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Laboratori Nazionali di Frascati

Submitted to Z. Phys. C

LNF-92/029 (P)
10 Aprile 1992

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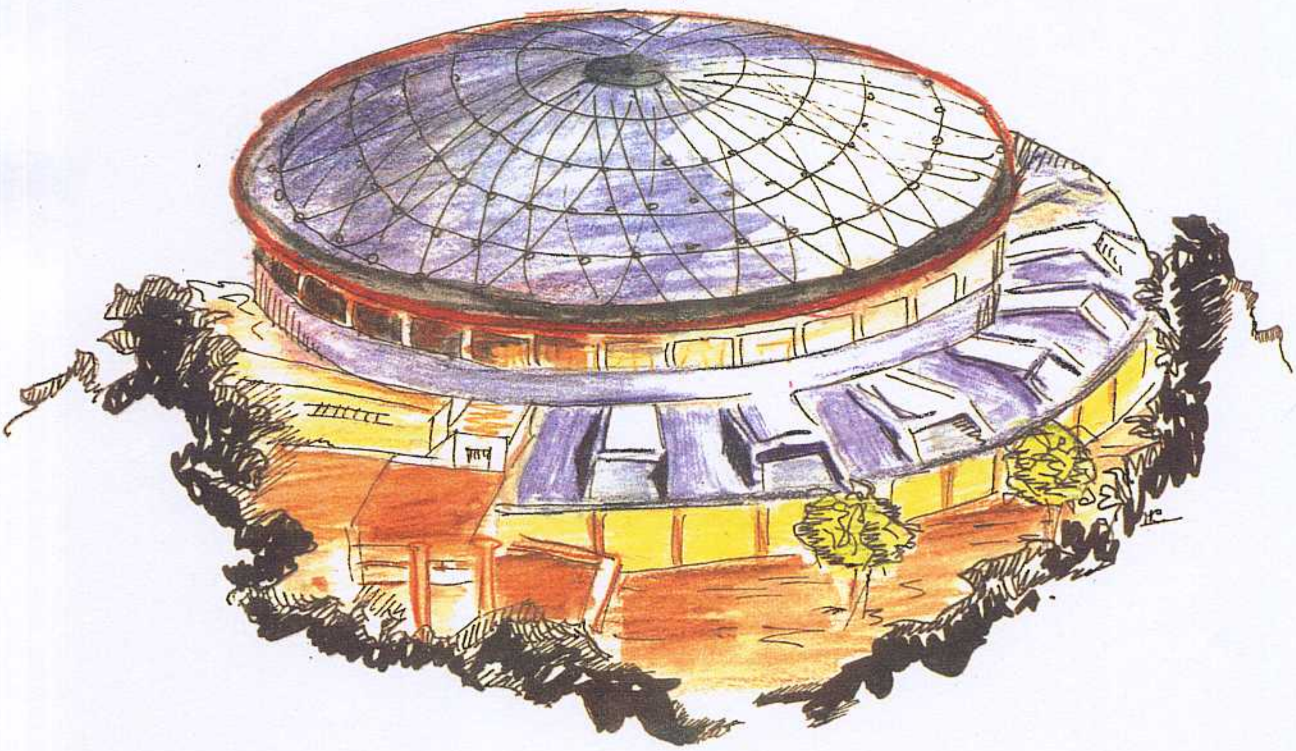

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BROKEN SUPERSYMMETRY IN THE MATRIX MODEL ON A CIRCLE

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ABSTRACT

We consider the discretization of a $D = 2$ surface using polygons. We map the surface onto superspace and integrate over surfaces of arbitrary genus, obtaining a discretized version of the Green-Schwarz string in $D = 1$. Taking an unusual critical limit of the supersymmetric matrix model involved, we construct exact solutions, to all perturbative orders, for the discretized superstring in one dimension, both when the target space is a real line and when the theory is represented in terms of matrix variables on a circle of finite radius. We comment on the behavior of the compactified perturbative expansion under duality transformations.

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The analysis contained in the present paper, is concerned with the nonperturbative solution of the supersymmetric matrix model. The study of string theories in low dimensions can be carried out by discretizing the worldsheet [1]. A connection to the matrix theory is so established. Carrying out random triangulations of arbitrary $D = 2$ surfaces one can recover $D = 2$ gravity [2]. Summing over arbitrary genus surfaces in the critical limit, which is equivalent to summing the series to all orders, is a consequence of the random matrix model representation of $D = 1$ string theory [3]. The case of one-dimensional matter coupled to gravity has been considered in ref. [4]. This case is equivalent to the theory of $D = 1$ strings. The connection to random matrix theory reduces the problem to that of N fermions in $D = 1$ quantum mechanics [1]. In the critical limit, this is exactly solvable. There appear logarithmically divergent terms due to massless modes [5].

In ref. [6], the supersymmetric theory is analyzed. Our main motivation in taking up the supersymmetric matrix model is that, in the supersymmetric case, one of the flaws that afflict the bosonic theory in $D > 1$, i.e. the presence of a tachyon in the spectrum, is taken care of by the invariance of the model. We consider arbitrary potentials, through supersymmetric quantum mechanics, in a certain critical limit, to all orders. We obtain an exact expression for the density of states and the critical coupling, by adopting a supersymmetric WKB (sWKB) method [7]. In this way, we are able to evaluate the spectral density in the double scaling limit. The density of states turns out to be different from the standard critical limit of the bosonic matrix model. In the present work, we also consider multicritical points and compute the contribution of various terms in the sWKB series, commenting on the multicritical cases that could arise.

Recent results on nonperturbative features of $D = 2$ quantum gravity have been obtained by carrying out numerical simulations [8]. The discretization of the surface gives us exact results. The matrix model is equivalent to $D = 2$ quantum gravity, in the continuum limit. The discrete theory allows to study nonperturbative phenomena, at least for the cases when embedding of the theory in a target space can be carried out in a calculable way. In this respect, numerical simulations can provide a complementary view. This gives rise, according to ref. [8], to quite surprising results, such as the formation of $D = 1$ baby universes, i.e. circles whose radiuses correspond to the time coordinate on the world-sheet, in $D = 2$ quantum gravity.

