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PREDICTING THE STATISTICAL ACCURACY OF AN EXPERIMENT

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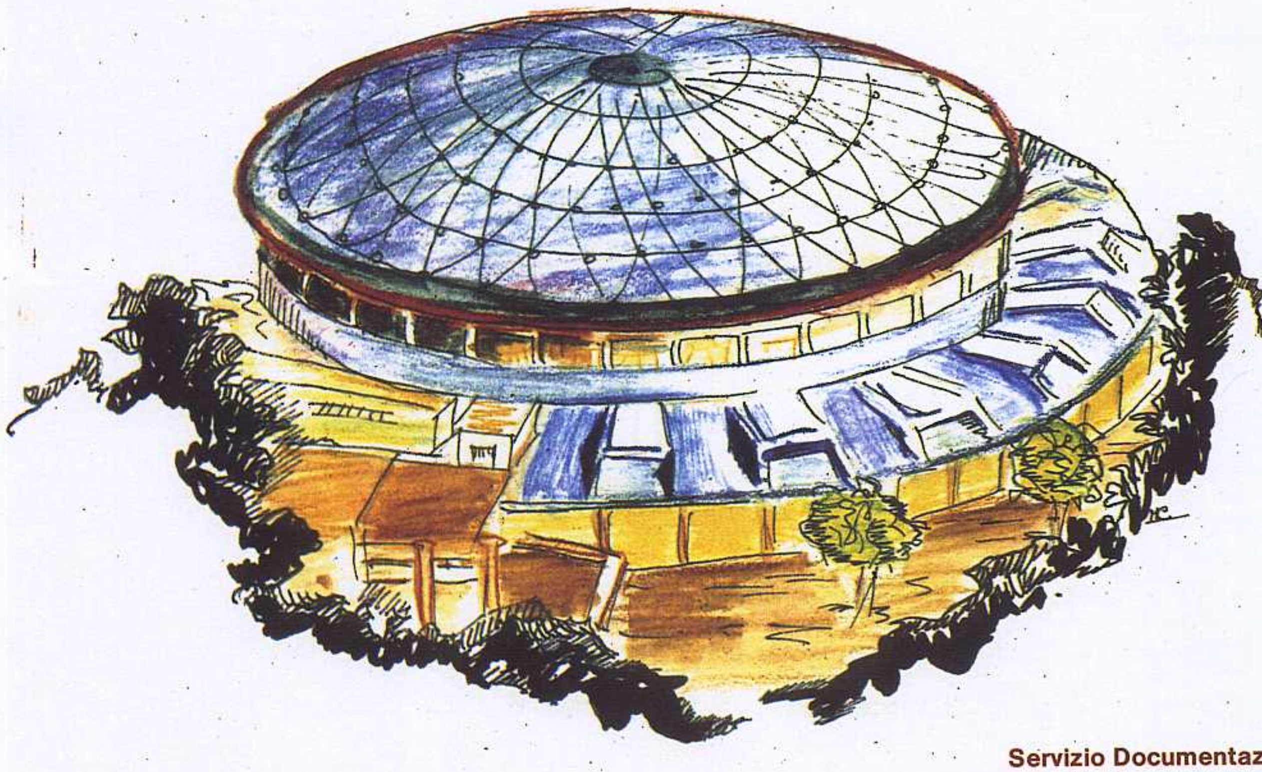

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PREDICTING THE STATISTICAL ACCURACY OF AN EXPERIMENT

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ABSTRACT

We discuss a general method for estimating the number of events required to obtain a given accuracy in the measurement of a parameter of a theory. Simple examples are given. We also discuss resolution effects.

1. Introduction

Consider a variable x, a physical measurable parameter, for which a theory predicts the distribution function $f(x;p)$, where p is a (set of) parameter(s) to be determined experimentally. $f(x;p)$ is called the probability density function, meaning that $\mathrm{d}P$, the probability of observing an event at x in the interval $\mathrm{d}x$, is $f(x;p)\,\mathrm{d}x$. The function f is normalized to unity over the whole $\{x\}$ interval where x is physical.

