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COMPACT AND NONCOMPACT GAUGE THEORIES ON A LATTICE

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ABSTRACT

A noncompact formulation of gauge theories on the lattice recently proposed makes use of auxiliary fields. We show that in such a formulation the representation of the gauge group in the case of SU(2) is completely reducible, one of the irreducible multiplets containing the Yang-Mills fields plus one auxiliary field. Elimination of this field by a gauge-invariant constraint leads to Wilson's formulation.

The only consistent regularization of gauge theories outside the perturbative domain is their definition on the lattice. In Wilson's formulation(1), however, gauge fields have to be identified with coordinates on a compact group manifold. This can give rise to artifacts in numerical calculations. Moreover the action of a gauge transformation is highly nonlinear even in a neighbourhood of the identity, resulting in a very complicated structure of the Faddeev-Popov terms which have to be introduced together with a gauge fixing to define a calculation procedure suitable for a perturbative treatment(2).

These difficulties could be avoided with an alternative regularization in which the action of gauge transformations is linear. This obviously requires noncompact gauge fields.

A noncompact formulation of gauge theories on a lattice has been proposed where exact gauge-invariance is obtained by introducing auxiliary fields[3]. The whole set of gauge plus auxiliary fields lives in the algebra of GL(M, c), where M is the dimension of the (matrix) representation.Suitable gauge-invariant terms can be added to the Action in order to insure that in the formal continuum limit the auxiliary fields vanish, with the exception of an abelian real field.

No attention was paid, formulating the new regularization, to the reducibility of the representation of the gauge group. In this note we show that for the gauge group SU(2) the representation breaks into 2 irreducible representations, one of which contains all the Yang-Mills fields plus one auxiliary field which can be made to vanish in the continuum limit. In this way the realization of the gauge group is linear at finite lattice spacing. But the auxiliary field can also be eliminated by a gauge-invariant constraint. Then the realization of the gauge group becomes nonlinear and one obtains Wilson's formulation.

To fix the notation and for the convenience of the reader let us sketch briefly the new lattice regularization.It is based on a discretized form of the continuum gauge tranformations

$$A'_\mu(x) = \frac{1}{i}\, g(x)\, \Delta_\mu\, g^+(x) + g(x)\, A_\mu(x)\, g^+(x+\mu). \tag{1}$$

In the above equation a is the lattice spacing, x a vector with integral components, μ a vector with components $\mu_\nu = \delta_{\mu\nu}$, g(x) an element of the group U(N) and Δ_μ the discrete derivative

$$\Delta_\mu f = \frac{1}{a}\left[f(x+\mu) - f(x)\right]. \tag{2}$$

The coupling constant has been set equal to unity.

In order that (1) transform into one another objects of the same type A_μ must be of the form

$$A_\mu = \left(\mathcal{A}_{\mu a} + i\, B_{\mu a}\right) T_a + \frac{1}{\sqrt{M}}\left(V_\mu + i\, W_\mu\right) I, \tag{3}$$

where the factor $1/\sqrt{M}$ has been introduced for convenience, T_a are the generators of the gauge group and I is the identity matrix. We use the normalization

$$\begin{aligned} \left[T_a, T_b\right] &= i\, f_{abc}\, T_c \\ \left\{T_a, T_b\right\} &= (2/M)\, \delta_{ab} + d_{abc} T_c. \end{aligned} \tag{4}$$

From the transformation of the components

$$\mathcal{A}'_{\mu a} = \mathcal{A}_{\mu a} + \Delta_\mu \vartheta_a - f_{abc} \mathcal{A}_{\mu b} \vartheta_c - (a/\sqrt{M})\ W_\mu \Delta_\mu \vartheta_a$$
$$- (a/2)\, d_{abc} B_{\mu b} \Delta_\mu \vartheta_c - (a/2)\, f_{abc} \mathcal{A}_{\mu b} \Delta_\mu \vartheta_c - a\, B_{\mu a} \Delta_\mu \vartheta_o$$