Next, we consider the mapping from a $D = 2$ surface to superspace [6]. The

$D = 2$ surface can be discretized using polygons of the same size and associating to each polygon a point in superspace. By integrating over all surfaces of arbitrary genus, imposing reparametrization invariance and supersymmetry, respectively, in parameter and target spaces one can define the superstring theory. In this way, one obtains a discrete version of the $D = 1$ Green-Schwarz string. The features of the latter can be reproduced in the large N limit (i.e. $N \to \infty$).

We define a superfield matrix ϕ

$$\Phi \equiv \phi + \theta\psi + \bar{\theta}\bar{\psi} + \theta\bar{\theta}F , \tag{1}$$

where ϕ, ψ, F are $N \times N$ hermitean matrices. The superfield action can be written as

$$S = \int dt\, d\theta\, d\bar{\theta}\, Tr\left[-\Phi D^2\Phi + W(\Phi) \right] . \tag{2}$$

Here we denote by D the covariant derivative and by $W(\Phi)$ the supersymmetric potential. The partition function $Z(\beta)$ is obtained by integrating eq. (2)

$$Z(\beta) = \int D\Phi\, e^{-\beta S} . \tag{3}$$

In order to obtain the vanishing fermion number hamiltonian, we integrate out the fermions ψ. This gives

$$H = Tr\left(p^2 + F^2 - \frac{1}{\beta}\sigma_3 F' \right) . \tag{4}$$

Here we introduce the notation $p^2 = -\frac{1}{\beta^2}\frac{d^2}{dx^2}$, $F = \frac{dW}{d\phi}$. Diagonalizing the matrix field and introducing an appropriate jacobian for the measure, the problem can be reduced to that of N fermions at zero temperature. Using a diagonal representation for σ_3 we can rewrite the hamiltonian as

$$H = \begin{pmatrix} H_B & 0 \\ 0 & H_F \end{pmatrix} = \begin{pmatrix} p^2 + F^2 - \frac{1}{\beta}F' & 0 \\ 0 & p^2 + F^2 + \frac{1}{\beta}F' \end{pmatrix} . \tag{5}$$

In the large N limit, near the critical point, with the levels becoming dense and the maximum of the potential just touching the Fermi level, the problem can be analyzed by studying the singularity structure of the density of states [4]

$$\rho(E) \equiv \frac{1}{\beta}\sum_n \delta(E_n - E) . \tag{6a}$$

One can introduce also the coupling constant

$$g = \frac{N}{\beta} = \int_0^{E_F} \rho(E)\, dE \; . \tag{6b}$$

Here N is the number of fermions and coincides with the order of the matrix.

One has the ground state energy

$$E_{gs} = \sum_n E_n = \beta^2 \int_0^{E_F} E\, \rho(E)\, dE \; . \tag{7}$$

Defining $\mu = V_{max} - E_F$ and letting $\mu \to 0$, the density of states and the coupling constant become divergent, reflecting a critical behaviour. We introduce the derivatives of g and E_{gs} [1]

$$\frac{\partial g}{\partial \mu} = -\rho \tag{8a}$$

and

$$\frac{\partial E_{gs}}{\partial \mu} = \beta^2 \mu \frac{\partial g}{\partial \mu} = -\beta^2 \mu \rho \; . \tag{8b}$$

In the WKB approximation, i.e. in the large N limit, we obtain to the zeroeth order

$$\rho = \frac{1}{\pi} \int \frac{dx}{\sqrt{(E-V)}} \; . \tag{9}$$

The singularity of the density of states at the turning points that correspond to maxima of the potential reads

$$\rho = -\frac{1}{2\pi} \, log\; \mu \; . \tag{10}$$

The $\frac{df}{dx}$ term in the potential for the supersymmetric case is $O(\frac{1}{\beta}) \sim O(\frac{1}{N})$. Expanding $\sqrt{E - F^2 + \frac{1}{\beta}F'}$ in eq. (9) one can calculate, in the leading N approximation, the maximum of the potential F^2 that contributes to the divergent part. Hence, to the leading order in N, we can compute the energy of the ground state

$$E_{gs} \sim -\frac{N^2 (\Delta g)^2}{log\; (\Delta g)} \; .$$

This expression exhibits a well-known logarithmic dependance on the renormalized cosmological constant.

Next, we will carry out a resummation of the contributions to all orders and take a double scaling limit different from that in ref. [4]. To this purpose, we are going to choose the origin of the energy variable at the maximum of the F^2 term, near the critical point. In doing so, we are not going to concern ourselves with the issue of maintaining the supersymmetry invariance of the theory, when the zero of the energy coincides with the maximum of the potential. The non-preservation of supersymmetry when the energy is measured from the new origin is carried out in an appendix.

The potential can be expanded in the vicinity of the maximum of F. Hence, recalling eq. (5), we obtain

$$H = p^2 + a^2x^2\Big(1+\sum_{n>0} c_n x^n\Big) - \frac{1}{\beta}\Big(a + \sum_{n>0} d_n x^n\Big) . \qquad (11)$$

Here we made use of the vanishing of the maximum of the potential. The contribution of the coefficients c_n and d_n to the hamiltonian (11) is suppressed by powers of $\frac{1}{N}$. In fact, if we scale μ as $\frac{1}{\beta}$ and x^2 as $\frac{1}{\beta}$, the hamiltonian scales as $\frac{1}{\beta} \sim \frac{1}{N}$. In this scaling limit, we remain with the potential of a supersymmetric harmonic oscillator, analytically continued to imaginary frequency. This is the same behaviour obtained in the bosonic case.

