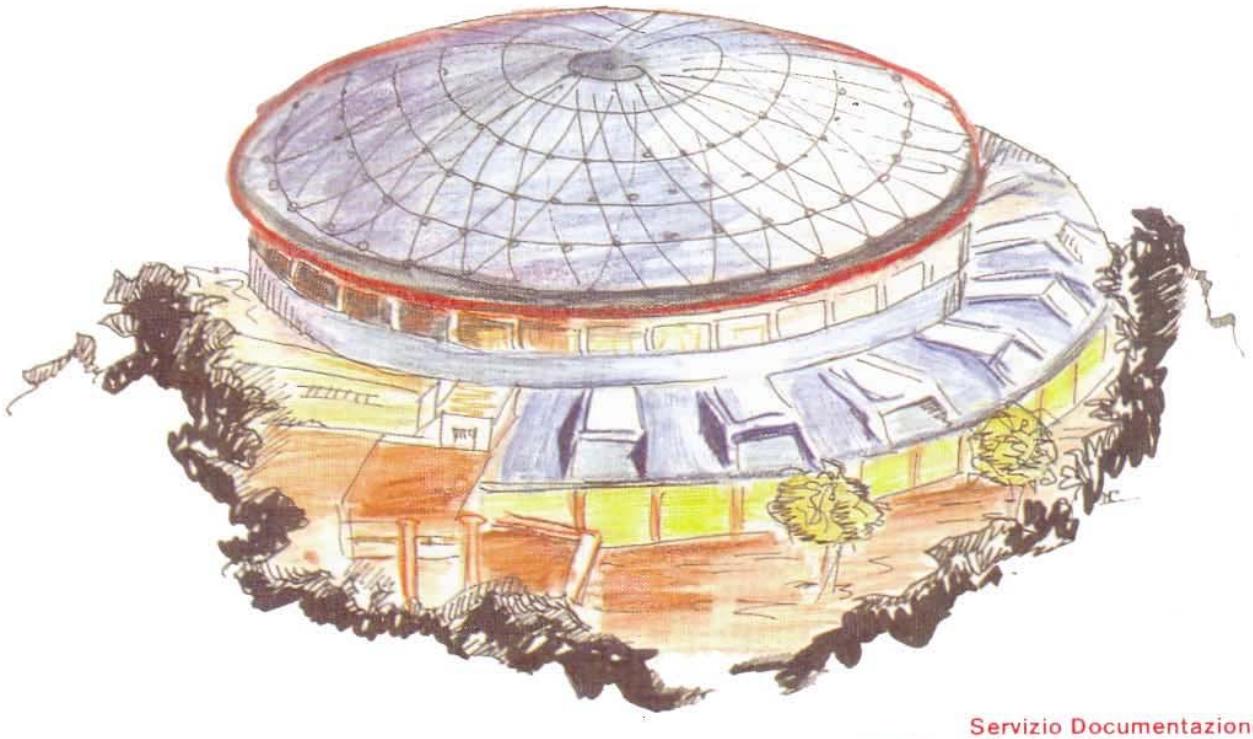
ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI FISICA NUCLEARE - ISTITUTO NAZIONALE DI

Laboratori Nazionali di Frascati

LNF-91/031 (R)
7 Giugno 1991

S. Guiducci:

CHROMATICITY



Servizio Documentazione
dei Laboratori Nazionali di Frascati
P.O. Box, 13 - 00044 Frascati (Italy)

INFN – Laboratori Nazionali di Frascati
Servizio Documentazione

LNF–91/031(R)
7 Giugno 1991

CHROMATICITY

S. Guiducci
Frascati National Laboratories - INFN, Frascati, Italy

ABSTRACT

A simple calculation of the contribution of quadrupoles and sextupoles to storage ring chromaticity is given. Problems arising from chromaticity correction are discussed. An accurate derivation of chromaticity formulae for a general bending magnet, exact also for small machines with small radius of curvature, is given.

1. INTRODUCTION

In the design of storage rings there are many similarities with geometrical light optics. In analogy to chromatic aberrations in light optics, in particle accelerators a parameter called chromaticity is introduced. In light optics rays of different wavelength find a different refraction index in a lens and therefore experience a different focal length. Similarly in a storage ring particles of different momentum see a different focusing strength in the quadrupoles and, as a consequence, have a different betatron oscillation frequency.

We define the chromaticity as the variation of the betatron tune Q with the relative momentum deviation δ ($\delta = \Delta p/p$):

$$Q' = \frac{dQ}{d\delta} . \tag{1}$$

Sometimes the relative chromaticity ξ is used:

$$\xi = \frac{Q'}{Q} . \tag{2}$$

Let us point out the importance of the chromaticity in circular accelerators. The chromaticity has a deleterious influence on the beam dynamics for two main reasons.