$$W'_\mu = W_\mu + (a/\sqrt{M})\, \mathcal{A}_{\mu a} \Delta_\mu \vartheta_a + a\, V_\mu \Delta_\mu \vartheta_o$$

$$B'_{\mu a} = B_{\mu a} - f_{abc} B_{\mu b} \vartheta_c + (a/\sqrt{M})\ V_\mu \Delta_\mu \vartheta_a$$
$$+ (a/2)\, d_{abc} \mathcal{A}_{\mu b} \Delta_\mu \vartheta_c - (a/2)\, f_{abc} B_{\mu b} \Delta_\mu \vartheta_c + a\, \mathcal{A}_{\mu a} \Delta_\mu \vartheta_o$$

$$V'_\mu = V_\mu - (a/\sqrt{M})\, B_{\mu a} \Delta_\mu \vartheta_a + \sqrt{M}\, \Delta_\mu \vartheta_o - a\, W_\mu \Delta_\mu \vartheta_o\,, \tag{5}$$

we see that for $a \to 0$ the $\mathcal{A}_{\mu a}$ transform like Yang-Mills fields so that the $B_{\mu a}$, V_μ and W_μ are auxiliary fields.

The basic ingredient to construct the Lagrangian is the covariant derivative

$$D_\mu = 1/a\ \ I + i\, A_\mu \tag{6}$$

which under (1) transforms according to

$$D_\mu(x) = g\,(x)\, D_\mu(x)\, g^+\ (x + \mu)\,. \tag{7}$$

The strength can be defined in analogy to the continuum

$$F_{\mu\nu}(x) = \frac{1}{i}\left[D_\mu\,(x)\, D_\nu(x{+}\mu) - D_\nu(x)\, D_\mu\,(x + \nu)\right]. \tag{8}$$

It is antisymmetric and transforms according to

$$F_{\mu\nu}\,(x) \to g\,(x)\, F_{\mu\nu}(x)\ \ g^+(x + \mu + \nu)\,. \tag{9}$$

The pure Yang-Mills Lagrangian density can therefore be written

$$\mathcal{L}_{YM} = \frac{1}{4} \beta\, T_r\, F^+_{\mu\nu} F_{\mu\nu}\ . \tag{10}$$

It is easy to check that $\mathcal{L}_{YM}$ becomes the Yang Mills Lagrangian density in the continuum limit, so that in this limit it is also parity invariant. But at finite lattice spacing it is not. Denoting by $\mathcal{I}_\mu$ the operator which performs the inversion with respect to the μ-axis ($x \to i_\mu x$) we have

$$I_\mu^{-1} D_\mu(x)\, I_\mu = D_\mu^+(i_\mu x - \mu)$$

$$I_\mu^{-1} D_\nu(x)\, I_\mu = D_\nu(i_\mu x), \qquad \text{for } \mu \neq \nu. \tag{11}$$

$\mathcal{L}_{YM}$ is not invariant under the above transformations, but it can be made invariant by addition of suitable terms that will be discussed later.

The gauge-invariant term which makes to vanish all the auxiliary fields but V_μ is

$$\mathcal{L}_C = \frac{a}{2\lambda} T_r \sum_\mu \left(D_\mu^+(x) D_\mu(x) - \frac{1}{a^2} \right)^2, \tag{12}$$

where λ is a parameter with the dimension of a length. In the formal limit $a \to 0$ it provides a term

$$\prod_{x,\mu} \delta\,[W_\mu(x)] \prod_a \delta\,[B_{\mu\, a}(x)] \tag{13}$$

in the measure of the partition function. Notice that the surviving field V_μ cannot be required to be zero by gauge fixing because the corresponding Faddeev-Popov determinant would vanish for $B_{\mu a}$=0.