Considering a supersymmetric harmonic oscillator and analytically continuing its frequency to imaginary values, taking into account that the energy levels are modified to be $n\hbar$ rather than $(n+\frac{1}{2})\hbar$, one can obtain the density of states

$$\rho(\mu) = \frac{1}{\pi} Re \sum_{n=0}^{\infty} \frac{1}{2n + i\beta\mu} . \qquad (12)$$

Modifying the singularity, in order to achieve agreement with the value obtained in (10) for $N \to \infty$, one introduces the redefinition

$$\rho(\mu) = -\frac{1}{2\pi}\left[Re\, \psi\Big(i\frac{1}{2}\beta\mu\Big) - log\, \Big(\frac{1}{2}\beta\Big)\right] . \qquad (13)$$

Here ψ is the digamma function

$$\psi(z) = \frac{d}{dz} log\, \Gamma(z) . \qquad (14)$$

The redefined density of states can be rewritten as a power series, with coefficients given in terms of the Bernoulli numbers

$$\begin{aligned}\rho(\mu) &= -\frac{1}{2\pi}log\ \mu + \frac{1}{\pi\beta\mu}Im\sum_{n=0}^{\infty}\frac{1}{1-i\frac{2n}{\beta\mu}} \\ &= -\frac{1}{2\pi}log\ \mu - \frac{1}{2\pi}\sum_{k=1}^{\infty}\frac{|B_{2k}|}{2k}\left(\frac{2}{\beta\mu}\right)^{2k} .\end{aligned} \tag{15}$$

Using the explicit value of the first few Bernoulli numbers

$$|B_2| = \frac{1}{6}\ ,\ |B_4| = \frac{1}{30}\ ,\ |B_6| = \frac{1}{42}\ ,$$

one can explicitly express the density of states in the form

$$\rho(\mu) = -\frac{1}{2\pi}\left(log\ \mu + \frac{1}{3\beta^2\mu^2} + \frac{2}{15\beta^4\mu^4} + \frac{16}{63\beta^6\mu^6} + ...\right) . \tag{16}$$

Next, we describe a calculation carried out using the method of sWKB [7]. It is quite interesting that this procedure leads to the same series obtained above using the expansion of the resolvent.

We start with the expression of the potential in supersymmetric quantum mechanics

$$V = f^2 - \hbar f' . \tag{17}$$

It is well-known that the conventional WKB approximation can be described as an expansion in powers of the parameter $\hbar$. The underlined assumption is that $\hbar$ is small. In our case $\hbar$ corresponds to $\frac{1}{\beta}$ or $\frac{1}{N}$, where $N \to \infty$. However, it seems that the f' term in the 1-particle potential is not suppressed by a power of $\frac{1}{N}$. Indeed, the factor $\frac{1}{N} \sim \hbar$, arising because this term comes from fermionic loops, cancels due to the presence of N fermions per eigenvalue in the matrix model, in agreement with refs. [6]. We take the integrand of sWKB to be f^2, rather than the full potential $f^2 - \hbar f'$ which includes the contribution of fermionic loop integrals. The second difference in the sWKB procedure, with respect to the conventional WKB method, arises in choosing the turning points. In fact, in sWKB, the turning points are given by $E - F^2 = 0$, rather than by $E - F^2 + \hbar F' = 0$, as is the case of the standard WKB approximation. Effectively, sWKB provides a resummation of terms in the conventional WKB series. There are cases where the energy spectrum can be calculated exactly and one can obtain analytic solutions. These are the supersymmetric quantum mechanical problems with exactly solvable potentials. In these cases, the solution receives contributions from an infinite number of terms of the conventional WKB series, i.e. from terms of all orders, while the result obtained using sWKB to first order is exact.

It is instructive to compare the series obtained by both approximation techniques. We begin by giving the conventional WKB series, up to $O(\hbar^6)$ terms

$$
\begin{aligned}
&\sqrt{2m}\int_{x_1}^{x_2}\sqrt{E-V}\,dx-\frac{\hbar^2}{24\sqrt{2m}}\frac{d}{dE}\int_{x_1}^{x_2}\frac{V''}{\sqrt{E-V}}\,dx\\
&\quad+\frac{\hbar^4}{2880(2m)^{\frac{3}{2}}}\frac{d^3}{dE^3}\int_{x_1}^{x_2}\frac{7(V'')^2-5V'V'''}{\sqrt{E-V}}\,dx\\
&\quad-\frac{\hbar^6}{725760(2m)^{\frac{5}{2}}}\left[\frac{d^4}{dE^4}\int_{x_1}^{x_2}\frac{216(V''')^2}{\sqrt{E-V}}\,dx\right.\\
&\quad\left.+\frac{d^5}{dE^5}\int_{x_1}^{x_2}\frac{93(V'')^3-224V'V''V'''+35(V')^2V''''}{\sqrt{E-V}}\,dx\right]\\
&\quad=N\pi\hbar\,,
\end{aligned}
\tag{18}
$$

where we denote by x_1 and x_2 two points where the expression $E-V$ vanishes. Following ref. [7], the series obtained using sWKB reads

$$
\begin{aligned}
&\sqrt{2m}\int_a^b\sqrt{E-f^2}\,dx-\frac{\hbar^2E}{6\sqrt{2m}}\frac{d^2}{dE^2}\int_a^b\frac{(f')^2}{\sqrt{E-f^2}}\,dx\\
&\quad+\frac{\hbar^4}{720(2m)^{\frac{3}{2}}}\left[\frac{d^2}{dE^2}\int_a^b\frac{30f'f'''}{\sqrt{E-f^2}}\,dx\right.\\
&\quad\left.+\frac{d^3}{dE^3}\int_a^b\frac{-8(f')^4-31f(f')^2f''+7f^2(f'')^2-5f^2f'f'''}{\sqrt{E-f^2}}\,dx\right]\\
&\quad+\frac{\hbar^6}{90720(2m)^{\frac{5}{2}}}\left[\frac{d^3}{dE^3}\int_a^b\frac{378(f''')^2}{\sqrt{E-f^2}}\,dx\right.\\
&\quad+\frac{d^4}{dE^4}\int_a^b\frac{[-2160ff'f''f'''+1674(f')^2(f'')^2-108f^2(f''')^2]}{\sqrt{E-f^2}}\,dx\\
&\quad+\frac{d^5}{dE^5}\int_a^b\frac{1}{\sqrt{E-f^2}}[96(f')^6-1119f(f')^4f''+729f^2(f')^2(f'')^2\\
&\quad\left.+399f^2(f')^3f'''-93f^3(f'')^3+224f^3f'f''f'''-35f^3(f')^2f'''']\right]\\
&\quad=N\pi\hbar\,.
\end{aligned}
\tag{19}
$$

The expression $E-f^2$ vanishes at the integration extrema a, b.