First, a momentum spread σ_p is always present in a particle beam, therefore the chromaticity produces a tune spread in the beam:

$$\Delta Q = Q' \sigma_p . \tag{3}$$

In large rings, with high tune values, this tune spread is so large that it is impossible to accommodate the beam in the space between the resonance lines in the tune diagram. How dangerous these resonances can be for beam stability has been described in a previous lesson [1].

Second, in the case of bunched beams the chromaticity produces a transverse instability called "head-tail effect" (see Ref. [2] for a detailed treatment). The wake field produced by the leading part of a bunch (the head) excites an oscillation of the trailing part (the tail) of the same bunch. In half a synchrotron period the head and the tail of the bunch interchange their positions and the oscillation can be anti-damped and may cause a beam loss. A complete mathematical treatment shows that the growth rate of this instability is much faster for negative than for positive chromaticity values and vanishes for zero chromaticity. It may be counteracted by a transverse feedback system, but this makes machine operation much more critical. Therefore most of the storage rings operate with zero or slightly positive chromaticity.

The "natural" chromaticity of a storage ring is that due only to the elements of the linear lattice, i.e. quadrupoles and dipoles. As it will be shown later the "natural" chromaticity of a strong focusing storage ring is always negative, therefore special elements have to be introduced in the lattice to correct it.

In strong focusing lattices the main contribution to the chromaticity is due to the quadrupoles, in particular, in large rings with very large radius the contribution of the dipoles can be neglected; for small rings, however, the dipole contribution is important and has to be carefully calculated.

In sections 2 and 3 it is shown how to calculate the chromaticity due to the quadrupoles and sextupoles respectively. Then, in section 4, the effects on beam dynamics due to the chromaticity correcting sextupoles are briefly discussed. Finally, in section 5, a detailed derivation of the chromaticity for a general bending magnet is given, following the approach given by M. Bassetti in Ref. [3], which is very simple and intuitive, avoiding long mathematical derivations.

2. CHROMATICITY CALCULATION FOR A QUADRUPOLE

Let us consider the motion in a quadrupole of a particle which obeys the equation:

$$y'' + k_y y = 0 \qquad (y = x \text{ or } z) \tag{4}$$

with

$$k_x = -k$$
$$k_z = k .$$

Now we consider the dependence of k on momentum:

$$k = \frac{e}{p}\frac{\partial B_z}{\partial x} = \frac{e}{p_0(1+\delta)}\frac{\partial B_z}{\partial x} = \frac{k_0}{(1+\delta)} \simeq k_0 (1- \delta + \delta^2 -...) \tag{5}$$

to first order in δ:

$$k \simeq k_0 (1-\delta) = k_0 - k_0\delta . \tag{6}$$

The chromatic variation has always the opposite sign with respect to the focusing strength, therefore a particle with a larger energy sees a weaker focusing strength. On the contrary, in light optics, the variation of the refraction index with the wavelength can be either positive or negative and the chromatic effect can be corrected to first order combining lenses of different material.

Keeping only the linear term in δ the equation of motion in a quadrupole yields:

$$y'' = -k_y (1-\delta)\, y \tag{7}$$

which is equivalent to adding to the focusing quadrupole a defocusing one with a strength $-k_y\delta$ and viceversa for the defocusing quadrupole.

In a thin section of a quadrupole of infinitesimal length ds the particle receives an angular kick

$$dy' = y'' ds = k_y\, \delta\, y\, ds \tag{8}$$

described by a thin lens (defocusing for the focusing quadrupole) matrix:

$$\begin{pmatrix} 1 & 0 \\ k_y\delta ds & 1 \end{pmatrix} \tag{9}$$

To compute the effect of this kick on the betatron tune, the one-turn matrix is obtained by multiplying the unperturbed matrix (δ = 0) by this thin lens.

$$M = \begin{pmatrix} \cos\mu_y & \beta_y \sin\mu_y \\ -\sin\mu_y/\beta_y & \cos\mu_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k_y\delta ds & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_y + \beta_y \sin\mu_y k_y \delta ds & \beta_y \sin\mu_y \\ -\sin\mu_y/\beta_y + \cos\mu_y k_y \delta ds & \cos\mu_y \end{pmatrix} \tag{10}$$

Then we compute the trace of M to get the new value of μ_y ($\mu_y = 2\pi Q_y$):

$$\frac{1}{2}\ \mathrm{Tr}\ M = \cos(\mu_y + d\mu_y) = \cos\mu_y + \frac{1}{2}\,\beta_y \sin\mu_y\, k_y \delta\, ds \tag{11}$$

since:

$$d(\cos\mu_y) = \cos(\mu_y + d\mu_y) - \cos\mu_y = -\sin\mu_y\, d\mu_y \tag{12}$$

we get:

$$d\mu_y = -\frac{1}{2}\,\beta_y\, k_y\, \delta\, ds \tag{13}$$

or

$$dQ_y = -\frac{1}{4\pi}\,\beta_y\, k_y\, \delta\, ds\ . \tag{14}$$

Integrating over all the ring circumference L, the formulae for the two planes, horizontal and vertical, are :

$$\frac{\partial Q_x}{\partial \delta} = -\frac{1}{4\pi} \int_0^L \beta_x(s)\, k_x(s)\, ds \tag{15}$$