Finally among other terms there is

$$\mathcal{L}_m = \frac{1}{2} m^2 \sum_\mu \mathrm{Tr} \left(D_\mu^+(x)\, D_\mu(x) - \frac{1}{a^2} \right), \tag{14}$$

which has the form of a mass term, and has in fact led to speculate(3) that the present regularization might provide a way to smuggle a mass for the vector bosons without making recourse to the Higgs mechanism. At the end of the paper we will show that actually this is not possible.

The gauge transformations within each multiplet are essentially homogeneous. They become exactely homogeneous introducing the field

$$C_\mu = \frac{\sqrt{2}}{a} - W_\mu. \tag{15}$$

We must emphasize, however, that the linear symmetry must be spontaneously broken according to the above equation, in order that the covariant derivative contain an ordinary derivative and hence the theory be a gauge theory. Such a breaking does occur in the presence of $\mathcal{L}_C$.

The case of U(2) is different from the others because of the reality of the SU(2) spinor representation which makes the symmetric constants d_{abc} to vanish. If we restrict ourselves to SU(2), by putting $\vartheta_o=0$ in Eqs. (5), we see that the multiplets $(\mathcal{A}_{\mu a}, W_\mu)$ and $(B_{\mu a}, V_\mu)$ do not mix with each other and are irreducibile representations

$$\mathcal{A}'_{\mu a} = \mathcal{A}_{\mu a} + \left(1 - \frac{a}{\sqrt{2}} W_\mu\right) \Delta_\mu \theta_a - f_{abc} \mathcal{A}_{\mu b} \left(1 + \frac{a}{2} \Delta_\mu\right) \theta_c$$

$$W'_\mu = W_\mu + \frac{a}{\sqrt{2}} \mathcal{A}_{\mu a} \Delta_\mu \theta_a$$

$$B'_{\mu a} = B_{\mu a} + \frac{a}{\sqrt{2}} V_\mu \Delta_\mu \theta_a - f_{abc} B_{\mu b} \left(1 + \frac{a}{2} \Delta_\mu\right) \theta_c$$

$$V'_\mu = V_\mu - \frac{a}{\sqrt{2}} B_{\mu a} \Delta_\mu \theta_a . \tag{16}$$

The explicit expression of $\mathcal{L}_{YM}$ for the multiplet $(\mathcal{A}_{\mu a}, W_\mu)$ is

$$\mathcal{L}_{YM} = \frac{1}{4} \beta \Big\{ \Big[\Delta_\mu W_\nu(x) - \Delta_\nu W_\mu(x) + \frac{1}{\sqrt{2}} \beta \big(\mathcal{A}_{\mu b}(x)\, \mathcal{A}_{\nu b}(x+\mu)$$

$$- W_\mu(x)\; W_\nu(x+\mu) - (\mu \leftrightarrow \nu)\big)\Big]^2 + \Big[\Delta_\mu\; \mathcal{A}_{\nu a}(x) - \Delta_\nu\, \mathcal{A}_{\mu a}(x)$$

$$- \frac{1}{\sqrt{2}} \big(\varepsilon_{abc}\, \mathcal{A}_{\mu b}(x)\, \mathcal{A}_{\nu c}(x+\mu) + W_\mu(x)\, \mathcal{A}_{\nu a}(x+\mu)$$

$$+ W_\nu(x+\mu)\mathcal{A}_{\mu a}(x) - (\mu \leftrightarrow \nu)\big)\Big]^2 \Big\} . \tag{17}$$

To have a renormalizable Lagrangian we must add all the other gauge-invariant terms of dimension not greater than 4. These terms are polynomials in the composite gauge invariant field t_μ

$$t_\mu(x)\, I = D^+_\mu(x)\, D_\mu(x) - \frac{1}{a^2} = D_\mu(x)\, D^+_\mu(x) - \frac{1}{a^2} = \left[\frac{1}{2} \vec{\mathcal{A}}^{\,2}_\mu + \frac{1}{2} W^2_\mu - \frac{\sqrt{2}}{a} W_\mu\right] I \tag{18}$$

where

$$\vec{\mathcal{A}}_\mu^2 = \sum_a \mathcal{A}_{\mu a}^2 \, . \qquad (19)$$

Here we denote all these polynomials by $\mathcal{P}$ of which we only use the normalization

$$\mathcal{P} = 0 \, , \quad \text{for} \;\; t_\mu = 0. \qquad (20$$

The expression of $\mathcal{P}$ will be given in a separate paper where the perturbative features of the present regularization are analyzed. $\mathcal{P}$ contains also the terms necessary to make $\mathcal{L}_{YM}$ parity invariant.