The joint probability density or likelihood of an observation (experiment) resulting in a set of N values x_i for the variable x is:

$$\mathcal{L} = \prod_{1}^{N} f(x_i;p). \tag{1.1}$$

The best estimate for p is the value $\overline{p}$ which maximizes the likelihood $\mathcal{L}$ of the observation or, equivalently of W=log$\mathcal{L}$. From the theorem above[1] and the theorem that for large N the likelihood function approaches a gaussian we obtain that the actual result of an experiment with N events will fluctuate around the true value with a variance $\langle(p-\overline{p})^2\rangle = \sigma_p^2$ given by:

$$\sigma = \sqrt{-\frac{1}{\left.\frac{\partial^2 W}{\partial p^2}\right]_{p=\overline{p}}}} \tag{1.2}$$

From eq. (1.1) the likelihood for an experiment in which one event is observed is $\mathcal{L}=f(x,p)$ and $W = \log\mathcal{L} = \log f$. If we want the accuracy with which we can determine the parameter p, we use the result in equation (1.2),[2] relating $\frac{\partial^2 W}{\partial p^2}$ to σ_p as:

$$\frac{\partial^2 W}{\partial p^2} = \frac{\partial^2 \log f(x;p)}{\partial p^2}. \tag{1.3}$$

The value of $\frac{\partial^2 W}{\partial p^2}$ averaged over repeated experiments of one event each is:

$$\left\langle\frac{\partial^2 W}{\partial p^2}\right\rangle = \int \frac{\partial^2 \log f(x;p)}{\partial p^2} f(x;p)\,\mathrm{d}x \tag{1.4}$$

and for N events

$$\left\langle\frac{\partial^2 W}{\partial p^2}\right\rangle = N\int \frac{\partial^2 \log f(x;p)}{\partial p^2} f(x;p)\,\mathrm{d}x. \tag{1.5}$$

Computing the derivative in the integral above gives:

$$\int \frac{\partial^2 \log f}{\partial p^2}\,\mathrm{d}x = -\int \frac{1}{f}\left(\frac{\partial f}{\partial p}\right)^2 \mathrm{d}x + \int \frac{\partial^2 f}{\partial p^2}\,\mathrm{d}x, \tag{1.6}$$

where the last term vanishes, exchanging integration and differentiation, since $\int f\,\mathrm{d}x=1$. Thus finally we obtain:

$$\sigma_p = \frac{1}{\sqrt{N}}\left(\int \frac{1}{f}\left(\frac{\partial f(x;p)}{\partial p}\right)^2 \mathrm{d}x\right)^{-\frac{1}{2}} \tag{1.7}$$

For the case of many parameters, the error matrix, which in general is non diagonal if the parameters are correlated, is given by:

$$\overline{(p_i-\bar{p}_i)(p_j-\bar{p}_j)} = \frac{1}{N}\left[\int \frac{1}{f}\left(\frac{\partial f(x;p)}{\partial p_i}\frac{\partial f(x;p)}{\partial p_j}\right)\mathrm{d}x\right]^{-1} \tag{1.8}.$$

Notice that the expression in brackets which has to be inverted is a square, symmetric matrix. While the integrals in (1.7) and (1.8) might appear ugly, they are trivially computed numerically, even if no closed form for f is given by theory. The errors, as expected, decrease as $1/\sqrt{N}$ and the accuracy of the error estimate itself does not depend on N but on the approximations in computing the integrals.

