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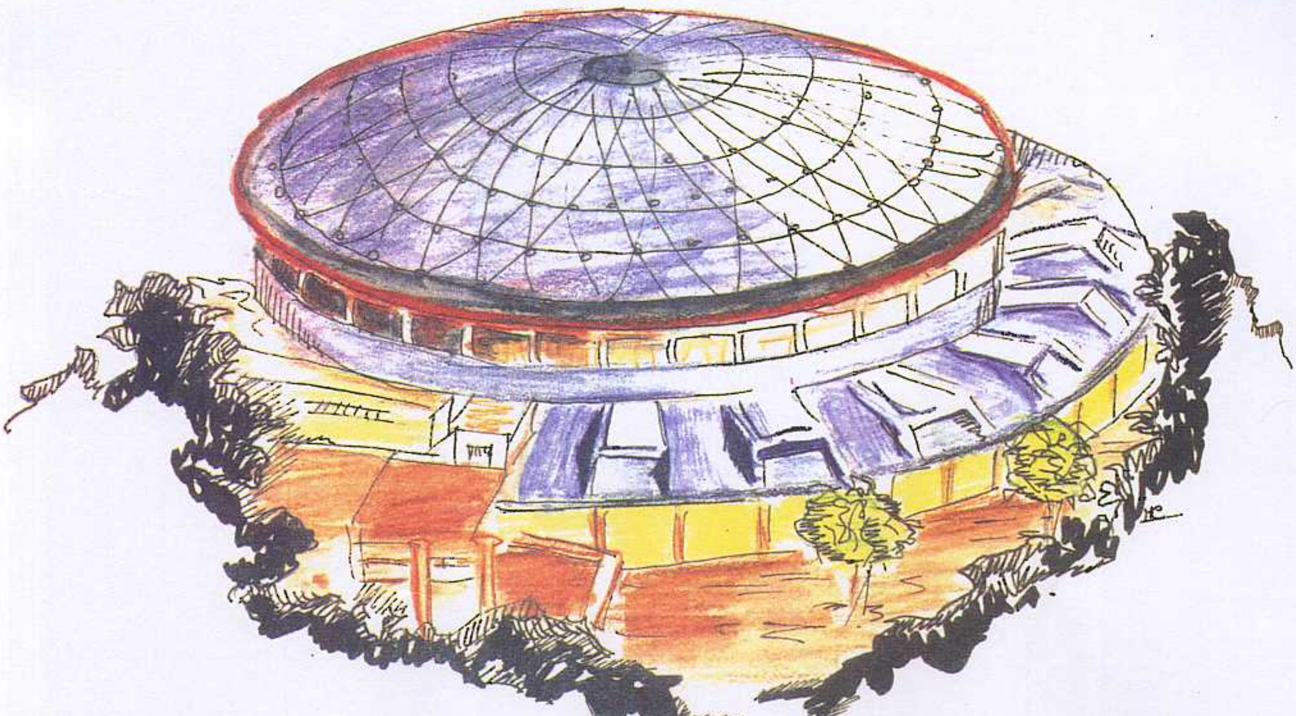

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ON THE FINITENESS OF THE INVERSE DIFFRACTION TRANSFORM KERNEL

E. Etim, L. Magro and M. Pallotta
INFN – Laboratori Nazionali di Frascati, P.O. Box 13, 00044 – Frascati (Roma), Italy

ABSTRACT

The diffraction transform solves both the direct and the inverse scattering problem for the Helmholtz equation. Its kernel is itself an integral. It is the representation of the evolution operator associated with translations of a constrained Cartesian coordinate. The corresponding momentum operator is, consequently, non–Hermitian. This fact threatens the inverse scattering problem with a divergence if the transform kernel is understood as a Cauchy integral. The kernel is, however, everywhere convergent if its integral representation is interpreted as a Cesàro summable integral. Taking a Bjorken type limit of the Cesàro representation leads to the summability of the divergent kernel with respect to a spectral measure. This accords well with the general solution of the scattering problem in which such spectral functions are admissible. The use of variously motivated cut–offs and other ad hoc filters to enforce convergence is therefore highly contrived.

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1. – INTRODUCTION

In classical physics, diffraction is essentially a boundary value problem associated with a linear wave equation [1]. According to this picture, a diffractive object (or target) is no more than a passive geometrical obstacle in the path of the incident wave (or projectile). The incident wave field is specified as a boundary condition over the target profile. Because of the linearity of the wave equation, the scattered field is obtained as an integral transform of the incident field. This is the so–called direct scattering problem. The inverse problem consists in the recovery of the incident field, given the scattered field configuration. This solution provides information about the target profile geometry [2–5]. Therein lies the importance of the classical inverse scattering problem. There are in fact important applications (e.g. diffraction tomography [6], optical information processing and, more generally, Fourier optics [7]) which make extensive use of classical diffraction theory. This theory is, however, saddled with a long–standing, apparently unyielding, divergence problem [8, 9]. The problem is evaded, especially in applications, by recourse to filtering methods [6, 8, 9]. These filters are usually nothing else but cut–offs studiedly chosen to enforce convergence. They are essentially contrived regularization methods. However well they are physically motivated, cut–offs obscure, at least, in part, the essential dynamics. To avoid this, the divergence problem has therefore, to be approached differently, particularly in view of the many applications of the theory. This is the purpose of this paper. We propose to show that the divergence in question is spurious. By this we mean, firstly, that if no simplifying assumption is made regarding certain spectral properties of the theory, then the divergence just does not occur. Secondly, even when the simplifying assumption is made, the divergence occurs only in a context where all integrals in the theory, independently of the formal manipulations which lead to them are naively assumed to be Cauchy summable. The latter is, in general, not true. In particular, it is not true of the inverse integral transform which solves for the incident field in terms of the scattered field in classical diffraction theory [8, 9]. It is the Cauchy divergence of the kernel of this integral transform which constitutes the unsolved problem of this theory. We shall show that if this and other integrals in the theory are interpreted as Cesàro integrals [10, 11, 12, 13], then the divergence does not exist. Put differently, the Cauchy divergence is regularized by means of Cesàro summability. Cauchy integrals form a sub–class of Cesàro summable integrals. By taking a Bjorken type limit [14] of the Cesàro representation we show that the transformation kernel is also summable with respect to a gaussian spectral measure. The general solution of the scattering problem admits a large class of such spectral functions.

The paper is organized as follows: in Section 2 we review briefly the classical diffraction theory in terms of solutions of the Helmholtz equation. In Section 3 we examine the symmetries of the problem and their consequences. Section 4 implements the regularization of the diffraction transform kernel using the Cesàro summability procedure. Section 5 contains conclusions.

2. – DIFFRACTION TRANSFORMS

Consider a scalar field $\psi(t, x)$, associated with a mass parameter m, and satisfying the Klein–Gordon equation [15]

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi(t, x) = 0 \tag{1}$$

where

$$\nabla^2 \equiv \sum_{j=1}^{3} \frac{\partial^2}{\partial t^2} \tag{2}$$

is the Laplace operator in 3–space with coordinate vector $\vec{x}$. It will be convenient in what follows to view this space in terms of two–dimensional plane slices orthogonal to the z–axis. A point on each such slice is described by the position z of the plane along the z–axis and a 2–

vector coordinate $\vec{b}$ (the impact parameter) on the plane. Thus, the 3–vector $\vec{x}$ is decomposed into $\vec{x} \equiv (\vec{b}, z)$, and will be so understood throughout the rest of this paper.

Now, let us look for monochromatic solutions

$$\psi(t, x) = e^{\pm i \omega t} \varphi(x) \tag{3}$$

of Eq. (1). Substituting (3) in (1) yields the Helmholtz equation.

$$\left(\nabla^2 + k^2\right) \varphi(x) = 0 \tag{4}$$

where

$$k^2 = \omega^2 - m^2 \tag{5}$$

Eq. (4) is to be solved with the boundary condition

$$\varphi(b, z=z_0) = \varphi_0(b) \tag{6}$$

The search for this solution in the half–space $z \geq z_0$ constitutes the direct exterior problem while in the half–space $z \leq z_0$ it becomes the direct interior problem. The exterior and interior solutions are related by a reflection principle (i.e. parity) as will emerge later (cf. Fig. 1). It is therefore sufficient to concentrate on the direct exterior problem.