Finally the gauge field Lagrangian density is

$$\mathcal{L}_G \; = \; \mathcal{L}_{YM} + \mathcal{L}_C + \mathcal{P} + \mathcal{L}_m \qquad (21)$$

and the partition function can be written

$$Z = \int \prod_{x,\mu} \; dW_\mu\,(x) \prod_a d\mathcal{A}_{\mu a}\,(x) \;\; e^{-\sum_y \mathcal{L}_G(\mathcal{A}_\mu, W_\mu)} . \qquad (22)$$

This concludes the discussion of the irriducible representation when the auxiliary field W_μ is retained.

One can obtain Wilson's formulation as a special case if W_μ is altogether eliminated by imposing the constraint

$$t_\mu = 0 \, , \qquad (23)$$

whose solutions are

$$W_\mu^{\sigma_\mu} = \frac{\sqrt{2}}{a} - \sigma_\mu \frac{\sqrt{2}}{a}\left[1 - \frac{1}{2}\, a^2\, \vec{\mathcal{A}}_\mu^2 \;(x)\right]^{\frac{1}{2}}, \qquad \sigma_\mu = \pm 1, \qquad (24)$$

with

$$\vec{\mathcal{A}}_\mu^2 \leq \frac{2}{a^2} \, . \qquad (25)$$

The constraint (23) has the effect of compactifying the gauge field.

As usual after elimination of auxiliary fields the representation becomes non linear

$$\mathcal{A}'_{\mu a} = \mathcal{A}_{\mu a} + \sigma_\mu \left(1 - \frac{1}{2} a^2 \vec{\mathcal{A}}_\mu^2\right)^{1/2} \Delta_\mu \vartheta_a - f_{abc} \mathcal{A}_{\mu b} \vartheta_c - (a/2) f_{abc} \mathcal{A}_{\mu b} \Delta_\mu \vartheta_c . \qquad (26)$$

The covariant derivative becomes

$$D_\mu = \frac{1}{a} U_\mu , \qquad (27)$$

where

$$U_\mu = \sigma_\mu \left(1 - \frac{1}{2} a^2 \vec{\mathcal{A}}_\mu^2\right)^{1/2} I + i a \mathcal{A}_\mu \qquad (28)$$

is a unitary matrix which can be identified with Wilson's link variable. All the terms in the Action now vanish with the exception of $\mathcal{L}_{YM}$ which becomes Wilson's Lagrangian

$$\mathcal{L}_{YM} = \mathcal{L}_W, \qquad \text{for } t_\mu = 0. \qquad (29)$$

The partition function can be written

$$Z_W = \int \prod_{x,\mu} dW_\mu(x) \prod_a d\mathcal{A}_{\mu a}(x)\, \delta\left[t_\mu(x)\right] e^{-\sum_y \mathcal{L}_{YM}(\mathcal{A}_\mu, W_\mu)} . \qquad (30)$$

Integrating over W_μ we get

$$Z_W \propto \int \prod_{x,m,a} d\mathcal{A}_{\mu a}(x) \left[1 - \frac{1}{2} a^2 \vec{\mathcal{A}}_\mu^2 (x)\right]^{-\frac{1}{2}} \sum_{\sigma_\mu} e^{-\sum_y \mathcal{L}_{YM}\left(\mathcal{A}_\mu, W_\mu^{(\sigma_\mu)}\right)} . \qquad (31)$$