2. Examples

2.1 Slope

The simplest application is estimating the number of events necessary do determine a *slope* parameter g defined as in $f(x;g) = (1+xg)/2$, with x in $\{-1,\ 1\}$. The integral in equation (1.7) is:

$$\int_{-1}^{1} \frac{x^2}{2(1+gx)}\,\mathrm{d}x = \frac{1}{2g^3}\Big(\log\Big(\frac{1+g}{1-g}\Big) - 2g\Big) \tag{2.1}$$

giving $\sigma_g = \sqrt{3/N}$, for g=0. For convenience I recall that $\log\big((1+x)/(1-x)\big) = 2(x - x^3/3\cdots)$. Measurements of a slope of $\sim5\times10^{-5}$ to 3σ requires therefore $\sqrt{3/N} \sim 5\times10^{-5}/3$ or $N \sim 10^{10}$.

2.2 Dalitz Plot Slope

The measurement of the slope of the Dalitz plot density for the odd pion from the 3π decays of K mesons is of particular interest here. The accuracy for this case is slightly worse than for the previous one because events at the edge of phase space which carry the most information about the slope are less frequent. The loss is however very small, $\sim$15%. The odd pion energy distribution is $(1 + g \times E_\pi) \times \phi(E_\pi)$, where g is the slope of interest and $\phi(E_\pi)$ is the phase space factor. We approximate the shape of the Dalitz plot with a circle centered at the origin. The probability density is then $f(x;g) = (2/\pi)(1+gx)\sqrt{1-x^2}$, with x in $\{-1,\ 1\}$. The error on g is given by:

$$(\sigma_g)^2 = \frac{1}{N}\Big(\frac{2}{\pi}\int_{-1}^{1} \frac{x^2\sqrt{1-x^2}}{1+gx}\,\mathrm{d}x\Big)^{-1} = \frac{\kappa^2}{N} \tag{2.2}$$

where for $g = 0.26$, the value for $K^{\pm}_{3\pi}$ decays, $\kappa = 1.98$ resulting in a fractional accuracy, $\sigma_g/g = 7.56/\sqrt{N}$.

Observation of 10^{10} K^+ and K^- 3π decay events allows one to measure σ_g/g to an accuracy of 7.6×10^{-5}. To observe a 3σ difference in the K^+ and K^- slopes, they must differ by $3\times\sqrt{2}\times0.26\times7.6\times10^{-5}$=$8.4\times10^{-5}$ or $\Delta g/g = 3.2\times10^{-4}$. This is probably the ultimate sensitivity which could be achieved at DAΦNE. Kaon beams at other facilities could provide smaller statistical errors, but will probably suffer from worse systematic uncertainties.

2.3 The $\pi\pi$ scattering length

The same method is easily applied to the question of what can DAΦNE do with respect to measuring the $\pi\pi$ scattering lengths, what kind of detector is required and whether tagging is necessary. Given the appropriate dependence of the measurable physical quantities, in various conditions, on the physical parameters of interest. one can estimate the number of $e^+e^- \to \pi\pi e^+e^-$ events which must be detected to obtain the required accuracy, following the procedure outlined above.

3. Including the effects of experimental resolution

We assume the resolution function is known, otherwise the case is, of course, hopeless. A common procedure is to try to unfold the resolution effects from the data. This procedure is ambiguous and can lead to incorrect results. The correct procedure is however very simple. Convolute the resolution r with the probability density f:

$$f'(x;p) = \int f(x - x';p)r(x')\,\mathrm{d}x', \tag{3.1}$$

make sure that f' is correctly normalized and proceed as above.[3] The loss in accuracy due to resolution can be obtained in this way, helping to define how to design an experiment.

REFERENCES

1. H. Cramer, *Mathematical Methods of Statistics*, Princeton University Press, 1946. For a more prosaic and simplified presentation see Probability, Statistics and Monte Carlo, in Review of Particle Properties, Particle Data Group, Phys. Lett. B **239**, p. III.31
2. Technically we are using here direct probabilities. See for instance J. Orear, UCRL-8417 (1958); CLNS/511 (1982). Both originate from E. Fermi lectures in Chicago, 1957
3. See for instance V. Patera, *Proceedings of the Workshop on Physics and Detectors for Daphne*, G. Pancheri Ed., Frascati, 1991, p. 499.