In each half–space $z \geq z_0$ or $z \leq z_0$, there are two linearly independent solutions $\varphi_\pm(x)$ of the Helmholtz equation: the one, e.g. $\varphi_+(x)$, is the regular solution in the half–space $z \geq z_0$ in the sense that it tends to zero for $z \to \infty$. We will also refer to it as the outward propagating solution. The other $\varphi_-(x)$ is the singular solution. It does not vanish for $z \to \infty$, but rather for $z \to -\infty$. It is the inward propagating solution. $\varphi_+(x)$ and $\varphi_-(x)$ interchange roles in the half–space $z \leq z_0$, i.e. $\varphi_-(x)$ is there regular and outward propagating while $\varphi_+(x)$ is singular and inward propagating (see Fig. 1). To obtain these solutions, substitute the two–dimensional Fourier transform

$$\varphi(b, z) = \int d^2q \, e^{i \vec{q} \cdot \vec{b}} f(q, z) \tag{7}$$

into Eq. (4) to get the equation

$$\left[\frac{d^2}{dz^2} + p^2(q)\right] f(q, z) = 0 \tag{8}$$

where

$$p^2(q) := k^2 - q^2 \tag{9}$$

like k^2 in Eq. (5) is not necessarily positive. Define the two real momentum variables

$$p_1(q) := \sqrt{k^2 - q^2} \quad p^2(q) \geq 0 \tag{10.i}$$

$$p_2(q) := \sqrt{q^2 - k^2} \quad p^2(q) < 0 \tag{10.ii}$$

The inverse

$$f(q, z) = \frac{1}{(2\pi)^2} \int d^2b \, e^{-i\vec{q}\cdot\vec{b}} \varphi(b, z) \tag{11}$$

of Eq. (7) will also be needed.

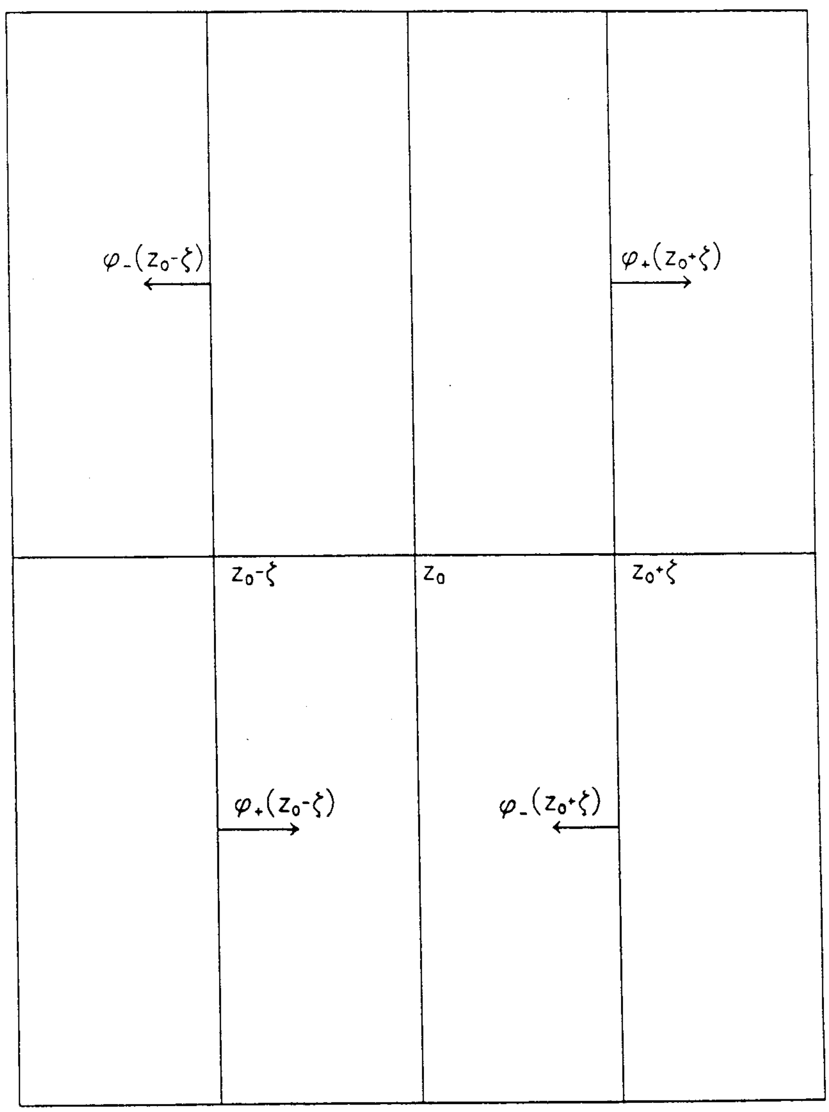

FIG. 1 – Illustration of the reflection principle (Eq. (37)). $\varphi_+(b, z_0 \pm \zeta)$ and $\varphi\_(b, z_0 \mp \zeta)$ are mirror images of each other with respect to the plane at $z = z_0$.

Two linearly independent solutions of Eq. (8) are

$$f_\pm(q, z) = \theta(k^2 - q^2)\, a_\pm(q)\, e^{\pm ip_1 z} + \theta(q^2 - k^2)\, b_\pm(q)\, e^{\mp p_2 z} \tag{12}$$

The coefficients $a_\pm(q)$ and $b_\pm(q)$ are arbitrary functions of $\vec{q}$. They may be fixed by boundary conditions. Since Eq. (8) is of second order, two boundary conditions are necessary to determine these coefficients. The one boundary condition in Eq. (6) is therefore not enough to determine them. This is in itself not bad because it leaves room for a very general treatment. The essential step in this treatment is to invert Eq. (12) and solve for $a_\pm(q)$and $b_\pm(q)$ in terms of $f_\pm(q, z)$. The general way to do this is to introduce a probability distribution or spectral function $\rho(q, z)$ [16] in the variable z and dependent, in general, on $\vec{q}$. The distribution function $\rho(q,z)$ is normalised, i.e.

$$\int_{-\infty}^{+\infty} dz\, \rho(q, z) = 1 \tag{13}$$

and is assumed to be such that its characteristic function

$$w(q, p) := \int_{-\infty}^{+\infty} dz\, \rho(q, z)\, e^{ipz} \tag{14}$$

exists for all complex p and is finite for $|p| < \infty$.

Now, taking the average of both sides of Eq. (12) with respect to $\rho(q, z)$ and making use of Eq. (14) and Eq. (11) for $f_\pm(q, b)$ yields the inversions

$$\theta(k^2 - q^2)\, a_\pm(q) = \frac{\theta(k^2 - q^2)}{w(q, \pm p_1)} \frac{1}{(2\pi)^2} \int d^2b\, e^{\mp i\vec{q}\cdot\vec{b}} \int_{-\infty}^{+\infty} dz\, \rho(q, z)\, \varphi_\pm(b, z) \tag{15.i}$$

$$\theta(q^2 - k^2)\, b_\pm(q) = \frac{\theta(q^2 - k^2)}{w(q, \pm ip_2)} \frac{1}{(2\pi)^2} \int d^2b\, e^{\mp i\vec{q}\cdot\vec{b}} \int_{-\infty}^{+\infty} dz\, \rho(q, z)\, \varphi_\pm(b, z) \tag{15.ii}$$

With the help of the spectral function $\rho(q, z)$, one thus expresses $a_\pm(q)$ and $b_\pm(q)$ in terms of the averages of $\varphi_\pm(x)$ and not in terms of the boundary conditions on them. Eqs. (15) therefore involve global constraints. $\rho(q, z)$ is introduced here on quite general grounds. Its role is, however, essentially that of a physical filter. It is not an ad hoc filter but rather a dynamical function which incorporates implicitly external constraints of which boundary conditions are special cases. Making use of Eqs. (15) in Eqs. (12) and (7), one arrives finally at the homogeneous integral equations

$$\varphi_{\pm}(x) = \int d^3x' \, D_{\pm}(x \,|\, x') \, \varphi_{\pm}(x') \tag{16}$$

where

$$D_{\pm}(b, z \,|\, b', z') = \frac{1}{(2\pi)^2} \int d^2q \, \rho(q, z') \, e^{\pm i \vec{q}(\vec{b} - \vec{b}')} \bullet$$

$$\bullet \left[\frac{\theta(k^2 - q^2)}{w(q, \pm p_1)} \, e^{\pm i p_1 z} + \frac{\theta(q^2 - k^2)}{w(q, \pm i p_2)} \, e^{\mp p_2 z} \right] \tag{17}$$

To arrive at Eq. (16) we have interchanged the order of the $\vec{b}$–integration coming from Eqs. (15) with that of the $\vec{q}$–integration coming from Eq. (7). This interchange is usually [8,9] held responsible for the divergence in $D_{\pm}(x \,|\, x')$ which arises when one makes the special choice

$$\rho_0(q, z) = \delta(z - z_0) \tag{18}$$

of the spectral function. This is an important point and will be taken up more fully in sect(4). $\rho_0(q, z)$ allows to relate $a_{\pm}(q)$ and $b_{\pm}(q)$ directly to the boundary values $\varphi_{\pm}(b, z_0)$ of $\varphi_{\pm}(x)$ on the plane $z = z_0$.The integral equation in Eq. (16) is equivalent to the Helmholtz equation.