$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi} \int_0^L \beta_z(s)\, k_z(s)\, ds\ . \tag{16}$$

From these formulae we can see why in a strong focusing lattice the chromaticity is always negative. In each plane a focusing quadrupole has a positive strength ($k_{x,z}>0$), and therefore a negative chromaticity, while a defocusing quadrupole has negative strength ($k_{x,z} < 0$) and positive chromaticity. In a strong focusing lattice the $\beta_{x,z}$-functions have the maximum values at the focusing quadrupoles, therefore the total chromaticity of a ring is dominated by the negative contribution of the focusing quadrupoles.

3. CHROMATICITY CALCULATION FOR A SEXTUPOLE

For off-momentum particles the closed orbit is displaced with respect to that of the reference particle by a quantity $D\delta$, where D is the dispersion function. This fact allows to correct the chromaticity through the insertion of special magnets, called sextupoles.

The field of a sextupole is given by :

$$B_x = g'xz$$
$$B_z = \frac{1}{2} g' (x^2 - z^2) \qquad (17)$$

with

$$g' = \frac{\partial B_z^2}{\partial x^2}$$

and the equations of motion become:

$$x'' + \frac{1}{2} r (x^2 - z^2) = 0 \qquad \text{where } r = \frac{e}{p_0} g' \qquad (18)$$
$$z'' - rxz = 0 .$$

In Fig. 1 an example of the pole shape of a sextupole magnet is given.

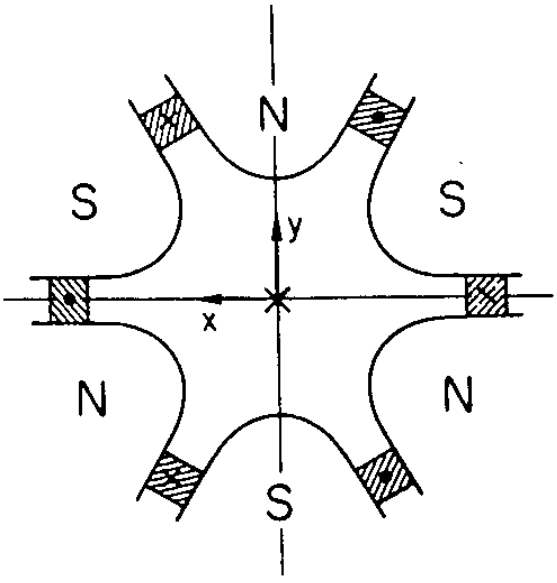


Fig. 1 - Schematic representation of a sextupole magnet cross section.

The sextupole kick is :

$$dx' = -\frac{1}{2}\ r\,(x^2-z^2)\,ds$$
$$dz' = r\,x\,z\,ds\ . \tag{19}$$

Substituting the total coordinates for the off-momentum particle

$$x_t = D\delta + x \tag{20}$$
$$z_t = z$$

it becomes:

$$dx' = -\left[D\delta\,x + \frac{1}{2}(D\delta)^2 + \frac{1}{2}(\,x^2 - z^2\,)\right] r\,ds$$
$$dz' = \left[\,D\delta\,z + xz\,\right]\ r\,ds\ . \tag{21}$$

The first term of Eqs. (21) is equivalent to the kick of a quadrupole with gradient $-rD\delta$ and, analogously to Eq. (14), gives a tune shift

$$\Delta Q = \frac{1}{4\pi}\ \beta\ rD\ \delta\ ds \tag{22}$$

and a contribution to the chromaticity:

$$\frac{\partial Q_x}{\partial \delta} = \frac{1}{4\pi}\int_0^L \beta_x(s)\ r(s)D(s)\ ds$$
$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi}\int_0^L \beta_z(s)\ r(s)D(s)\ ds\ . \tag{23}$$

4. CHROMATICITY CORRECTION

The most efficient way to correct chromaticity is to perform a localized correction, i.e. to insert sextupoles just close to each quadrupole, where the chromatic effect is produced. For each quadrupole is inserted a sextupole with a strength:

$$r\,l_S = -\frac{k\,l_Q}{D} \tag{24}$$

where l_S and l_Q are the lengths of the quadrupole and sextupole respectively.

