 J\mathcal{D}^3 J_{xx} + A_2 J\mathcal{D}^i J\mathcal{D}^i J_{xx} + A_3 J J_x J_{xx} + A_4 J\mathcal{D}^3 J\mathcal{D}^3 J \\ &\quad + A_5 J J_x \mathcal{D}^i J\mathcal{D}^{3-i} J + A_6 J\mathcal{D}^i J\mathcal{D}^i J\mathcal{D}^3 J + J\left(\mathcal{D}^i J\mathcal{D}^i J\right)^2\} \quad . \end{aligned} \tag{9}$$

The coefficients are then fixed by requiring the integral to be time-independent when $J(Z)$ is subjected to the equation (8),

$$\left(H_5\right)_t = 0 \, .$$

This also must fix the values of parameters a, a_2, a_6 in (8).

After tedious though straightforward calculations one finds that *all coefficients* in the integral (9) and in eq. (8) are fixed at the unique values

$$A_1=-5\ ,\quad A_2=-\frac{5}{2}\ ,\quad A_3=\frac{5}{2}\ ,\quad A_4=10\ ,\quad A_5=\frac{5}{3}\ ,\quad A_6=\frac{20}{3} \tag{10}$$

$$a=1\ ,\quad a_2=0\ ,\quad a_6=0\quad , \tag{11}$$

thus implying that in the $N=3$ supersymmetric case there exists only one superfield extension of the KdV equation which possesses a nontrivial higher order conservation law

$$J_t=-J_{xxx}+3\left(J\mathcal{D}^3J\right)_x+\frac{3}{2}J\left(\mathcal{D}^iJ\mathcal{D}^iJ\right)_x\quad . \tag{12}$$

It is curious that after reduction to the $N=2$ case this equation goes over just to the exceptional $N=2$ super KdV equation with parameter $a=1$.

For completeness we write also the first two lower order conserved quantities of eq. (12)

$$\begin{aligned} H_1 &= \int dx d^3\theta J \\ H_3 &= \int dx d^3\theta \left(J\mathcal{D}^3J+\frac{1}{3}J\mathcal{D}^iJ\mathcal{D}^iJ\right) \end{aligned} \tag{13}$$

Passing to the discussion, let us first stress that we have started from the most general $N=3$ superfield equation (4) with the only extra demand of a proper reduction to the $N=2$ case. It seems very intriguing that under such general assumptions we were eventually left with the unique candidate for the integrable $N=3$ KdV equation.

Secondly, recall that even for the $N=2$ super KdV equation the integrability at $a=1$ is an open problem due to lacking of the standard Lax representation in this case. The problem of proving integrability remains, of course, in our case too. Up to now we know only the first non-trivial conservation law for the equation (12). Let us point out, however, that the set of equations that must be satisfied by the coefficients a, a_i, A_i is highly overdetermined. There are about five times as many equations compared to the unknowns. So the very existence of this first nontrivial conserv ation law is a strong indication of the complete integrability of the corresponding equation.

Finally, we write down the bosonic core of our $N=3$ super KdV equation (12) (by putting all fermions equal to zero)

$$\begin{aligned} u_t &= -u_{xxx}+3\left(u^2-v^iv^i_{xx}+uv^iv^i\right)_x \\ v^i_t &= -v^i_{xxx}+3\left(uv^i\right)_x+3v^iv^jv^j_x\ , \end{aligned} \tag{14}$$

where

$$v^i=\mathcal{D}^iJ|\quad ,\quad u=\mathcal{D}^3J|\quad .$$

We see that the bosonic subsector of our $N=3$ super KdV equation consists of the two coupled equations – the KdV equation for the scalar field u and a three-component generalization of the mKdV equation, both with the extra mixed terms in the r.h.s. These equations cannot be decoupled by a redefinition of u. While the first equation is a kind of the perturbed KdV equation, the second one can be viewed as a perturbation of the equation

$$v^i_t = -v^i_{xxx} + 3v^i(v^2)_x , \quad (15)$$

which is a particular case of the general $SO(3)$ matrix mKdV equation

$$v_t = -v_{xxx} + A\frac{i}{2}[v, v_{xx}] + Bv_x\,(v^2) + Cv\,(v^2)_x \;, \quad v \equiv v^i\tau^i \;, \quad (16)$$

τ^i being Pauli matrices and A, B, C arbitrary numerical coefficients. Eq. (3.11) arises under the choice

$$A = B = 0 \;, \quad C = \frac{3}{2} \,. \quad (17)$$

Note that in ref. [11] the integrability has been shown for another particular case of eq.(16) corresponding to the option

$$A = 1 \;, \quad B = -C = \frac{1}{6} \,.$$

Our consideration suggests that, being extended to a coupled system including a KdV-type equation, this matrix mKdV equation can be as well integrable for the choice of parameters as in (17).

3. Our last topic will be the discussion of how $N = 3$ super KdV equation (12) can be reproduced in a Hamiltonian approach.