Note the following properties of the kernels $D_{\pm}(x \,|\, x')$:

$$\int_{-\infty}^{+\infty} dz \, D_{\pm}(b, z \,|\, b', z) = \delta^{(2)}(\vec{b} - \vec{b}') \tag{19}$$

$$\int d^3x' \, D_{\pm}(x \,|\, x') \, D_{\pm}(x' \,|\, x'') = D_{\pm}(x \,|\, x'') \tag{20}$$

Eqs. (19) and (20) hold independently of the explicit form of $\rho(q, z)$. Eq. (20) makes it clear that the matrices $D_{\pm}(x \,|\, x')$ have no inverse in the sense that there do not exist matrices $\hat{D}_{\pm}(x \,|\, x')$ such that the relations

$$\int d^3x' \, D_{\pm}(x \,|\, x') \, \hat{D}_{\pm}(x' \,|\, x'') = \delta^{(3)}(x - x'') \tag{21}$$

hold. In other words, the operators $D_{\pm}$, with matrix elements $D_{\pm}(x \,|\, x')$, are projection operators. With the choice of $\rho(q, z)$ in Eq. (18), $D_{\pm}(x \,|\, x')$ become

$$D_{\pm}(b, z \,|\, b', z') = \delta(z' - z_0) \, G_{\pm}(b, z \,|\, b', z_0) \tag{22}$$

where

$$G_{\pm}(b, z \mid b', z') = \frac{1}{(2\pi)^2} \int d^2q \, e^{\pm i \vec{q}(\vec{b} - \vec{b}')} \bullet$$
$$\bullet \left[\theta(k^2 - q^2)\, e^{\pm i p_1 (z-z')} + \theta(q^2 - k^2)\, e^{\mp p_2 (z-z')} \right] \tag{23}$$

But there is more: $G_{\pm}(b, z \mid b', z')$ can be interpreted as the matrix elements $G_{\pm}(z \mid z')_{b\,b'} := G_{\pm}(b, z \mid b', z')$ of the forward (+) and backward (–) evolution operators $G_{\pm}(z \mid z')$. From Eqs. (22) and (19), $G_{\pm}(z \mid z)$ reduce to the identity operator i.e.

$$G_{\pm}(z \mid z) = 1 \tag{24.i}$$

or in matrix form

$$G_{\pm}(b, z \mid b', z) = \delta^{(2)}(\vec{b} - \vec{b}') \tag{24.ii}$$

Secondly, $G_{\pm}^{-1}(z \mid z') := G_{\pm}(z' \mid z)$ are the inverses of $G_{\pm}(z \mid z')$ i.e.

$$G_{\pm}(z \mid z')\, G_{\pm}(z' \mid z) = 1, \tag{25.i}$$

equivalently

$$\int d^2b'\, G_{\pm}(b, z \mid b', z')\, G_{\pm}(b', z' \mid b'', z) = \delta^{(2)}(\vec{b} - \vec{b}'') \tag{25.ii}$$

We also have the group closure property

$$G_{\pm}(z \mid z')\, G_{\pm}(z' \mid z'') = G_{\pm}(z \mid z'') \tag{26.i}$$

In terms of matrix elements, (26.i) becomes [17]

$$\int d^2b'\, G_{\pm}(b, z \mid b', z')\, G_{\pm}(b', z' \mid b'', z'') = G_{\pm}(b, z \mid b'', z'') \tag{26.ii}$$

In Eqs. (24)–(26) , the parameters z, z', z" are not subject to any ordering restrictions such as $z \geq z' \geq z''$ or $z \leq z' \leq z''$. It is this fact which allows for the existence of the inverses of $G_{\pm}(z \mid z')$ and ensures that one is dealing with groups rather than semi–groups.

Substituting from Eq. (22) into (16), one obtains the so–called diffraction transforms

$$\varphi_\pm(b, z) = \int d^2 b' \, G_\pm(b, z \mid b', z') \, \varphi_\pm(b', z') \tag{27}$$

On account of Eq. (25), for any fixed pair of parameters (z, z') these transforms have an inverse, i.e.

$$\varphi_\pm(b', z') = \int d^2 b \, G_\pm(b', z' \mid b, z) \, \varphi_\pm(b, z) \tag{28}$$

The structure of Eqs. (27) and (28) coincide on account of the equality $G_\pm^{-1}(z \mid z') := G_\pm(z' \mid z)$. Thus, for any fixed pair of parameters (z,z') the diffraction transform operators $G_\pm(z, z')$ are involutory, that is, they satisfy Eq. (25.i). This fact has, in the past [8,9], received little or no attention. This is so because, according to Eq. (23), the kernel $G_+(b', z' \mid b, z)$ in Eq. (28) is divergent for z' < z. This behaviour is so different from the convergence of $G_+(b, z \mid b', z')$ in Eq. (27), for z' < z, that Eqs. (27) and (28) have been regarded as different. They are not. First of all, the divergence can be eliminated, so that both Eqs. (27) and (28) can be understood to hold for all values of z – z'. There is also a symmetry underlying the equivalence between Eqs. (27) and (28). This equivalence ensures that the diffraction transform solves both the direct and the inverse scattering problem for the Helmholtz equation. This is another statement of Eq. (25). We next proceed to examine the symmetries of the system.

3. – SYMMETRIES AND THEIR CONSEQUENCES

The Helmholtz equation is invariant under rotations in 3–space, as well as under translations

$$T_\lambda : \vec{x} \to \vec{x}' := \vec{x} + \vec{\lambda} \tag{29}$$

and parity

$$P : \vec{x} \to \vec{x}' := -\vec{x} \tag{30}$$

This means that if φ(x) is a solution of Eq. (4) then the functions

$$\varphi_\lambda(x) := T_\lambda \varphi(x) = \varphi(x - \lambda) \tag{31}$$

and

$$\varphi_P(x) := P\varphi(x) = \varphi(-x) \tag{32}$$

are also solutions. We shall restrict attention to only these two symmetries. They are the most relevant in the situation of overall axial symmetry of the coordinate system $\vec{x} \equiv (\vec{b}, z)$. In particular, we shall consider the actions of these transformations mainly on the variable z and on functions thereof.

To see the consequences of these symmetries, first note, from Eq. (14), that if the probability distribution is z–parity invariant [18] that is,

$$\rho(q,z) = \rho(q, -z) \tag{33}$$

then

$$w(q,p) = w(q, -p) \tag{34}$$

Consequently, from Eq. (17) one finds that

$$D_+(x \mid x') = D_-(x' \mid x) \tag{35}$$

Making use of (35) in (16) then leads to the relationship

$$P\varphi_\pm(x) = \varphi_\mp(x) \tag{36}$$

An equivalent way of expressing Eq. (36) is in terms of the reflection principle

$$\varphi_\pm(b,\ z + \zeta) = \varphi_\mp(b, z - \zeta); \ -\infty < \zeta < +\infty \tag{37}$$

which represents the effect of a parity operation about the plane at z.
In terms of the operators $D_\pm$, one rewrites Eq. (35) as

$$PD_+P = D_- \tag{38}$$

A further restriction of ρ (q, z) is required so that $D_\pm(x \mid x')$ would be translation invariant not only in the 2–vector $\vec{b}$ but also in z. Making use of (29) in (17) for infinitesimal $\vec{\lambda}$, the restriction is easily found to be

$$\frac{\partial D_\pm(b, z \mid b', z')}{\partial z} - \frac{\partial D_\mp(b', z' \mid b, z)}{\partial z'} = 0 \tag{39}$$

To arrive at Eq. (39) we have also made use of the equality

$$\tilde{D}_\pm = D_\mp \tag{40}$$

where the symbol (~) stands for the transpose operation. Translation invariance implies that $D_\pm(x \mid x') = D_\pm(x - x')$ is a function of only the difference $\vec{x} - \vec{x}'$.

With the choice of ρ (q, z) in Eq. (18) we now have, in place of $D_\pm$, the operators $G_\pm(z \mid z')$ (cf. Eqs. ((24)–(26)) and in place of Eqs. (38) and (40), respectively, the relations

$$PG_\pm(z \mid z')P = G_\mp(z \mid z') \tag{41.i}$$

$$G_\pm(z' \mid z) = G_\pm^{-1}(z \mid z') \tag{41.ii}$$

The two components (±) of Eq. (27) are therefore parity transforms of each other. Alternatively, given Eqs. (41), Eq. (28) can also be read directly from Eq. (27) and vice versa, whence their equivalence. Lastly, note that the evolution operators $G_\pm(z \mid z')$ satisfy the equation

$$\frac{\partial^2 G_\pm(z \mid z')}{\partial z^2} = H\, G_\pm(z \mid z') \tag{42.i}$$

where, in $\vec{b}$–space, the Hamiltonian H is given by

$$H(b)) = -\left(\nabla_b^2 + k^2\right) \tag{42.ii}$$

with ∇_b^2 the two–dimensional Laplacian. On the basis of Eqs. (24) – (26), Eq. (42) may be interpreted as a reversible evolution equation for the transition probability amplitudes $G_\pm(b\,,\ z \mid b',\ z')$. In such an interpretation z acts as a random sampling parameter and

$$\vec{B}(z) := \vec{b} \tag{43}$$

as the value of the random variable $\vec{B}(z)$ at parameter value z. The function defined by

$$G_\pm(b\,, z \mid b', z') \equiv G_\pm(B(z) = b\,, z \mid B(z') = b', z') \tag{44}$$

is then the conditional probability amplitude of finding $\vec{B}(z)$ at z, conditional on its value $\vec{B}(z')$ at z'. Eq. (24) gives the initial condition for $G_\pm$ (B(z), z | B(z'), z). Eq. (26), the group closure property, corresponds to the Chapman–Kolmogorov equation. Translation invariance (in z) means that the random process is stationary i.e.

$$G_\pm(B(z+\lambda)\,, z+\lambda \mid B(z'+\lambda)\,, z'+\lambda) = G_\pm(B(z)\,, z \mid B(z')\,, z') \tag{45}$$

The behaviour of $G_\pm$(B (z), z | B(z'), z') under parity (cf. Eq. (41)) expresses only reversibility, i.e.

$$G_\pm(B(-z)\,, -z \mid B(-z')\,, -z') = G_\pm(B(z')\,, z' \mid B(z)\,, z) \tag{46}$$

and not parity invariance.