Unfortunately, in most cases localized correction is not possible. For example collider storage rings have low-β insertions with very strong quadrupoles and zero dispersion function. Similarly, storage rings for synchrotron light production have many zero dispersion straight sections for insertion devices, like wigglers and undulators, and strong focusing quadrupoles to get low emittances. In these cases a strong chromaticity produced in the insertions has to be corrected in the arcs. If the arcs are built up by regular cells, two sextupoles are inserted in each cell, one in a high β_x place, to correct horizontal chromaticity, and the other in a high β_z position to correct the vertical one. The sextupole intensities are obtained by solving the following linear system of equations, where r_H and r_V are respectively the strengths of the horizontal and vertical chromaticity correcting sextupoles and N is the number of periodic cells:

$$\begin{aligned} r_H l_S \beta_x^H D^H + r_V l_S \beta_x^V D^V &= \frac{Q_x'}{N} \\ -r_H l_S \beta_z^H D^H - r_V l_S \beta_z^V D^V &= \frac{Q_z'}{N}. \end{aligned} \tag{25}$$

It has to be that

$$r_H < 0 \quad \text{and} \quad r_V > 0$$

since Q_x' and Q_z' are, usually, both negative. Therefore, in order to reduce the sextupole strengths, it is important to place them where the dispersion is high and the β functions are well separated. In fact, if $\beta_x >> \beta_z$ at the horizontal sextupole and viceversa at the vertical sextupole location, each sextupole corrects the chromaticity in one plane, without affecting it in the other plane. Often the vertical sextupole has a strength higher than the horizontal one because the dispersion function follows the behaviour of the horizontal β-function and therefore D is high at the horizontal sextupole and low at the vertical one. This is specially true for collider storage rings, where, due to the low value of β_z at the interaction region, the vertical chromaticity is generally higher than the horizontal one.

The sextupoles necessary to correct the chromaticity introduce unwanted effects due to the other two terms in Eqs. (21):

- the chromatic aberration term $(D\delta)^2$
- the geometric aberration term $(x^2 - z^2)$, xz .

The geometric aberration term introduces higher order terms in the equations of motion. In fact each sextupole inserted into the linear lattice, also in thin lens approximation, doubles the order of the polynomial which links the initial and final coordinates for one turn in the ring. With N sextupoles in the ring the final coordinates depend on the 2^N-th power of the initial one.

$$x(L) = a_{11}\, x(0) + a_{12}\, x'(0) + a_{13}\, \delta + \ldots a_{1j}\, x(0)^{2^N} .$$

The worst consequence of this is that in the case of nonlinear motion the stability of the trajectories is not given anymore from the knowledge of the one-turn matrix ($|\mathrm{tr}\, M| = |2\cos\mu| < 2$), but depends on the amplitude of the betatron and synchrotron oscillations.

In some very simplified cases an analytical calculation of the stability region is possible, for example in the unidimensional case (x,x' or z,z' phase plane) in the vicinity of a single resonance. In this case a closed curve, called separatrix, can be found which divides the phase plane in two regions, a stable one inside the separatrix and an unstable region outside.

In more general cases tracking is used, i.e. a computer code which, given the initial coordinates for a particle in phase space, follows the evolution of a trajectory with the mathematical model chosen for the ring. A trajectory is considered stable if it remains confined in a certain phase space region for a given number of turns. Changing the initial coordinates of the particles tracking is performed to determine the largest region of phase space which contains all stable trajectories. This region is called dynamic aperture.

This procedure is limited by computer time and precision, in fact the range of initial co-ordinates which can be explored in a six-dimensional phase space is very poor and the number of turns is always much smaller with respect to the beam lifetime or damping time.

After the linear lattice design a dynamic aperture optimization has to be done by choosing the distribution and the strengths of the sextupoles , the working point in the tune diagram, and even modifying the linear lattice to reduce chromaticities and sextupole strengths.

5. CHROMATICITY CALCULATION FOR A GENERAL BENDING MAGNET

In a bending magnet the betatron motion is given by the following equations:

$$y'' + k_y(s)y = 0 \qquad (y = x \text{ or } z) \tag{27}$$

with

$$k_x = -k+h^2$$

$$k_z = k$$

$$h = -\frac{e}{p} B_z \quad ; \quad k = \frac{e}{p}\frac{\partial B_z}{\partial x} .$$

The solution of these equations is represented, in each plane, by the two-by-two betatron matrix A. This matrix can be written as the product of N matrices A_i:

$$A = \prod_{i=1}^{N} A_i\left(\frac{l_B}{N}\right) \tag{28}$$

where l_B is the length of the bending magnet. We choose N large so that

$$\Delta s = \frac{l_B}{N} \longrightarrow 0 .$$

This is equivalent to subdividing the magnet into N thin pieces of length Δs. To first order in Δs, A_i can be written as the product of a thin lens and a drift space:

$$A_i = \begin{pmatrix} 1 & 0 \\ -k_y(s)\Delta s & 1 \end{pmatrix} \begin{pmatrix} 1 & \Delta s \\ 0 & 1 \end{pmatrix} \tag{29}$$

Now we consider the changes that occur in the betatron motion (i.e. in the matrix A_i) for a particle with a relative momentum deviation δ oscillating around the off momentum closed orbit.