We recall that we are approximating using an inverted supersymmetric harmonic oscillator. Hence, in our case, $f \equiv i\sqrt{2}x$ and $f' = i\sqrt{2}$. We note the presence of the imaginary factor i in the potential, although the final result is real, due to the appearance of only even powers of f. The appearance of imaginary factors is a consequence of the shift in the origin of the energy variable that we introduced when setting a vanishing maximum of the F^2 term in the potential.

The method of sWKB allows us to obtain the density of states. All one needs is taking one derivative of the sWKB series with respect to E. Differentiating eq. (19) yields the divergent part for $\rho(\mu)$

$$\rho(\mu) \;=\; -\frac{1}{2\pi} log\ \mu - \frac{1}{6\pi(\beta\mu)^2} - \frac{1}{15\pi(\beta\mu)^4} - \frac{8}{63\pi(\beta\mu)^6} - \cdots \,. \qquad (20)$$

This matches exactly the terms of the series in eq. (16). Thus, we have obtained the result that the modification of the WKB approximation adopted here coincides with the nonperturbative solution for the discretized $D = 1$ superstring we constructed by analytically continuing the supersymmetric harmonic oscillator. Next, we turn our attention to the exact expression that can be derived for the critical coupling.

In order to compute Δg entering the result to leading order in N for the ground state energy, we integrate eq. ($8a$) using the exact expression for the density of states given in (13), (14)

$$\Delta g \;=\; \frac{1}{\pi\beta}\left[- Im\ log\ \Gamma\left(i\frac{1}{2}\beta\mu\right) + \frac{\beta\mu}{2}\ log\ \left(\frac{\beta\mu}{2}\right)\right] - \frac{1}{2\pi}\mu\ log\ \mu \,. \qquad (21)$$

This represents an exact solution that can be asymptotically expanded, yielding the same result one obtains by integrating the asymptotic expansion of the density of states (15), modulo terms that vanish when the planar limit or the double scaling limit are considered. The integral representation of (21) allows the investigation of the non-perturbative aspects of the corresponding string theory. This includes the possibility of obtaining useful information on the ground state energy of the $D = 1$ Green-Schwarz superstring, by discretizing the worldsheet and establishing the connection to the matrix model we described. Thus, our exact results bear important consequences for the nonperturbative description of $D = 2$ quantum gravity through the supersymmetric string. Our nonperturbative solution can be elaborated, following the same procedure adopted in accord with the WKB expansion in ref. [9], as quantum oscillations on top of smooth distributions obtained using sWKB, instead of the conventional WKB solution. We note that all the correlation functions, which are related to $< Tr\phi^{2k} >$, can be calculated by differentiating the density of states ρ with respect to β [4].

One can consider next the case of potentials whose k-th derivative vanishes at the maximum. The critical behaviour turns out to be the same as in the case of the bosonic string theory [4]. The scaling law of Δg is given by $\mu^{(2+k)/2k}$. The ground state energy E_{gs} scales as $N^2(\Delta g)^{2+\gamma_{st}}$, where $\gamma_{st} = \frac{k-2}{k+2}$. We can treat the exact solution obtained by analyzing these potentials using sWKB. For this purpose, we take $f = ix^{k/2}$. In carrying out the nonperturbative analysis, we assume

$$\mu \sim x^k \sim \frac{1}{\beta^2}\partial_x^2 \, . \tag{22}$$

After differentiating the sWKB series (19) with respect to E, we obtain the behaviour of the nth order term contributing to the density of states

$$\rho^{(n)} \propto \frac{1}{\beta^n}\int \frac{(f')^n}{(E-f^2)^{\frac{1}{2}+n}}dx \, . \tag{23}$$

Hence

$$\rho = \mu^{\frac{2-k}{2k}} \sum_{n \ even} \frac{C_n}{\beta^n}\, \mu^{-n(\frac{k+2}{2k})} \, . \tag{24}$$

Plugging this expression in eq. (8a) and integrating, we find

$$\Delta g = \mu^{\frac{k+2}{2k}} \sum_{n \ even} \frac{D_n}{\beta^n}\, \mu^{-n(\frac{k+2}{2k})} \, . \tag{25}$$

We can also obtain the behaviour of the ground state energy. Integrating eq. (8b) using (24), yields

$$E_{gs} = \frac{1}{g_{st}^2}\left[1 + \sum_{n=1}^{\infty} \frac{a_{2n}}{\beta^{2n}(\Delta g)^{2n}}\right] , \tag{26}$$

where

$$g_{st}^2 = \frac{1}{\beta^2(\Delta g)^{\frac{3k+2}{k+2}}} .$$

Note that $\beta(\Delta g) \sim O(1)$, as can be seen by inspecting eq. (22). Hence, all the terms in the sum (26) are of the same order.

As we have seen in our model, string theory in $D = 1$, which is equivalent to gravity coupled to matter in $D = 1$, is linked to the random matrix models, through the discretization of the worldsheet. The random matrix theory representation of the one dimensional string allows to derive a solution to all perturbative orders, for both the case in which the target space is a real line [4] and the string theory compactified on a circle of finite radius R [10]. Our present analysis elucidates the solution to arbitrary genus of the supersymmetric string on the real line, in the critical limit. Next, following ref. [11], we consider the representation of the supersymmetric string in terms of matrix variables on a circle.