A stochastic process, reversible and stationary in time, was first invoked by Nelson [20] to derive the Schrödinger equation from classical mechanics and probability theory. Our interpretation of Eqs. (24), (26) and (42) is in agreement with Nelson's idea regarding the intimate relationship between quantum mechanics and classical mechanics combined with probability theory. The direct exterior problem for the Helmholtz equation in the half-space $z \geq z_0$ is in fact easily seen to be equivalent to the Schrödinger equation with an infinite potential barrier at $z=z_0$. The solution of the classical problem can therefore also be obtained quantum mechanically. The difference with the approach of Nelson is that one starts here with the Schrödinger equation itself (Eqs. (42)) and then arrives at its classical probability interpretation through Eqs. (24) – (26).The principal consequence of the constraint $z \geq z_0$ for the quantum mechanical problem is that the generator of translations in z is not a Hermitian operator. Classically, this is manifested in Eq. (12) by the existence of the two real momentum variables $p_1(q) = \sqrt{k^2 - q^2}$ ($k^2 \geq q^2$) and $p_2(q) = \sqrt{q^2 - k^2}$ ($k^2 < q^2$) in the solutions $f_{\pm}(q, z)$ of Eq. (8). The component of the fields $f_{\pm}(q,z)$ with momentum $p_1(q)$ is referred to as the homogeneous wave and the component with the momentum $p_2(q)$ as the inhomogeneous or evanescent wave. The combination of these two components in the solutions $f_{\pm}(q, z)$ leads to the fact that, although the evolution operator $G_+(z | z')$ possesses the inverse $G_+^{-1}(z | z') = G_+(z' | z)$, it is not unitary, that is, $G_+^{-1}(z | z') \neq G_+^{\dagger}(z | z')$, where (†) stands for the adjoint operation. This is easily checked from Eq. (23). The lack of unitarity of $G_+(z | z')$ does not mean, however, that the dynamics is necessarily irreversible. In fact, the equation of motion (the Helmholtz equation) is z–reversible. Microscopic reversibility is represented, in terms of the two independent solutions $\varphi_{\pm}(z)$ of the equation of motion, by the reflection principle in Eq. (37). We illustrate this schematically in Fig. (1) [18]. Eq. (37) ensures that the field configuration at the plane $z_0 + \zeta$ is the same as that of its mirror image at the plane $z_0 - \zeta$ $(-\infty < \zeta < +\infty)$, with the image fields propagating in directions opposite to those of the object fields. The overall field configuration (object plus image) is therefore parity invariant. The symmetry between object and image fields is, of course, broken if one is interested in the scattered fields only in the half space $z \geq z_0$. In particular, if the interest is only in the outward $(\varphi_+(z))$ or inward $(\varphi_-(z))$ propagating field in the half–space $z \geq z_0$, then the evolution of this field is irreversible. Irreversibility corresponds, in this situation, to the break–down of parity invariance by the boundary constraints. In this case, the original group defined by Eqs. (24)–(26) splits into two isomorphic semi–groups generated by $G_+(z | z')$ and $G_-(z | z')$.

4. – CESÀRO SUMMABILITY OF THE DIFFRACTION TRANSFORM KERNEL

We return, in this section, to the principal concern of this paper, viz, the elimination of the divergence in the diffraction transform kernels $G_{\pm}$ (b, z | b', z'). We recall that the divergence occurs in G_+(b, z | b', z') for z < z' and in G_-(b, z | b', z') for z > z'. We have elaborated on the claim that this divergence is spurious. The claim is based on the observation that it is the special choice of the spectral function ρ(q,z) in Eq. (18) that is responsible for the divergence. From Eq. (12) alone, this special choice is not given a priori, either by dynamics or by symmetry arguments. The general inversion formulae which follow from Eq. (12) are as given by Eqs. (15). An argument for the position in Eq. (18) would be simplicity. But the simplicity argument soon encounters the problem of whether the interchange of the order of integration leading to Eq. (16) is legitimate or not. In order to arrive at Eqs. (16) and (27) one has to face up to this problem. It is not this interchange by itself which is responsible for the divergence but rather the interchange togheter with the particular form of ρ (q, z) in Eq. (18). To understand this, note that there is enough freedom in the choice of the spectral function to allow for this interchange and consequently guarantee the convergence of the resulting integral. Put differently, the spectral function ρ(q, z), present in the theory on general grounds, also defines the summability of the integral representation in Eq. (23). From this point of view, the special choice of ρ (q, z) in Eq. (18) which leads to Eq. (23) is not completely innocent: it is precisely that element in the class of spectral functions ρ (q, z) which corresponds to Cauchy summability. Formal manipulations of Cauchy integrals (e.g. interchange of order of integration) are notoriously suspect because they cannot always be expected to yield resultant integrals which are summable in the same way as the composite integrals. More to the point, no integral has a value (finite or infinite) unless one has been assigned to it by means of a consistent definition. This is the fundamental thesis of summability theory [10]. Eq. (17) is, for this reason, much more than the generalization, in the physical sense, of Eq. (23). It is also the summability of the latter defined by the spectral function ρ (q, z). The spectral class is adapted to define a summability procedure by the requirement that ρ_0 (q, z) in Eq. (18) be also the limit of any given ρ (q, z) when certain control parameters in the latter are allowed to approach zero. The non–vanishing of these parameters defines a summability method different from that of Cauchy. The limit of these summable integrals when the control parameters tend to zero defines the regularization of the divergent Cauchy integral corresponding to ρ_0 (q, z). In the case of the kernels $G_{\pm}$ (b, z | b', z'), it emerges that their regularizations obtained in this way coincide with their representations as Cesàro integrals. We now proceed to show this. To this end, we make use of translation invariance and rewrite Eq. (23) as

$$G_{\pm}(b\,,z \mid b', z') \equiv G_{\pm}(r\,,\zeta) = \int_0^{\infty} dq\, q\, J_0(qr)\, u_{\pm}(q, \zeta) \tag{47}$$

where $\vec{r} := \vec{b} - \vec{b}'$, $\zeta := z - z'$ and

$$J_0(qr) := \frac{1}{2\pi} \int_0^{2\pi} d\varphi\, e^{iqr\cos(\varphi)} \tag{48}$$

is the Bessel function of order zero. The functions $u_{\pm}(q, \zeta)$ are defined by

$$u_\pm(q,\zeta) = u_\pm^{(1)}(q,\zeta) + u_\pm^{(2)}(q,\zeta) \tag{49}$$

where

$$u_\pm^{(1)}(q,\zeta) := \frac{1}{2\pi}\ \theta\,(k^2-q^2)\, e^{\pm i p_1 \zeta} \tag{50.i}$$

$$u_\pm^{(2)}(q,\zeta) := \frac{1}{2\pi}\ \theta\,(q^2-k^2)\, e^{\mp p_2 \zeta} \tag{50.ii}$$

The integrals in Eq. (47) over $u_\pm^{(1)}(q,\zeta)$ are convergent and will, therefore, not be discussed further. We concentrate on the contributions of $u_\pm^{(2)}$ (q, ζ) to these integrals and restrict attention only to $G_+(r, \zeta)$ since $G_-(r, \zeta) = G_+(r, -\zeta)$. To lighten the notation, we represent this contribution as

$$G_2(r,\zeta) := \frac{1}{2\pi}\int_k^\infty dq\, q\, J_0(qr)\, e^{-p_2\zeta} = \frac{k^2}{2\pi}\int_0^\infty dt\, t\, J_0(R\sqrt{1+t^2}\,)\, e^{-t\, Y} \tag{51}$$

where $R := kr$ and $Y := k\,\zeta$.