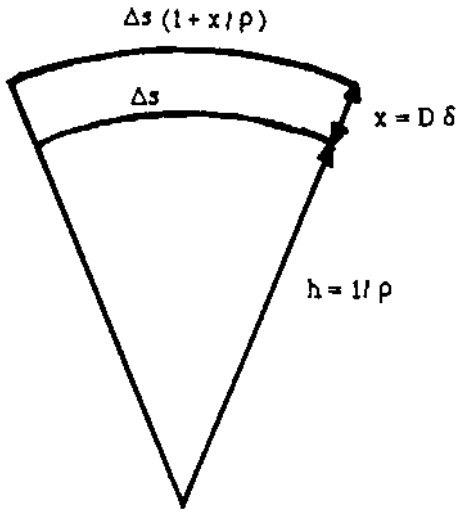


Fig. 2 - Orbit lengthening.

Two changes occur in the matrix A_i:

i) an orbit lengthening (see Fig.2)

$$\Delta s(\delta) = \Delta s(1 + hD\delta) \tag{30}$$

ii) a change in the focusing strength of the thin lens due to:

- momentum dependence of the focusing functions

- variation of the length

$$- k_y(s,\delta)\, \Delta s(\delta) = - [\, k_y(s) + k_{1y}(s)\delta + k_y(s)hD\delta]\, \Delta s \tag{31}$$

where k_{1y} is the derivative : $k_{1y} = \partial k_y/\partial\delta$.

As already seen, a change Δk in the focusing function at the position s gives a tune shift :

$$\Delta Q = - (1/4\pi)\, \beta(s)\, \Delta k \tag{32}$$

and, similarly, a change Δs in the length of a drift space gives:

$$\Delta Q = (1/4\pi)\, \gamma (s) \Delta s \tag{33}$$

where γ (s) is the Twiss function.

Integrating over all the circumference gives

$$\frac{\partial Q_y}{\partial \delta} = \frac{1}{4\pi} \int_0^L \{\, \beta_y [\, k_{1y} + k_y hD] + \gamma_y hD \,\}\, ds \,. \tag{34}$$

This formula is a generalization of that for a quadrupole, in fact for a quadrupole we have h = 0 and $k_{1y} = dk_y/d\delta = -k_y$ and we obtain again the formulae of Eqs. (15) and (16).

In order to calculate k_{1y} for the general bending magnet we need to know the fields seen by an off-momentum particle . First we write the second-order magnetic field expansion in the reference system of the design orbit for zero momentum deviation. The formulation of the field equations is that given by K. Steffen [4] with the only difference that h(s) has the opposite sign and its dependence on s is explicitly given, i.e:

$$h(s) = [h + h's + \frac{1}{2}\, h''s^2 + 0(3)] \,. \tag{35}$$

As it will be useful in the following to distinguish the second-order terms they have been enclosed in square brackets:

$$B_z = \frac{p}{e}\left\{-h - h's + kx + \left[-\frac{1}{2} h''s^2 + k'xs + \frac{1}{2} rx^2 + \frac{1}{2}(h'' - hk - r)\, z^2\right] + 0(3)\right\}$$

$$B_x = \frac{p}{e}\left\{kz + [\, k'zs + rxz \,] + 0(3)\right\} \tag{36}$$

$$B_s = \frac{p}{e}\left\{-h'z + [\, (hh' + k')xz \,] + 0(3)\right\} .$$

The previous equations are completely general, they are only based on the assumption of a field symmetry with respect to the median plane (z = 0). Therefore, if we change the momentum of the particle, the origin and the orientation of the axis in the z = 0 plane, the magnetic field has always the same form, but different values of the coefficients.

Now we make a transformation to the reference system of the off-momentum particle, as shown in Fig. 3:

$$\begin{aligned} p &= p^*/(1+\delta) \\ z &= z^* \\ x &= d + x^*\cos\theta + s^*\sin\theta \\ s &= - x^*\sin\theta + s^*\cos\theta \end{aligned} \tag{37}$$

where $d = D\delta$ and $\theta = D'\delta$.

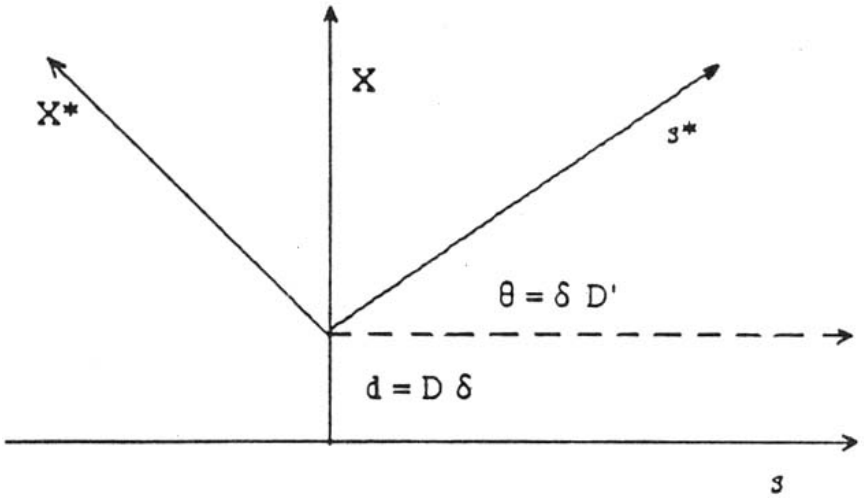


Fig. 3 - Transformation of the reference system.