We begin with the topological expansion of the discretized string partition function

$$Z = \sum_{G} N^{2(1-G)} \sum_{S_N} g^A Z_N , \tag{27}$$

where G denotes the genus and N is the number of points in the discretization S_N. Here A represents the area of the discretization and g is the coupling constant. The latter can be expressed in terms of the cosmological constant Λ

$$g = e^{-\Lambda} . \tag{28}$$

In the string path integral (27) we introduced the term

$$Z_N = \int [dx_k] exp\Big(-\sum_{<ij>} E_{ij}{}^{(N)}\Big) , \tag{29}$$

where the sum is over the nearest neighbour vertices of S_N. We recall that the target space is one dimensional. Hence, there are two cases to be analyzed. First, we considered the case of a real line, where we integrated in the range $(-\infty, +\infty)$. In this case, we took a gaussian link factor

$$E_{ij} \sim \frac{1}{2}(x_i - x_j)^2 . \tag{30}$$

Now, we take into consideration the case of a circle, where the integrations are carried out in the range $[0, 2\pi R)$. In this case, the link factor is assumed to be periodic with period $2\pi R$

$$E_{ij} \sim R(x_i - x_j) . \tag{31}$$

The string partition function is related to that of a matrix model

$$Z = \int [d\Phi_{ab}] exp\left[- N \int Tr\Big(\frac{1}{2}\dot{\Phi}^2 + V(\Phi)\Big) dt \right] , \tag{32}$$

where Φ_{ab} is a $N \times N$ hermitean matrix and the potential can be expressed as

$$V(\Phi) = \frac{1}{2}\Phi^2 + \frac{1}{3}g\Phi^3 + \dots . \tag{33}$$

The action of the supersymmetric matrix model in (2) can be expressed, by recalling the covariant derivative D

$$D = \frac{\partial}{\partial\theta} + \bar{\theta}\frac{\partial}{\partial t} \tag{34}$$

and carrying out the integration over the fermionic coordinates θ, $\bar{\theta}$ and eliminating the auxiliary field F, in terms of the components of the $D = 1$ superfield matrix $\Phi_{ab}(t, \theta, \bar{\theta})$, as follows:

$$S = \int dt Tr\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\bar{\psi}\dot{\psi} + \left(\frac{\partial W(\phi)}{\partial\phi_{ab}}\right)^2 + \bar{\psi}_{ab}\frac{\partial^2 W(\phi)}{\partial\phi_{ab}\partial\phi_{cd}}\psi_{cd}\right] . \tag{35}$$

Going to the hamiltonian formalism and noticing that $\frac{1}{N}$ plays the role of $\hbar$, we have

$$[p_{ab}, \phi_{cd}] = \frac{1}{N}\delta_{ad}\delta_{bc} , \tag{36}$$

$$\{\psi_{ab}, \bar{\psi}_{cd}\} = \delta_{ad}\delta_{bc} . \tag{37}$$

As $N \to \infty$, this reproduces the $D = 1$ superstring. The problem is reduced to that of finding the ground state of the hamiltonian

$$H = -\frac{1}{2N^2}\frac{\partial^2}{\partial{\phi_{ab}}^2} + \left(\frac{\partial W}{\partial\phi_{ab}}\right)^2 - \frac{1}{N}\frac{\partial^2 W}{\partial{\phi_{ab}}^2} . \tag{38}$$

We take as an ansatz for the wavefunction of the ground state $\chi(\{\lambda_i\})|0>$, where

$$\psi_{ab}|0> = 0 , \tag{39}$$

and $\{\lambda_i\}$ are the eigenvalues of ϕ_{ab}

$$\phi_{ab} = \Omega^{-1}\Lambda\Omega . \tag{40}$$

Here Ω denotes a unitary matrix, whereas Λ is a diagonal matrix. The change of variables introduces the Van der Monde determinant

$$[d\phi_{ab}] = [d\lambda_i][d\Omega]\Delta^2(\lambda) , \tag{41}$$

where

$$\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) \ . \tag{42}$$

Then, we can write, for the ground state energy,

$$E_{gs} \ = \ \frac{1}{N^2}\min_{\tilde{\chi}} \frac{\int[d\lambda](\tilde{\chi}H\tilde{\chi})}{\int[d\lambda](\tilde{\chi}\tilde{\chi})} \ , \tag{43}$$

where we have introduced the redefinition of the ground state wavefunction due to the Van der Monde determinant

$$\tilde{\chi}(\lambda) \equiv \Delta(\lambda)\chi(\lambda) \ . \tag{44}$$

The problem becomes that of N fermions in a one dimensional supersymmetric potential

$$V(\lambda) = [W'(\lambda)]^2 - \frac{1}{N}W''(\lambda) \ . \tag{45}$$

In the first part of our work, where we considered the discrete version of the supersymmetric string on the real line, we took, for the superpotential W

$$(W')^2 = -2\lambda^2 \ , \ W'' = i\sqrt{2} \ . \tag{46}$$

In analogy with the bosonic case, we solved the inverted supersymmetric harmonic oscillator. We define $\mu = V_{max} - E_F$. The density of states and the coupling constant are singular functions of μ. Accordingly with this critical behaviour as $\mu \to 0$, the ground state energy depends non-analytically on the coupling constant. The renormalized cosmological constant is identified with the nonanalytic component of $E_{gs}(\Delta)$, where $\Delta = 1 - g$ [4]. The function $E_{gs}(\Delta)$ can be calculated from the equation

$$\frac{\partial E_{gs}}{\partial \Delta} \ = \ \beta^2(\mu \ - \ V_{max}) \ , \tag{47}$$

plugging the function $\mu(\Delta)$ obtained by solving

$$\frac{\partial \Delta}{\partial \mu} \ = \ \rho(E_F) \ . \tag{48}$$

The high degree of divergence of the asymptotic expansion of the density of states (15) is expressed by the fast growth of the Bernoulli numbers, reflecting a typical stringy behaviour [12].