The integral in Eq. (51) is convergent for $\zeta > 0$; for $\zeta = 0$ it becomes

$$G_2(r,\zeta) := \frac{1}{(2\pi)}\ \delta\,(r^2) - \frac{1}{(2\pi)}\int_0^k dq\, q\, J_0(qr) \tag{52}$$

where

$$\int_0^\infty dq\, q\, J_0(qr) = \delta(r^2) \tag{53}$$

is the Dirac delta function. For $\zeta < 0$, the integral in (51) is Cauchy divergent. We propose to regularize it by means of Cesàro summability. To this end, we associate with Eq. (51) the sequence of partial sums [10–13]

$$C_0(r,\zeta;Q) := \int_0^Q dq\Big[\, qJ_0(qr)\, u_+^{(2)}(q,\zeta)\Big] \tag{54.i}$$

$$C_n(r,\zeta;Q) := \int_0^Q dq\, C_{n-1}(r,\zeta;q)\,; \qquad n \geq 1. \tag{54.ii}$$

Carrying out the implied iteration in Eq. (54.ii) and making use of (54.i) one finds that

$$C_n(r, \zeta; Q) := \frac{1}{n!} \int_0^Q dq\,(Q-q)^n \left[q\, J_0(qr)\, u_+^{(2)}(q, \zeta)\right] \tag{55}$$

Next, we define the n–th Cesàro mean by

$$C_2^{(n)}(r,\zeta;Q) := \frac{n!}{Q^n} C_n(r,\zeta;Q) = \int_0^Q dq\,(1-\frac{q}{Q})^n \left[q\, J_0(qr)\, u_+^{(2)}(q, \zeta)\right] \tag{56}$$

The limit

$$\text{Reg}\, G_2(r,\zeta) := \lim_{Q\to\infty} C_2^{(n)}(r,\zeta;Q) \qquad (C, n) \tag{57}$$

when it exists, is said to define Reg $G_2(r, \zeta)$ as a Cesàro summable integral of order n [21]. This is the significance of the symbol (C, n) in Eq. (57). By this definition, n=0 corresponds to Cauchy summability, whence the latter is a special case of Cesàro summability. If the limit in (57) exists for $n=N\geq 0$ but not for $n<N$, then it exists for all $n=N+m$, $m \geq 0$. For $\zeta < 0$, this limit does not exist for n=0; in other words, the original integral in Eq. (51) is, as we already know, not Cauchy summable. We have checked that for all finite $\zeta < 0$, there exists a finite $n \geq 1$ for which the limit exists. In other words, the integral in Eq. (51) is Cesàro summable for all finite $\zeta < 0$. We illustrate this convergence graphically in Fig. (2) where Reg $G_2(r, \zeta)$ is plotted against ζ for various values of r. For comparison we have also computed

$$G_1(r, \zeta) := \frac{1}{2\pi} \int_0^k dq\, q\, J_0(qr)\, e^{ip_1(q)\zeta} = \frac{k^2}{2\pi} \int_0^k d\,t\, t\, J_0(R\sqrt{1-t^2}\,)\, e^{it\,Y} \tag{58}$$

The real and imaginary parts of $G_1(r, \zeta)$ are plotted in Figs. (3) and (4) as functions of ζ. The real part of

$$G(r, \zeta) := G_1(r, \zeta) + \text{Reg}\, G_2(r, \zeta) \tag{59}$$

is plotted in Fig. (5).

The limit on the right hand side of Eq. (57) defines the Cesàro integral representation of $G_2(r, \zeta)$. We wish next to show that this representation coincides with the regularization of $G_2(r, \zeta)$, defined as a limit over the set of summable integrals $G_{2\sigma}(r, \zeta)$, characterized by spectral functions $\rho\,(q, z) \equiv \rho_\sigma\,(q, z)$, when the control parameters $\sigma \equiv (\sigma_1.....\sigma_N)$ tend to zero. It is sufficient for our purpose to consider the gaussian distribution $\rho_\sigma\,(q, z) \equiv \rho_\sigma\,(z, z_0)$ about the central point z_0 and having width σ, i.e.

$$\rho_\sigma(z, z_0) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-(z - z_0)^2/2\sigma^2\right] \tag{60}$$

For simplicity, we assume that both z_0 and σ are independent of $\vec{q}$. The distribution $\rho_\sigma(z, z_0)$ is normalised to unity in agreement with Eq. (13). The control or regularization parameter is σ. In fact, we re–obtain Eq. (18) as the limit

$$\lim_{\sigma\to 0} \rho_\sigma(z, z_0) = \delta(z-z_0) \tag{61}$$

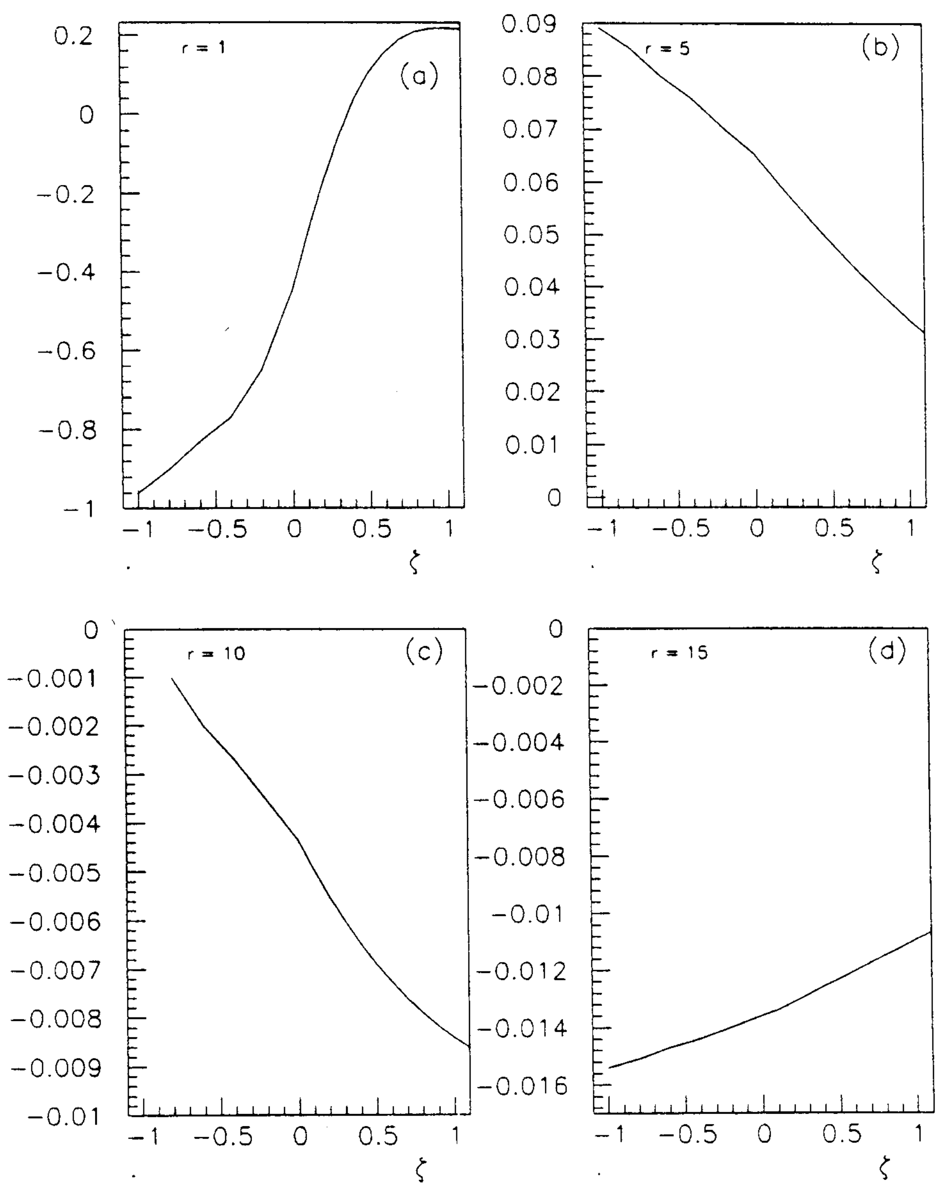

FIG. 2 – Plots of the Cesàro integral Reg $G_2(r, \zeta)$ as a function of ζ ($-1 \le \zeta \le +1$) for various values of r (r = 1, 5, 10, 15) and the parameter k = 1.