The field equations change in the following way:

$$\begin{aligned} B_z^* &= B_z[\vec{x}(\vec{x}^*)\,] \\ B_x^* &= B_x[\vec{x}(\vec{x}^*)\,]\cos\theta - B_s[\vec{x}(\vec{x}^*)\,]\sin\theta \\ B_s^* &= B_x[\vec{x}(\vec{x}^*)\,]\sin\theta + B_s[\vec{x}(\vec{x}^*)\,]\cos\theta\ . \end{aligned} \tag{38}$$

We are interested in the first-order field expansion, therefore we take only the first-order terms in Eqs. (36) and make the substitution :

$$\begin{aligned} B_z^* &= \frac{p}{e(1+\delta)}\left[-h + kd + x^*(k\cos\theta + h'\sin\theta) - s^*(h'\cos\theta - k\sin\theta)\right] \\ B_x^* &= \frac{p}{e(1+\delta)}z[k\cos\theta + h'\sin\theta] \\ B_s^* &= \frac{p}{e(1+\delta)}z[k\sin\theta - h'\cos\theta]\ . \end{aligned} \tag{39}$$

As already said, the various terms in the field equations have to be the same as in Eqs. (36), therefore equating the corresponding first-order terms we get the new coefficients:

$$h^* = \frac{h-kd}{1+\delta}$$

$$k^* = \frac{k\cos\theta + h'\sin\theta}{1+\delta} \qquad (40)$$

$$h'^* = \frac{h'\cos\theta - k\sin\theta}{1+\delta} \quad .$$

Using

$$\sin\theta \sim D'\delta \; ; \qquad \cos\theta \sim 1$$

and keeping only first-order terms in δ we get:

$$\begin{aligned} k_x^* &= h^{*2} - k^* = (h^2 - k) + \delta(-2h^2 - 2hkD + k - h'D') \\ k_z^* &= k^* = k + \delta(-k + h'D') \; . \end{aligned} \qquad (41)$$

We obtain the values of $k_{1y}(s)$ as:

$$\begin{aligned} k_{1x} &= \frac{\partial k_x^*}{\partial\delta} = k - 2h^2 - 2hkD - h'D' \\ k_{1z} &= \frac{\partial k_z^*}{\partial\delta} = -k + h'D' \; . \end{aligned} \qquad (42)$$

Inserting these values into Eq. (34) we obtain the final formulae:

$$\begin{aligned} \frac{\partial Q_x}{\partial\delta} &= \frac{1}{4\pi}\int_0^L \{\beta(k - 2h^2 - 2hkD - h'D') + \beta hD(h^2 - k) + \gamma hD\}ds \\ \frac{\partial Q_z}{\partial\delta} &= \frac{1}{4\pi}\int_0^L \{\beta(-k + hkD + h'D') + \gamma hD\}ds \; . \end{aligned} \qquad (43)$$

As we used only first order terms in this derivation the contribution of the sextupole term βrD, calculated in section 3, does not appear in Eqs. (43). In Appendix I a similar derivation using the second order field expansion is given. The final formulae contain the same terms as Eqs. (43) plus the sextupolar terms coming from the second order terms in the field expansion which are linear in x and, applying the translation $x = x^* + D\delta$ of Eqs. (37), produce linear terms.

5. END-FIELD EFFECTS

From Eqs.(43) it is possible to calculate the contribution of the fringing fields to the chromaticity, once known an expression for h'(s). In Ref. [5] the same formulae are obtained with a different derivation, moreover a detailed calculation of the fringing field effects is given. For completeness we report here the final formula. The schematization of the end fields used there is shown in Fig. 4 and the parameters definition is the following:

s_1	beginning of the central part
s_2	end of the central part
"1"	entrance of the fringing region
"2"	exit of the fringing region
θ	entrance or exit angle of the trajectory
$\frac{1}{\tau \cos^3\theta}$	radius of curvature of the end faces
$h = \frac{1}{\rho}$	curvature of the reference orbit
$k = -\frac{1}{B\rho}\frac{\partial B_z(0,0,s)}{\partial x}$	quadrupole component
$r = -\frac{1}{B\rho}\frac{\partial^2 B_z(0,0,s)}{\partial x^2}$	sextupole component
D, D'	dispersion function and its derivative
α, β, γ	Twiss functions.