In the jacobian arising from the integration we recognize the invariant measure on SU(2), so that we can rewrite Z_W in the standard form

$$Z_W \propto \int \prod_{x,\mu} dU_\mu(x)\, e^{-\sum_y \mathcal{L}_W} \qquad (32)$$

We conclude the paper by showing that even with the present regularization pure gauge theories are incompatible with massive vector bosons. To this end we derive the Ward identities for the effective action Γ, under the hypotesis that it is stationary for $\mathcal{A}_{\mu a}=0$, and $W_\mu=\overline{W}$

$$\frac{\partial\Gamma}{\partial\mathcal{A}_{\mu a}(x)} = \frac{\partial\Gamma}{\partial W_\mu(x)} = 0 \qquad \text{for } \mathcal{A}_{\mu a}(x)=0\,,\ W_\mu(x)=\overline{W}_\mu\,. \tag{33}$$

Because of gauge-invariance

$$\delta\Gamma = \sum_{\mu,x} \delta\, C'_\mu(x) \frac{\partial\Gamma}{\partial C_\mu(x)} + \sum_a \delta\mathcal{A}_{\mu a}(x) \frac{\partial\Gamma}{\partial\mathcal{A}_{\mu a}(x)} = 0\,, \tag{34}$$

where

$$C'_\mu = \frac{\sqrt{2}}{a} - \left(W_\mu - \overline{W}\right)\,. \tag{35}$$

By the definition

$$\frac{\partial\Gamma}{\partial\mathcal{A}_\mu(x)} \stackrel{\text{def}}{=} \sum_a T_a \frac{\partial\Gamma}{\partial\mathcal{A}_{\mu a}(x)} \tag{36}$$

the above equation can be rewritten

$$\delta\Gamma = T_r \sum_{\mu,x} \left\{ -\sqrt{2}\, a\, \mathcal{A}_\mu(x)\, \Delta_\mu\, \theta(x) \frac{\partial\Gamma}{\partial C_\mu(x)} + 2\,\delta\, \mathcal{A}_\mu(x) \frac{\partial\Gamma}{\partial\mathcal{A}_\mu(x)} \right\}. \tag{37}$$

Integrating by parts we get

$$\delta\Gamma = \sum_{\mu,x} T_r\, \theta(x) \left\{ \sqrt{2}a\, \Delta_\mu^{(-)} \left[\mathcal{A}_\mu(x) \frac{\partial\Gamma}{\partial C_\mu(x)} + \left(\overline{W} - C'_\mu(x)\right) \frac{\partial\Gamma}{\partial\mathcal{A}_\mu(x)} \right] - i\left[\mathcal{A}_\mu(x-\mu), \frac{\partial\Gamma}{\partial\mathcal{A}_\mu(x-\mu)}\right] - i\left[\mathcal{A}_\mu(x), \frac{\partial\Gamma}{\partial\mathcal{A}_\mu(x)}\right] \right\} = 0\,, \tag{38}$$

$\Delta_\mu^{(-)}$ being the left derivative

$$\Delta^{(-)}_{\mu} f(x) = \frac{1}{a}\left[f(x) - f(x-\mu)\right] . \tag{39}$$

Due to the arbitrariness of $\theta(x)$ the above equation implyies the vanishing of the quantity in curly brakets at each point x. Evaluating the derivative w.r. to $\mathcal{A}_\nu(y)$ of such a quantity at $\mathcal{A}_\mu$=0, $W_\mu = \overline{W}$ we get

$$\sum_\mu \left[\overline{W} - C'_\mu(x)\right] \frac{\partial}{\partial \mathcal{A}_\nu(y)} \frac{\partial}{\partial \mathcal{A}_\mu(x)} \Gamma = 0 , \qquad \text{for } \mathcal{A}_\mu = 0, \qquad C'_\mu = \frac{\sqrt{2}}{a} . \tag{40}$$

As already noticed we must exclude the possibility $\overline{W} = \sqrt{2}/a$, for which the covariant derivative would no longer contain an ordinary derivative. It then follows that there can be no effective mass term in the effective action.

REFERENCES

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