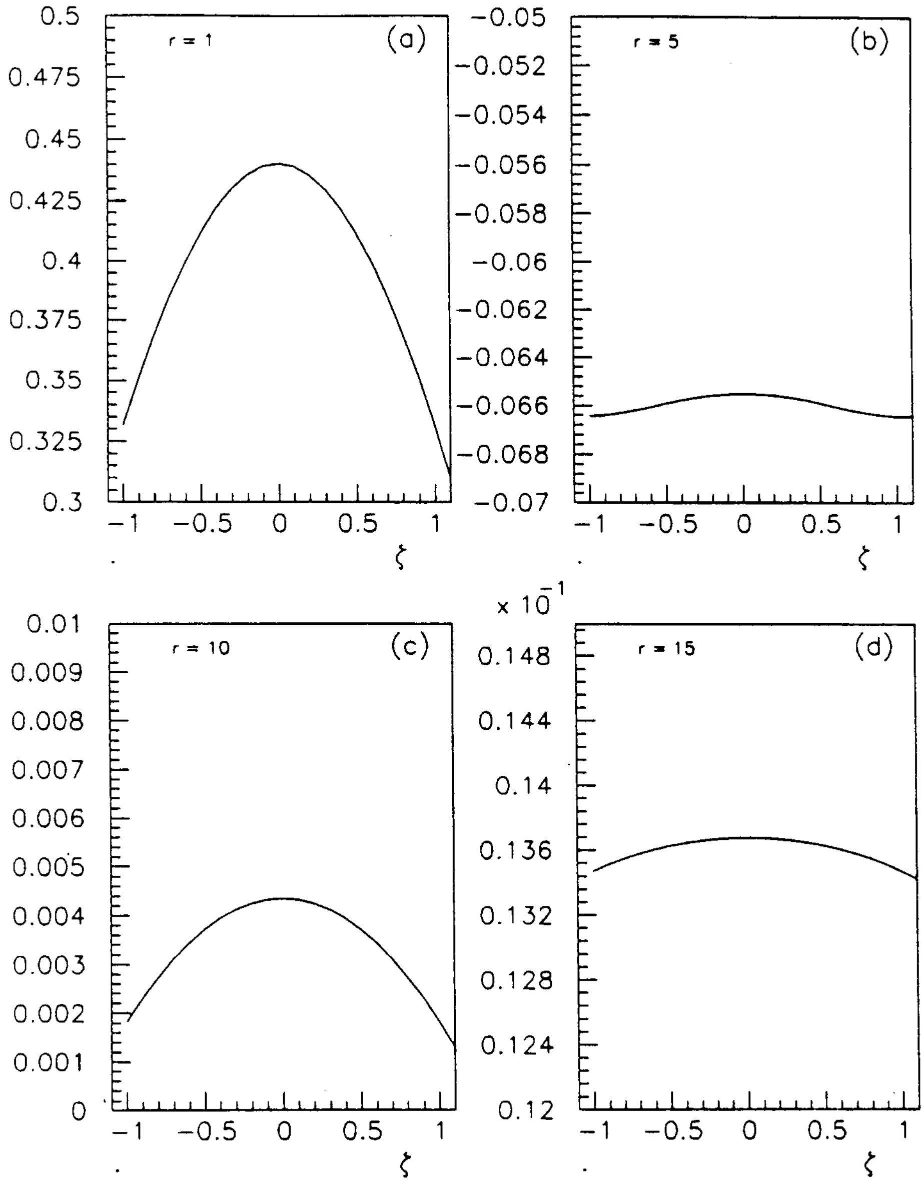

FIG. 3 – Plots of the real part $\mathrm{Re}G_1(r, \zeta)$ of $G_1(r, \zeta)$ (cf. Eq. (58)) as a function of ζ $(-1 \leq \zeta \leq +1)$ for various values of r (r=1, 5, 10, 15) and the parameter k = 1.

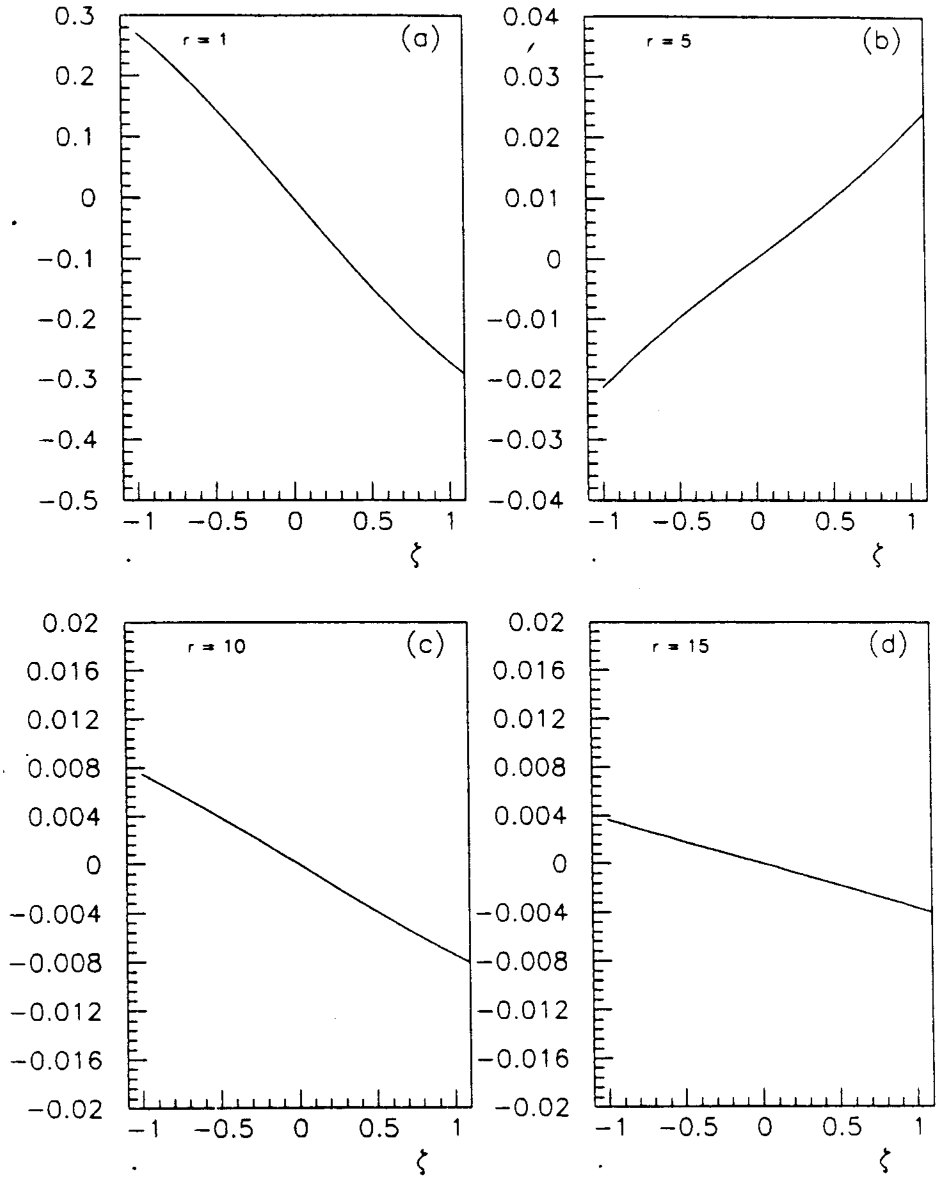

FIG. 4 –Plots of the imaginary part $\mathrm{Im}G_1(r, \zeta)$ of $G_1(r, \zeta)$ (cf. Eq. (58)) as a function of ζ ($-1 \leq \zeta \leq +1$) for various values of r ($r = 1, 5, 10, 15$) and the parameter $k = 1$.

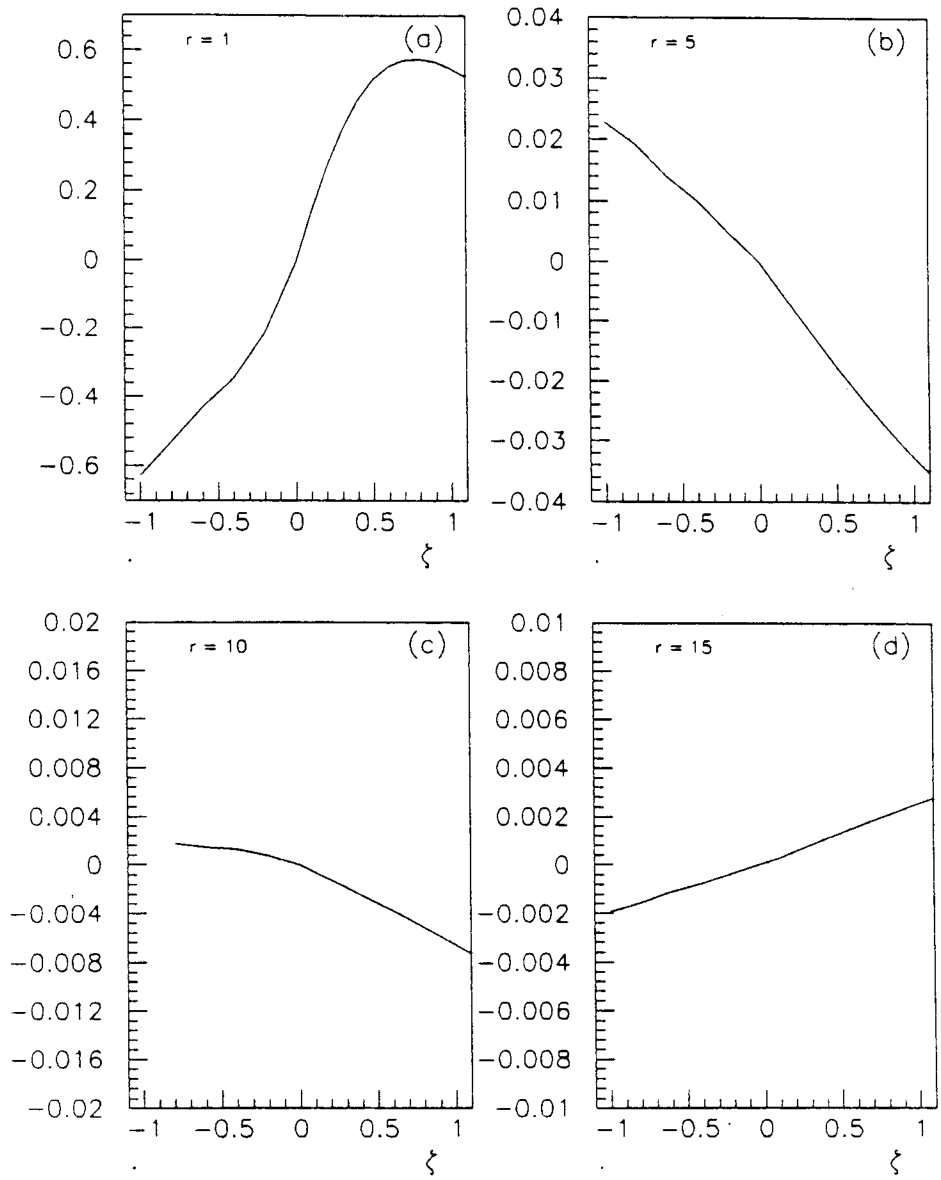

FIG. 5 – Plots of $\mathrm{ReG}_1(r, \zeta) + \mathrm{RegG}_2(r, \zeta)$ of (cf. Eq. (59)) as a function of ζ ($-1 \leq \zeta \leq +1$) for various values of r (r =1, 5, 10, 15) and the parameter k = 1.