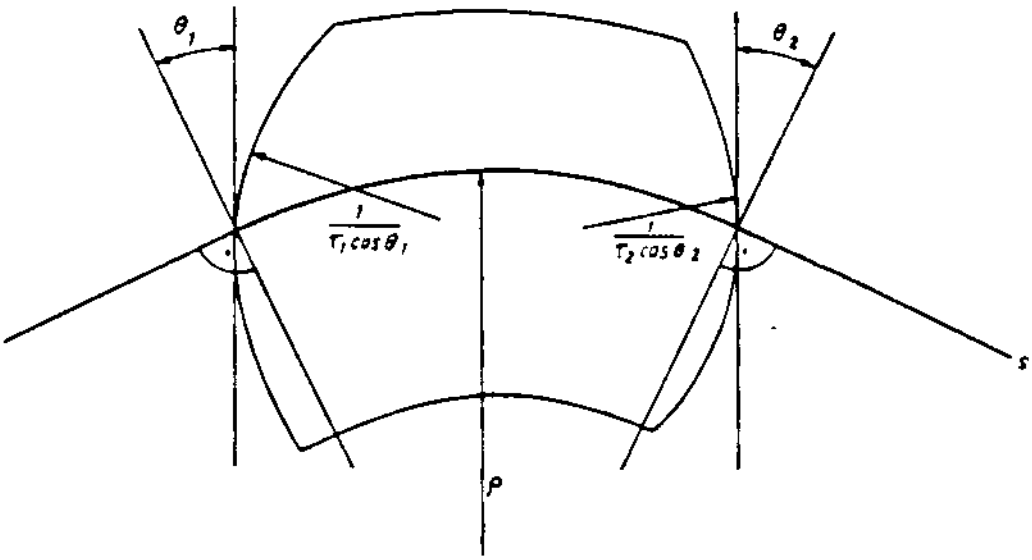


Fig.4 - Field boundaries for a bending magnet.

The formulae to calculate the chromaticity of a magnet in terms of the lattice functions at the reference orbit are:

$$\frac{\partial Q_x}{\partial \delta} = -\frac{1}{4\pi}\int_{s_1}^{s_2}[\,(h^2-k)\beta + rD\beta + h\,(2kD\beta + 2D'\alpha - D\gamma)]ds$$

$$+\left[-tg\theta(h\beta + 2Dk\beta) + htg^2\theta(\beta D' - 2\alpha D + hD\beta tg\theta) + \tau h\beta D\right]_{"1"}$$

$$+\left[-tg\theta(h\beta + 2Dk\beta) - htg^2\theta(\beta D' - 2\alpha D - hD\beta tg\theta) + \tau h\beta D\right]_{"2"} \qquad (44)$$

$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi}\int_{s_1}^{s_2}[\,k\beta - rD\beta - h\,(kD\beta + D\gamma)]ds$$

$$+\left[tg\theta(h\beta + 2Dk\beta) - htg^2\theta(\beta D' - 2\alpha D - hD\beta tg\theta) - \beta hD' - \tau h\beta D\right]_{"1"}$$

$$+\left[tg\theta(h\beta + 2Dk\beta) + htg^2\theta(\beta D' - 2\alpha D + hD\beta tg\theta) + \beta hD' - \tau h\beta D\right]_{"2"}$$

ACKNOWLEDGEMENTS

This lecture is based on a paper written by M. Bassetti. I wish to thank him for the suggestions and helpful discussions during the preparation of this note.

REFERENCES

[1] E.J.N. Wilson, Non-linearities and resonances, Proceedings CERN Accelerator School, Gif-sur-Yvette, Paris, Sept. 1984, CERN 85-19 (1985).

[2] J.-L. Laclare, Introduction to coherent instabilities, Ibid.

[3] M. Bassetti, A simplified derivation of chromaticity formulae, CERN LEP Note 504 (1984).

[4] K. Steffen, High energy beam optics, *Wiley* (1965) and also Basic course on accelerator optics, CAS, Gif-sur-Yvette, Sept. 1984, CERN 85-19 (1985).

[5] W. Hardt, J. Jä ger, D. Mö hl, A general analytical expression for the chromaticity of accelerator rings, CERN PS/LEA/82-5 (1982).

APPENDIX 1

CHROMATICITY FOR A BENDING MAGNET TAKING INTO ACCOUNT SECOND-ORDER TERMS

A1.1 DERIVATION OF THE CHROMATICITY FORMULAE

The chromaticity formulae in section 5 are obtained using the first-order expansion of the magnetic field. To obtain an expression for the chromaticity which contains also the sextupolar terms, the same derivation is repeated here using also the second-order terms in the field expansion given by Eqs.(36). To get the new expressions for the magnetic field we apply the reference system transformation described in section 5 to the second-order field expansion of Eqs. (36).