The exact result we obtained can be rewritten as an integral representation

$$\frac{1}{\beta}\frac{\partial \rho}{\partial \mu} = \frac{1}{2\pi\beta\mu} \, Im \int_0^\infty dt \; exp\,(-it)\; exp\left(\frac{t}{\beta\mu}\right) \frac{t}{\beta\mu} \left[sinh\left(\frac{t}{\beta\mu}\right)\right]^{-1} . \quad (49)$$

This should be regarded as an attempt to provide a nonperturbative definition of the supersymmetric string theory, in terms of the integral over the Borel transform of the asymptotic expansion (15), supplemented by a principle value prescription for integrating about the infinite number of poles on the real axis. We will have to keep in mind that the potential ambiguities associated with the infinite number of arbitrary parameters introduced by the principle value prescription, make the validity of the integral representation (49) beyond its asymptotic expansion (15) unclear.

Integrating eq. (48) using the perturbative series (15), we obtain

$$\Delta = \frac{\mu}{2\pi}\left[- \, log\,\mu + \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k(2k-1)} \left(\frac{2}{\beta\mu}\right)^{2k} \right] . \quad (50)$$

This expression can be inverted to write μ as a function of Δ

$$\mu = - \frac{2\pi\Delta}{log\,\Delta}\left[1 + \frac{1}{log\,\Delta}\sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k(2k-1)} \left(\frac{log\,\Delta}{\pi\beta\Delta}\right)^{2k} \right] , \quad (51)$$

where we neglect double logarithms, as well as terms that are suppressed by powers of $log\,\Delta$. Using $\mu(\Delta)$ from (51) and integrating eq. (47), yields the leading-logarithmic series for the ground state energy

$$E_{gs} = - \frac{\pi^2\beta^2\Delta^2}{log\,\Delta} - \frac{1}{6\pi} log\,\Delta + \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{|B_{2k+2}|}{k(k+1)(2k+1)} \left(\frac{log\,\Delta}{\pi\beta\Delta}\right)^{2k} . \quad (52)$$

For comparison, note that in the bosonic case one has

$$E_{gs} = - \frac{\pi^2\beta^2\Delta^2}{log\,\Delta} + \frac{1}{12\pi} log\,\Delta - \frac{1}{4\pi}\sum_{k=1}^{\infty} \left(2^{2k+1} - 1\right) \frac{|B_{2k+2}|}{k(k+1)(2k+1)} \left(\frac{log\,\Delta}{2\pi\beta\Delta}\right)^{2k} . \quad (53)$$

In writing (53), we are correcting an error in the result given in eq. (3.13) of ref. [10]. The appearance of logarithmic divergences in our result deserves two comments. Firstly, these divergences can only be detected by the discretized version of the theory, though agreement is found with the scaling of the free energy like Δ^2 obtained from conformal field theory [13]. Secondly, the logarithmic dependance on the renormalized cosmological constant has been linked, in the bosonic string case, to the presence of massless modes [5].

As in the bosonic case [10], we define the matrix variable $\phi(t)$ on a circle of radius R, with the partition function

$$Z \;=\; Tr\; exp(-2\pi R\beta H)\ . \tag{54}$$

Note that $2\pi R$ can be interpreted as the inverse temperature. The coupling constant is defined in terms of the chemical potential μ_F as follows:

$$g \;=\; \int_0^\infty \rho(E)\{1 \;+\; exp[2\pi R\beta(E \;-\; \mu_F)]\}^{-1}\, dE\ . \tag{55}$$

In order to calculate the free energy F as a function of Δ, one needs to consider the equations

$$\frac{\partial\Delta}{\partial\mu} \;=\; \int d\lambda\; \rho(\mu_c \;-\; \lambda)\frac{\partial}{\partial\mu}\{1 \;+\; exp[2\pi R\beta(\mu \;-\; \lambda)]\}^{-1}\ , \tag{56}$$

$$\frac{\partial F}{\partial\Delta} \;=\; \beta^2(\mu \;-\; \mu_c)\ . \tag{57}$$

Here we have introduced $\mu \equiv \mu_c \;-\; \mu_F\ ,\ \lambda \equiv \mu_c \;-\; E$. First, one uses (56) to determine the function $\mu(\Delta)$. Then, $F(\Delta)$ can be calculated by integrating eq. (57).

Next, we proceed by differentiating (56)

$$\frac{\partial^2\Delta}{\partial\mu^2} \;=\; \int d\lambda\frac{\partial\rho}{\partial\lambda}\,\frac{1}{2}\pi R\beta\;\{cosh[\pi R\beta(\mu \;-\; \lambda)]\}^{-2}\ . \tag{58}$$

It is convenient to use the integral representation of ρ given in eq. (49) that provides the exact result for the supersymmetric string theory. The integral in λ can be carried out introducing the variable $x \;=\; \frac{t}{\beta\mu}$

$$\begin{aligned}\frac{1}{2}\pi R\beta\int d(\mu \;-\; \lambda)\; exp[i\beta(\mu \;-\; \lambda)x]\;\{cosh[\pi R\beta(\mu \;-\; \lambda)]\}^{-2}\\ =\; \frac{x}{2R}\left[sinh\left(\frac{x}{2R}\right)\right]^{-1}\ ,\end{aligned} \tag{59}$$

and yields the result

$$\frac{1}{\beta}\frac{\partial^2\Delta}{\partial\mu^2} = \frac{1}{2\pi} Im \int_0^\infty dx\ x e^x \frac{x}{2R} e^{-ix\beta\mu} \left[sinh(x)\ sinh\left(\frac{x}{2R}\right)\right]^{-1} . \quad (60)$$

Integrating in μ we find

$$\frac{\partial\Delta}{\partial\mu} = \frac{1}{2\pi} Re \int_\mu^\infty dt\ e^{-it} \frac{t}{2R\beta^2\mu^2} exp\left(\frac{t}{\beta\mu}\right)\left[sinh\left(\frac{t}{\beta\mu}\right) sinh\left(\frac{t}{2R\beta\mu}\right)\right]^{-1} , \quad (61)$$

where we fix the integration constant, in order to agree with the WKB approximation.