Eq. (23) too follows as a limit when $\sigma \to 0$. To see this, first substitute Eq. (60) into (14) to get

$$w_\sigma(p) := \int_{-\infty}^{+\infty} dz\, \rho_\sigma(z, z_0)\, e^{ipz} = \exp\left[ipz_0 - \frac{p^2\sigma^2}{2} \right] \tag{62}$$

On making use of (60) and (62) in (17) then yields

$$D_{\pm\sigma}(b, z \mid b', z') = \rho_\sigma(z, z_0) \cdot G_{\pm\sigma}(b, z \mid b', z_0) \tag{63}$$

where

$$G_{\pm\sigma}(b, z \mid b', z') = \frac{1}{2\pi} \int d^2q\, e^{\pm i \vec{q}(\vec{b} - \vec{b}')}\, u_{\pm\sigma}(q, z-z') \tag{64}$$

and

$$u_{\pm\sigma}(q,z-z') = \frac{1}{2\pi}\left[\theta(k^2-q^2)\, e^{\pm ip_1(z-z')+p_1^2\sigma^2/2} + \theta(q^2-k^2)\, e^{\mp p_2 (z-z')-p_2^2\sigma^2/2} \right] \tag{65}$$

Eqs. (63) and (64) are to be compared with Eqs. (22) and (23), respectively. Note that Eq. (65) is in the form of Eq. (12). From Eqs. (64)and (65) one finds that $G_\pm(b , z \mid b', z')$ in Eq. (23) follows as the limit

$$G_\pm(b, z \mid b', z') = \frac{1}{2\pi} \int d^2q\, e^{\pm i \vec{q}(\vec{b} - \vec{b}')} \cdot \lim_{\sigma\to 0} u_{\pm\sigma}(q, z-z') \tag{66}$$

Now for $\sigma \neq 0$, the $G_{\pm\sigma}(b , z \mid b', z')$ are finite for all values of $z-z'$. One states this by saying that the original integrals $G_\pm(b , z \mid b', z')$ are σ-summable. The limit

$$\mathrm{Reg}\, G_\pm(b , z \mid b', z') := \lim_{\sigma\to 0} G_{\pm\sigma}(b, z \mid b', z') \tag{67}$$

defines the regularization of $G_\pm(b, z \mid b', z')$ through the spectral function $\rho_\sigma(q, z_0)$. The claim is that the limits in Eqs. (57) and (67) [22] define the same functions. We illustrate this equality with the regularization of the function $G_2(r, \zeta)$ in Eq. (51).

The σ-summable integral corresponding to it is

$$G_{2\sigma}(r,\zeta) := \frac{1}{2\pi} \int_k^\infty dq\, q\, J_0(qr)\, e^{-p_2\zeta - p_2^2\sigma^2/2} = \frac{k^2}{2\pi} \int_0^\infty dt\, t\, J_0(R\sqrt{1+t^2})\, e^{-t\, Y - t^2\sigma_k^2/2} \tag{68}$$

Where $R := kr$, $Y := k\zeta$ and $\sigma_k := k\sigma$. The approach of $G_{2\sigma}(r, \zeta)$ to the limit

$$\text{Reg}\, G_2(r, \zeta) := \lim_{\sigma \to 0} G_{2\sigma}(r, \zeta) \tag{69}$$

as σ tends to zero is illustrated in Fig. (6). Comparing Fig. 6 with Fig. 2b one finds that $G_{2\sigma}(r, \zeta)$ tends to the Cesàro representation of $G_2(r, \zeta)$ for $\sigma \to 0$. There are two parts to this result. The first part is straightforward and states only that Reg $G_{\pm}(b\,,z \mid b', z')$ is different from $G_{\pm}(b\,,z \mid b', z')$. It is instructively expressed by the non–commutativity

$$\lim_{\sigma \to 0} \int d^2q\, e^{\pm i\, \vec{q}(\vec{b} - \vec{b}')}\, u_{\pm\sigma}(q, z-z') \neq \int d^2q\, e^{\pm i\, \vec{q}(\vec{b} - \vec{b}')} \cdot \lim_{\sigma \to 0} u_{\pm\sigma}(q, z - z') \tag{70}$$

of the operations of integrating over the two–vector $\vec{q}$ and taking the limit $\sigma \to 0$. It is therefore the interchange of the order of these operations which gives rise to a divergence. The second part of the result is much more important theoretically. It consists in the fundamental equality

$$\text{Reg}\, G_{\pm}(b\,, z \mid b', z') := \lim_{\sigma \to 0} G_{\pm\sigma}(b\,, z \mid b', z') \qquad (C, n). \tag{71}$$

Eq. (71) identifies the procedure of regularization achieved through the control parameter σ with Cesàro summability. The latter method is well defined and unambiguous. It is logically and systematically constructed as the generalization of the Cauchy integral. The introduction of regularization parameters, on the contrary, is arbitrary. The choice of these parameters is practically unlimited and is dictated by no logical procedure. One is led to it by intuition and experience. The gaussian distribution $\rho_\sigma(q, z_0)$ was chosen essentially on this basis, guided, albeit, by the known relationship between it and the Dirac delta function. Eq. (71) is important because it allows to dispense with these ad hoc regularization procedures. The right hand side of Eq. (71) can in fact be obtained much more directly from the left hand side for $n \to \infty$. Recall that if an integral is summable $(C,\ n = N)$, then it is summable for all $n \geq N$, including $n \to \infty$. In the limit $n \to \infty$, Cesàro summability goes over into summability with respect to a spectral measure. To see this and to determine the measure, consider the divergent integrals in Eq. (47) and their Cesàro representations at fixed n

$$\text{Reg}\, G_{\pm}(r, \zeta) := \lim_{Q \to \infty} \int_0^Q dq\, \left(1 - \frac{q}{Q}\right)^n q\, J_0(qr)\, u_{\pm}(q, \zeta) \qquad (C, n) \tag{72}$$

Now send both Q and n to infinity at fixed ratio (Bjorken type limit [14])

$$\lambda := \frac{n}{Q} \tag{73}$$

and then let $\lambda \to 0$ in the end.