Following the derivation given in section 5 and using :

$$\sin\theta \sim D'\delta \qquad ; \qquad \cos\theta \sim 1$$

we get the expressions for the magnetic field in the new reference system:

$$B_z^* = \frac{p}{e(1+\delta)} \; [-h - h'(-x^*D'\delta + s^*) + k(D\delta + x^* + s^*D'\delta) + k'(D\delta + x^* + s^*D'\delta)(-x^*D'\delta + s^*)$$

$$+ \frac{1}{2} r (D\delta + x^* + s^*D'\delta)^2 + \frac{1}{2}(h'' - hk - r) z^2]$$

$$B_x^* = \frac{p}{e(1+\delta)} \; z\{k + k'(-x^*D'\delta + s^*) + r (D\delta + x^* + s^*D'\delta) - [-h' - h''(-x^*D'\delta + s^*)$$

$$+ (hh' + k') (D\delta + x^* + s^*D'\delta)] D'\delta\} \qquad (45)$$

$$B_s^* = \frac{p}{e(1+\delta)} \; z\{[k + k'(-x^*D'\delta + s^*) + r (D\delta + x^* + s^*D'\delta)] D'\delta - h' - h''(-x^*D'\delta + s^*)$$

$$+ (hh' + k')(D\delta + x^* + s^*D'\delta)\} \; .$$

Neglecting the second-order terms, except for the chromatic ones, i.e. the terms $x\delta$, $z\delta$ and $s\delta$, we obtain:

$$B_z^* = \frac{p}{e(1+\delta)} \; [-h + kD\delta + x^*(h'D'\delta + k + rD\delta) + s^*(-h' + kD'\delta + k'D\delta)]$$

$$B_x^* = \frac{p}{e(1+\delta)} \; z[k + rD\delta + h'D'\delta] \qquad (46)$$

$$B_s^* = \frac{p}{e(1+\delta)} \; z[-h' + kD'\delta + (hh' + k')D\delta].$$

Comparing these equations with Eqs. (36) and equating the corresponding first-order terms, we get the new coefficients:

$$h^* = \frac{h - kD\delta}{1+\delta}$$

$$k^* = \frac{k + h'D'\delta + rD\delta}{1 + \delta} \qquad (47)$$

$$h'^* = \frac{h' + kD'\delta + (hh'+k')D\delta}{1 + \delta} \quad .$$

Now, following the same procedure as in section 5, we use the coefficients h^* and k^* given by Eqs. (47) to obtain the values of the focusing strength for the off-momentum particle:

$$\begin{aligned} k_x^* &= h^{*2} - k^* = (h^2 - k) + \delta\,(-2h^2 - 2hkD + k - h'D' - rD) \\ k_z^* &= k^* = k + \delta\,(-k + h'D' + rD). \end{aligned} \qquad (48)$$

Then, we get the variation of the focusing strength with momentum, $k_{1y}(s)$:

$$\begin{aligned} k_{1x} &= \frac{\partial k_x^*}{\partial \delta} = k - 2h^2 - 2hkD - h'D' - rD \\ k_{1z} &= \frac{\partial k_z^*}{\partial \delta} = -k + h'D' + rD \; . \end{aligned} \qquad (49)$$

The variation of the orbit length with momentum has been already taken into account in Eq. (34), therefore inserting Eqs. (49) into (34) we obtain the final formulae for the chromaticity, which are more complete than that of Eqs. (43) because they contain also the sextupolar terms.

$$\begin{aligned} \frac{\partial Q_x}{\partial \delta} &= \frac{1}{4\pi} \int_0^L \{\beta\,(k - 2h^2 - 2hkD - h'D' - rD) + \beta hD\,(h^2 - k) + \gamma hD\}ds \\ \frac{\partial Q_z}{\partial \delta} &= \frac{1}{4\pi} \int_0^L \{\beta\,(-k + hkD + h'D' + rD) + \gamma hD\}ds \; . \end{aligned} \qquad (50)$$

A1.1.1 An observation on Eqs.(47)

Let us notice that the coefficient h'* given by Eqs. (47) is obtained as the coefficient of the variable z in the equation (46) for B_s, and that it is different from the coefficient of s which appears in the expression for B_z. This ambiguity comes from the fact that, while for Eqs. (36) the relation:

$$\frac{\partial B_z}{\partial s} = \frac{\partial B_s}{\partial z} \tag{51}$$

is valid, this is not true for Eqs. (46), for which it is:

$$\frac{\partial B_s^*}{\partial z} \neq \frac{\partial B_z^*}{\partial s^*} . \tag{52}$$

Equations (46) are anyway correct, but the new variable s* has to be modified. In cylindrical coordinates (z,x,ϕ), the radial component of the Maxwell equation is written:

$$\frac{1}{\rho}\frac{\partial Bz}{\partial \phi} = \frac{\partial B_\phi}{\partial z} . \tag{53}$$

When making the transformation given by Eqs. (37), which is essentially a translation in the radial direction, in Eqs. (53) ρ has to be replaced by ρ+Dδ. As a consequence, the Maxwell equation is written:

$$\frac{1}{\rho+D\delta}\frac{\partial B_z^*}{\partial \phi} = \frac{1}{1+hD\delta}\frac{\partial B_z^*}{\partial s^*} = \frac{\partial B_s^*}{\partial z} \quad ; \quad h = \frac{1}{\rho} . \tag{54}$$

This relation is in effect verified by Eqs. (46) to first order in δ.