It is a crucial novel feature of this equation compared to the $N = 1$ and $N = 2$ super KdV ones that it cannot be obtained as a Hamiltonian flow on the relevant, i.e. $N = 3$, superconformal algebra. Indeed, the only conserved quantity which has the appropriate dimension for being the Hamiltonian in the case at hand is H_3 defined in eq. (13). The equation produced for J by this Hamiltonian via the Poisson structure forming an $N = 3$ superconformal algebra [10]

$$\{J(Z), J(Z')\}_+ = \left[\frac{1}{2}\mathcal{D}^3 - \frac{1}{2}J\partial + \frac{1}{2}\mathcal{D}^iJ\mathcal{D}^i + \partial J\right]\Delta(Z - Z') \quad , \quad (18)$$

where we denoted

$$\Delta(Z - Z') = \frac{1}{6}\epsilon^{ijk}(\theta^i - \theta^{i'})(\theta^j - \theta^{j'})(\theta^k - \theta^{k'})\delta(z - z') \;,$$

is as follows

$$J_t = -J_{xxx} + 3\left(J\mathcal{D}^3J\right)_x + \mathcal{D}^3\left(J\partial J\right) + \left(J\mathcal{D}^iJ\mathcal{D}^iJ\right)_x \quad . \quad (19)$$

But this does not coincide with (12) and is just one of the non-integrable cases of $N = 3$ super KdV: though H_3 is still conserved quantity for eq. (19) (as well as for eq. (12)), H_5 is certainly not.

Thus in order to give a Hamiltonian interpretation to the equation (12) we have to examine the question of existence of another Hamiltonian structure for this system.

Our proposal is to introduce one more dimension 2 spinor $N = 3$ superfield $\tilde{J}$ and to re-obtain (12) as a closed subsector of some Hamiltonian system of equations for the extended set of superfields J, $\tilde{J}$.

It can be easily checked that under the following Poisson structure

$$\begin{aligned} \{J(Z),\, J(Z')\}_+ &= 0 \\ \left\{\tilde{J}(Z),\, J(Z')\right\}_+ &= \left[\frac{1}{2}\mathcal{D}^3 - \frac{1}{2}J\partial + \frac{1}{2}\mathcal{D}^i J \mathcal{D}^i + \partial J\right]\Delta(Z-Z') \\ \left\{\tilde{J}(Z),\, \tilde{J}(Z')\right\}_+ &= \left[\frac{\tilde{c}}{12}\mathcal{D}^3 - \frac{1}{2}\tilde{J}\partial + \frac{1}{2}\mathcal{D}^i \tilde{J} \mathcal{D}^i + \partial \tilde{J}\right]\Delta(Z-Z') \end{aligned} \qquad (20)$$

the Hamiltonian

$$H = \int dx d^3\theta \left(2\tilde{J}\mathcal{D}^3 J - \tilde{J}\mathcal{D}^i J \mathcal{D}^i J - 4\tilde{J}JJ_x\right) \qquad (21)$$

gives rise for J to the $N=3$ super KdV equation(12).

So we have succeeded in interpreting our $N=3$ super KdV equation as a Hamiltonian equation in the framework of an extended system which includes the additional superfield $\tilde{J}$. It is worthwhile to emphasize that in this approach the KdV superfield J generates a commutative translation superalgebra instead of $N=3$ superconformal algebra; the crucial point in deducing eq.(12) from the Hamiltonian (21) is that J behaves as a quasi-primary superfield with respect to an extra $N=3$ superconformal algebra generated by $\tilde{J}$. This manifests itself as the presence of a nonvanishing central charge in second of the relations (20). It is easy to show that (20) can be obtained as a contraction of a direct sum of two $N=3$ superconformal algebras with independent central charges.

Let us finally note that almost all known systems with $N=3$ supersymmetry respect as well $N=4$ supersymmetry. Thus, the above doubling of fields could perhaps be interpreted as an extension of our $N=3$ multiplet of currents to the $N=4$ one or at least as coming from a contraction of the second Hamiltonian structure for $N=4$ super KdV equation. This question certainly warrants further investigation.

4. In this letter we have demonstrated that in the case of the $N=3$ super KdV equation the standard second Hamiltonian structure based on $N=3$ superconformal algebra results in a non-integrable system. We have deduced another $N=3$ super KdV equation by considering the most general $N=3$ superextension of the KdV equation and checking the existence of the higher order non-trivial superfield conserved quantity for it. It is remarkable that there exists a unique $N=3$ superextension of the KdV equation which possesses such a non-trivial conserved quantity. After reduction to the $N=2$ case this equation turns into the exceptional $N=2$ super KdV equation (with parameter $a=1$) the integrability of which is under investigation [8]. Thus the $N=3$ superfield equation constructed is a good candidate for integrable $N=3$ super KdV equation. Respectively, its bosonic core (a new system of coupled KdV and matrix mKdV equations) has also a great chance to be integrable.

We have proposed the Hamiltonian structure for our $N=3$ super KdV equation. It appears as some contraction of the direct sum of two $N=3$ superconformal algebras. It is an open question whether this structure can be somehow related to $N=4$ superconformal algebras. So it seems very interesting to construct possible integrable $N=4$ superextensions of the KdV equation.

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