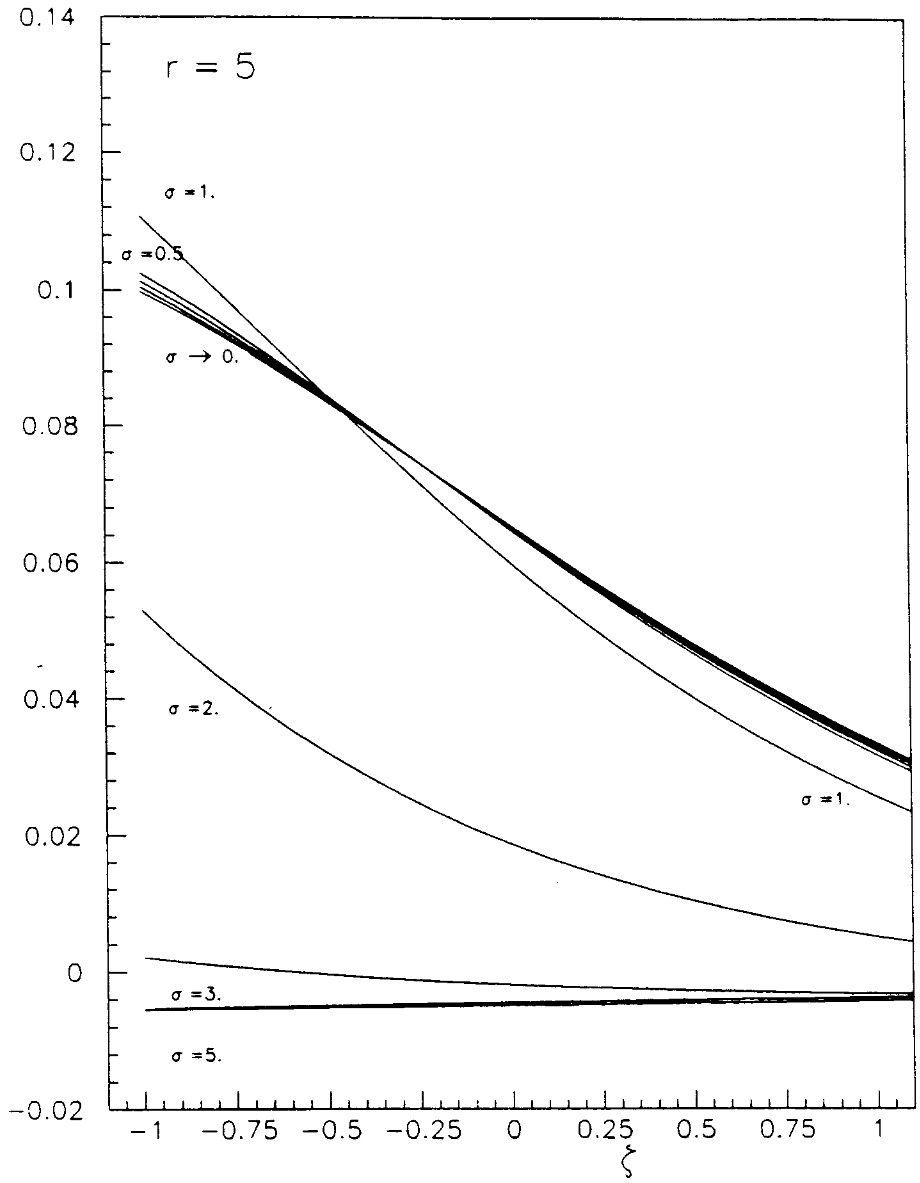

FIG. 6 – Plots of $G_{2\sigma}$ (r, ζ) (cf. Eq. (68)) as a function of ζ ($-1 \leq \zeta \leq +1$) for r = 5 and values of σ tending to zero (σ = 5, 4, 3, 2, 1, 0.5, 0.4, 0.3, 0.2, 0.1, 0). The parameter k is set equal to one.

Carrying out these operations in Eq. (72) yields

$$\text{Reg}\, G_{\pm}(r, \zeta) = \lim_{\lambda \to 0} \int_0^{\infty} dq\, q\, J_0(qr)\, u_{\pm}(q, \zeta)\, e^{-\lambda q} \tag{74}$$

where we have made use of the definition

$$\lim_{n \to \infty} \left(1 - \frac{\lambda q}{n}\right)^n = e^{-\lambda q} \tag{75}$$

Cesàro summability is therefore equivalent to summability with respect to the spectral measure exp(–λq). Eq. (74) agrees with Eq. (71) upon making the identification

$$\lambda = p_2^2\, \sigma^2 / q \tag{76}$$

In z–space, the gaussian distribution $\rho_\sigma(z, z_0)$ therefore corresponds to Cesàro summability, whence the legitimacy of Eq. (71).

5. – CONCLUSIONS

The problem solved in this paper was first clearly formulated by Sherman [8] and by Shewell and Wolf [9]. These authors seemed to have been mostly concerned in pointing out the existence of the divergence in the matrix elements of the forward propagator $G_+(z \mid z')$ for $z < z'$. Sherman laboriously constructed the inverse $G_+^{-1}(z \mid z') = G_+(z' \mid z)$ of $G_+(z \mid z')$ for $z > z'$ to arrive at the divergence. Shewell and Wolf, on the other hand, noticed much more easily the relationship (cf. Eq. (41.ii)) between $G_+(z \mid z')$ and its inverse $G_+^{-1}(z \mid z')$ and hence the divergence in the latter for z>z'. The divergence, according to these authors, arises as a result of the interchange of the order of the $\vec{q}$–and $\vec{b}$–integrations, in the passage from Eqs. (7) and (15) to (16), which defines the kernels $G_\pm(b\,,\, z \mid b'\,,\, z')$ when $\rho(q, z) \equiv \rho_0(q, z) := \delta(z - z_0)$. Compare this point of view with Eq. (70). Eq. (70) expresses the real problem and the operations involved. Shewell and Wolf propose to regularize the singularity most simply by means of a cut–off, a so called band–width limitation. The cut–off eliminates the higher frequencies ($q^2 > k^2$) (i.e. the evanescent waves) in Eqs. (17) and (23), leaving only the homogeneous waves. The arguments in support of this cut–off procedure are not theoretical but rather they make recourse to the behaviour of frequency detectors. The arguments claim that no detector can resolve frequencies that are arbitrarily high. True, therefore it should be the detectors which must confirm this rather than it being built into the mathematics. Sherman is much more sophisticated in his regularization programme. He suggests to interpret the divergent integral as a distribution. The suggestion, unfortunately, does not go further to specify the space of test functions on which the distribution is to act. In any case, the suggestion implies no more than using test functions to operate cut–offs. The work of Sherman and of Shewell and Wolf is widely used in applications. And in doing so serious theoretical problems do in fact arise. We quote, in this regard, the otherwise interesting paper of Devaney on diffraction tomography [6]. One encounters here too the divergent kernel $G_+(b\,,\, z \mid b'\,,\, z')$ (z<z'). To invert the diffraction transform and recover the object field from a given scattered field configuration, Devaney resorts to the construction of a set of filters upon which are imposed various band–width limitations. The operators corresponding to these filters are then expected to combine and yield a "good approximation" to the unit operator in function space. With this approximation one inverts the diffraction transform and recovers the required object field. The latter is then compared with the experimentally deduced target geometrical profile. It is to be feared that, under these circumstances, a theory of the combination of specially constructed filters is being used to simulate experimental cuts. There is no more to this approach than that. Filtering operations, when not required, as here, by the theory are

irrelevant encumbrances. They obscure the dynamics with details of filter properties which one would better do without. To ensure good object recostruction, it is important that the inversion process should be as uncluttered as possible by otherwise avoidable approximations and facile expediencies, especially in the face of mathematical difficulties. Our aim in this paper has been to show that Cesàro summability offers a clean and uncluttered procedure for the inversion of the diffraction transform.The divergence problem in the inversion of the diffraction transform resides in a force of habit. The habit is to assume that all integrals encountered in physical problems, independently of the manipulations which give rise to them, are invariably Cauchy summable. This is far from being true. Integrals, as a rule, have no value until they have been consistently defined [10]. The integral representation of the diffraction transform kernel is not everywhere Cauchy summable. It becomes therefore necessary to interpret it within a more general context which includes the Cauchy integral as a special case. The context we propose is that of Cesàro summability. This interpretation eliminates the divergence in the theory. One describes this phenomenon commonly, but improperly, by saying that the divergence of the corresponding Cauchy integral has been regularized. The regularization is, in this case, equivalent to the regularization achieved by specifying the spectral function present in the general solution of the Helmholtz equation. The corresponding spectral function is a gaussian. The Helmholtz equation should be viewed much more generally. The parameter k^2 in Eq. (4) could be any complex number not necessarily a positive real one. In particular, k^2 could be real and negative. Eq. (5) shows how, starting from the Klein–Gordon equation the latter situation may arise. The solution of the scattering problem would then consist of only evanescent waves. Band–width limitations cannot be invoked to eliminate these waves. The interpretation of the integral representation of the diffraction transform kernel as a Cesàro integral then becomes a necessity. One may apply the same procedure to the solution of the classical scattering problem for the Klein–Gordon equation with $m^2 < 0$.

The scattering problem for the Helmholtz equation involves essentially the solution of the one dimensional Schrödinger equation with a potential barrier at the boundary plane $z = z_0$. The text book solution of this quantum mechanical problem is well known. One may approach the problem differently by taking issue with the non–Hermiticity of the translation operator conjugate to the constrained variable z. We propose to re–consider the problem from this point of view elsewhere.

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[15] It is useful to consider this equation not only for $m^2 > 0$ but also for $m^2 < 0$ or, better, for $m^2 > 0$, in general, complex. The considerations which follow, for the Helmholz equation, would then apply also to the Klein–Gordon equation.

[16] $\rho(q,z)$ need not be a real function let alone a probability distribution so that $\rho(q,z) \geq 0$. However, to give it a physical meaning we interpret it, for simplicity, as a probability distribution.

[17] Sherman advertises Eq. (26.ii) as the product of convolutions; cf. Ref. (8) Eq. (51).

[18] It is also implicitly assumed that ρ (q, z) depends only on the magnitud of $\vec{q}$.

[19] Shewell and Wolf refer to this reflection principle as a reciprocity theorem; cf. Ref. (9) Eqs. (4.11) and (4.12).

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[21] In the mathematical literature, it is usual to write G_2 (r, ζ) for Reg G_2 (r, ζ) in Eq. (57). We deviate from this convenient notation in favour of the usage in quantum field theory in order to emphasize that Reg G_2 (r, ζ) is different from the Cauchy integral representation for G_2 (r, ζ).

[22] In Eq. (67) restricted to the function $G_{2\sigma}$(b , z | b', z').