The result (61) is not symmetric under the duality transformations

$$2R \to \frac{1}{2R}\ ,\ \beta \to 2R\beta\ . \quad (62)$$

This is the opposite of what occurs in the bosonic case [10], whose solution can be obtained from (61) by simply dropping the factor $exp\left(\frac{t}{\beta\mu}\right)$ on the *r.h.s.* of (61)

$$\frac{\partial\Delta}{\partial\mu} = \frac{1}{2\pi} Re \int_\mu^\infty dt\ e^{-it} \frac{t}{2R\beta^2\mu^2} \left[sinh\left(\frac{t}{\beta\mu}\right)\ sinh\left(\frac{t}{2R\beta\mu}\right)\right]^{-1} . \quad (63)$$

Remarkably, eq. (63) has a duality symmetry under the transformations (62). Although this is not a symmetry of the relation (61) holding for the supersymmetric theory, the transformations (62) still play a special role. Firstly, note that the exact supersymmetric solution (49) is obtained from (60), in the limit $R \to \infty$. Secondly, let us carry out a duality transformation of (60) and follow the fate of the exponential factor

$$exp\left(\frac{t}{\beta\mu}\right) \to exp\left(\frac{t}{2R\beta\mu}\right) . \quad (64)$$

Then, taking the limit $R \to \infty$, this exponential factor goes to one and we recover the nonperturbative solution of the bosonic theory [4]

$$\frac{1}{\beta}\frac{\partial\rho}{\partial\mu} = \frac{1}{2\pi\beta\mu} Im \int_0^\infty dt\ exp\left(-it\right) \frac{t}{\beta\mu}\left[sinh\left(\frac{t}{\beta\mu}\right)\right]^{-1} . \quad (65)$$

Hence, we conclude that duality is not a symmetry, rather it maps the nonperturbative solution of the $R = \infty$ supersymmetric string into the corresponding solution of the bosonic theory. One way to interpret this result is by speculating that the exact solution we found in the case of the supersymmetric matrix model on the real line suggests itself as a candidate for a dual point of string theory. Summarizing, that is what we learned, having modified string theory to put it on a circle. Although the validity of (61) is based upon the assumption that the integral representation (49) that summarizes the asymptotic expansion of the density of states is correct, the asymptotic expansion of (61) holds unambiguously, just as the asymptotic expansion of ρ given in (15).

The $R \to \alpha'/R$ duality of the bosonic solution [10] is broken for $T > T_{KT}$, where T_{KT} denotes the Kosterlitz-Thouless transition temperature [14]. We can study the Kosterlitz-Thouless phase transition on random surfaces. In this respect, it is useful to consider the theory of quantum mechanics with a discrete time step ϵ [15]. This problem can be reduced to finding the hamiltonian $H(\epsilon)$ such that

$$< x|exp\Big(-\epsilon N H(\epsilon)\Big)|y> = K(x,y) \ , \tag{66}$$

where $K(x,y)$ is the transfer matrix. The latter coincides with the propagator for the inverted harmonic oscillator. Hence, the free energy reads

$$E(\epsilon,\Delta) = \frac{1}{2}\omega(\epsilon)E(\Delta) \ , \tag{67}$$

where $E(\Delta)$ coincides with the N-fermion ground state energy of matrix quantum mechanics found in eq. (52). For $\epsilon > 1$, both $\omega(\epsilon)$ and $H(\epsilon)$ have complex values. This can be interpreted as an indication that the $c = 1$ phase of string theory is unstable. As $\epsilon \to 0$, we have $\omega(\epsilon) \to 2$. Hence, one recovers matrix quantum mechanics from the infinite chain of matrices with nearest neighbour couplings.

In conclusion, we recall that we evaluated the spectral density of the target-space supersymmetric matrix model in a double scaling limit different from that in ref. [4]. In the case of the random matrix model on the real line, we found that expanding the resolvent yields the same result as the calculation carried out using sWKB. The origin of the logarithmic divergences that appear again, as in the universal result of ref. [4], may be linked to the presence of massless modes. The divergent series obtained in our solution is not summable. However, we showed that the series can be analytically continued. We discussed the case of multicritical points and gave the scaling behaviour of the sWKB result. We considered a $D = 1$ theory which is effectively equivalent to a $D = 2$ model with the inclusion of gravity fluctuations [16]. Our procedure can be followed in studying the $D = 0$ theory, in order to find the changes induced in the nonlinear equations obeyed by the susceptibility. The scaling behaviour for this model changes with respect to the ordinary bosonic case of ref. [4]. Hence, the density of states turns out to be different.

We want to finish with a few clarifying remarks about the comparison of the results obtained here with related results in the literature and their physical significance. Had we considered an arbitrary potential and expanded around the critical point, we would have obtained

$$(W')^2 = V_{max} - \lambda^2 , \quad W'' = -\lambda(V_{max} - \lambda^2)^{-\frac{1}{2}} . \tag{68}$$

Substituting these expressions into the sWKB expansion (19) yields the asymptotic expansion of the density of states for the bosonic random matrix model of ref. [4]

$$\begin{aligned} \rho(\mu) &= -\frac{1}{2\pi} log\ \mu + \frac{1}{\pi\beta\mu} Im \sum_{n=0}^{\infty} \frac{1}{1 - i\frac{2n+1}{\beta\mu}} \\ &= -\frac{1}{2\pi} log\ \mu - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{|B_{2k}|}{2k} \left(\frac{2}{\beta\mu}\right)^{2k} (2^{1-2k} - 1) , \end{aligned} \tag{69}$$

rather than the asymptotic series (15) of the supersymmetric case. In order to elucidate further what appears to be a deep relationship between the two nonperturbative solutions, one may ask what happens in the limit $V_{max} \to \infty$. In this sense it may prove useful to investigate the entire complex parameter space of the couplings, where one may be able to establish a relationship to the complex solutions in $D = 0$ by using stochastic quantization. Note that the model we solved differs from the Marinari-Parisi one-dimensional supersymmetric matrix model, because of the target-space supersymmetry-breaking procedure we adopted in taking the scaling limit. We recall that, assuming zero fermion content of the ground state, the energy of the Marinari-Parisi model is the sum of the first N levels in a one-particle potential $V = W'W' - W''$. Below the critical point, the minimum and local maximum of this potential lie respectively below and above zero. In this regime, the sum of the first N levels vanishes to all orders in the WKB expansion, consistent with a perturbatively unbroken supersymmetry invariance of the matrix model. Above the critical point, the local maximum disappears, supersymmetry is spontaneously broken and the scaling properties are identical to the $D = 0$ (pure gravity) case. On the other hand, we find a nonvanishing energy already in the leading planar order, but with the scaling properties of the $D = 1$ bosonic string, in apparent contradiction with both of the abovementioned behaviours. This discrepancy is due to the fact that the energy shift we used to bring the local maximum to zero does not respect supersymmetry (see the appendix). Thus, the model we solved is equivalent to a bosonic $D = 1$ matrix model, with a potential of the form: $W'W' - \frac{1}{N}W''$, which we studied by the sWKB approximation. A very interesting feature of our solution is that the answer differs from the a-priori universal analysis of ref. [4]. The reason for this difference is very simple: because N enters explicitly in the potential, $\frac{1}{N}$ loses its geometric interpretation as a loop-counting parameter, or put differently, the scaling limit is different from that in ref. [4]. We hope that a future investigation will clarify the geometric interpretation for taking the unusual scaling limit we introduced in this work. In this sense, it appears natural to consider the matrix model involved as the representation of the $D = 1$ supersymmetric string with broken target-space supersymmetry.

Finally, following González and Vozmediano [6], we may choose as an ansatz for the wavefunction of the ground state $\chi(\{\lambda_i\})\ \bar{\psi}_{11}\bar{\psi}_{22}\cdots\bar{\psi}_{NN}|0>$ or $\chi(\{\lambda_i\})\ det(\bar{\psi}_{ab})|0>$. As a consequence, we obtain a different hamiltonian. In this case, the ground state energy is given, in terms of the ground state energy of the one particle-hamiltonian e_{gs}, by $E'_{gs} = \beta e_{gs}$. This should be compared with the solution (52), which exhibits a different behavior $E_{gs} \sim \beta^2$. This may indicate that, as β goes to infinity, the theory undergoes a sort of phase transition from the essentially bosonic choice for the ground state wavefunction $\chi(\{\lambda_i\})|0>$, to the fermionic ansatz $\chi(\{\lambda_i\})\ det(\bar{\psi}_{ab})|0>$. An investigation of the features of the latter would probably require the use of collective coordinates for ψ. Evaluating e_{gs} will also prove to be problematic, with sWKB not quite adequate, and with issues of nonperturbative breaking of supersymmetry.

Appendix

The formalism for dealing with a supersymmetric quantum mechanical theory is described in ref. [17], which we refer to throughout this appendix. Here, we focus on the discussion of the shift of the zero of the energy away from the minimum of the potential, introduced in writing the hamiltonian (11) as well as the sWKB series (19). The original supersymmetric hamiltonian appears in the algebra, as a result of anticommuting the supersymmetric charge Q and its hermitian conjugate

$$\{Q^{\dagger}, Q\} = 2H . \tag{70}$$

Next, we show that introducing in the hamiltonian a constant shift K, according to

$$H \to \tilde{H} = H + K , \tag{71}$$

implies the redefinition of the supersymmetric charge

$$\tilde{Q} = e^{i\phi(x)} Q e^{-i\phi(x)} . \tag{72a}$$

Note that we are working with a formalism where Q is complex, rather than expressing our result in terms of the real and imaginary part of Q. The redefinition of the hermitian conjugate yields

$$\tilde{Q}^\dagger = e^{i\phi(x)} Q^\dagger e^{-i\phi(x)} . \tag{72b}$$

For the time being, ϕ represents any real function of x.

Indeed, given any real function $\phi(x)$ in (72), the redefined charge $\tilde{Q}$ satisfies the same supersymmetry algebra (70) as Q. However, the expression of the hamiltonian obtained from the anticommutator of $\tilde{Q}$ and its hermitian conjugate reads

$$\tilde{H} = e^{i\phi(x)} H e^{-i\phi(x)} . \tag{73}$$

We can expand this expression for small values of ϕ

$$\tilde{H} = H - [p^2, \phi] . \tag{74}$$

This can be represented as

$$\tilde{H} = H + i\hbar [\phi''(x) + 2\phi'(x) \frac{d}{dx}] , \tag{75}$$

where we reintroduced explicitly the $\hbar$ parameter. Hence, the requirement that supersymmetry is preserved by the redefinition of the charges would force us to relate the second derivative of the function $\phi(x)$ to the constant shift K defined in (71)

$$\phi''(x) = - \frac{i}{\hbar} K . \tag{76}$$

The requirement $\phi'' = \mathit{constant}$ corresponds to a potential such that $f' = i\sqrt{2}\hbar$, which is what we introduced in the text, after eq. (19). Note, however, the first derivative term in eq. (75). This forces us to restrict the function $\phi(x)$ to be a constant and K to vanish. We conclude that the definition of the zero of the energy cannot be changed without affecting the validity of the supersymmetry algebra.

Another important issue we must consider is the possible modification of the energy spectrum, as a consequence of shifting the hamiltonian according to (71). The eigenvalue equation

$$H\psi = E\psi \,, \tag{77}$$

becomes, after the redefinition,

$$\tilde{H}\tilde{\psi} = \tilde{E}\tilde{\psi} \,. \tag{78}$$

The redefined wave function of the eigenstate is related to the original one by the transformation

$$\tilde{\psi} = e^{i\phi}\psi \,. \tag{79}$$

This result shows that nothing has been done that would modify the spectrum.

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