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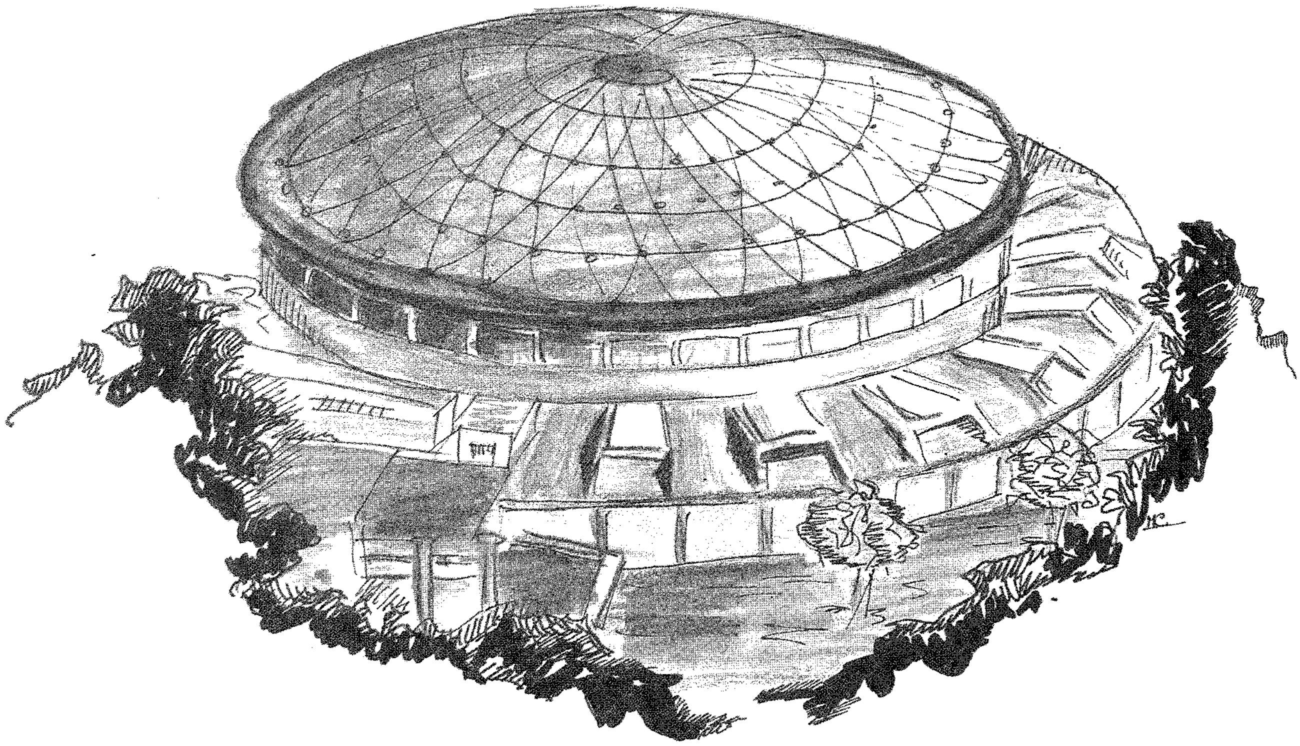

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GAUGE INVARIANCE ON THE LATTICE WITH NONCOMPACT GAUGE FIELDS

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ABSTRACT

An action is constructed which is gauge invariant on the lattice with noncompact gauge fields. It requires the introduction of auxliary fields which disappear in the continuum limit.

The linear potential which arises in the strong coupling limit of compact gauge theories is an artifact. It is due to the fact that in the phase factor which makes gauge-invariant the product of matter fields at different sites, in addition to the pure gauge degrees of freedom, there are also the gauge-invariant ones which are not necessary(1). Let us illustrate this point in the abelian case. Here the standard definition of the gauge-invariant product of matter fields ψ at neighbouring sites is:

$$\psi^{+}(x) \exp\left[i\, a\, g\, A_{\mu}(x)\right] \psi(x+\mu) \quad (1)$$

where A_{μ} is the gauge field, a the lattice spacing, x a vector with integral components and μ a vector with components $\mu_{\nu} = \delta_{\mu\nu}$.

Expression (1) is a periodic function of ($a\, g\, A_{\mu}$), which is therefore a compact variable. As it is well known with such an interaction one gets a linear potential in strong coupling.

Now the gauge field can be splitted into its longitudinal and transverse parts

$$A_{\mu} = A_{L\,\mu} + A_{T\,\mu} \quad (2)$$

and expression (1) can be replaced by

$$\psi^{+}(x) \exp[i\, a\, g\, A_{L\mu}(x)]\, \psi(x+\mu) \qquad (3)$$

which is also gauge-invariant but not periodic with respect to ($a\, g\, A_\mu$) because of the longitudinal projection operator. The variable ($a\, g\, A_\mu$) is noncompact and with (3) one gets a Coulombic potential for all couplings. The linear potential with (1) is due to the unnecessary transverse field.

The artful nature of such a potential is made clearer by the fact that even with (1) by a proper choice of boundary conditions (of the state functional with respect to $A_\mu(x)$) one can get a Coulombic potential[2].

The situation for non abelian theories is similar. Recently a lattice formulation with noncompact fields has been proposed[1] which makes use of gauge invariant variables. These are polar variables. In strong coupling confinement results from a requirement of equivalence between curvilinear and Cartesian coordinates in the sense that the eigenstates of the Hamiltonian in polar coordinates should also be eigenstates of the Hamiltonian in Cartesian coordinates. But the confinement obtained in this way is not given by a linear potential.

All the above does not imply of course that the continuum limit obtained by Wilson's formulation should not be the correct one. The artful enforcement of confinement in strong coupling might indeed play a role analogous to the equivalence requirement in the polar formulation, like an "initial condition" to select the right states in the continuum. It is difficult to investigate this possibility by comparing to the polar formulation, because we do not know how to formulate the equivalence requirement in the path integral in order to study the continuum limit. In the absence of this condition there is no confinement in the continuum[3], implying that exact gauge invariance alone is not sufficient to enforce it.

The obvious way to clarify the situation is to study noncompact gauge theories with Cartesian fields. These theories have been first formulated on the lattice by direct discretization of the continuum and no confinement has been found[4]. This negative result, however, can be attributed to the explicit breaking of gauge-invariance by the discretization procedure.

In this letter we propose an approach with noncompact Cartesian fields where gauge invariance on the lattice is exact.

Let us first establish our notations. We denote by latin letters quantities on the lattice and by the corresponding script letters the corresponding quantities in the continuum. So for instance $A_\mu(x)$ is the gauge field on the lattice and $\mathcal{A}_\mu(x)$ its continuum limit. The generators of the gauge group T_a are hermitean and normalized according to

$$[T_a, T_b] = i\, f^{c}_{ab} T_c \quad ; \quad T_r T_a T_b = C\, \delta_{ab} \qquad (4)$$

where f^{c}_{ab} are the structure constants and C the value of the Casimir operator. The elements of the group are denoted by U(x).

We define transformations on the lattice by discretization of the continuum gauge transformations

$$A'_\mu(x) = \frac{1}{ig}\; U^+(x)\,\frac{1}{a}\Big[U(x+\mu) - U(x)\Big] \;+ U^+(x)\,A_\mu(x)\,U(x+\mu)\,. \tag{5}$$

The form of the first term is obvious while in the second one the appearence of $U(x+\mu)$ instead of $U(x)$ will be explained below.

With such a definition $A'_\mu(x)$ is not hermitean nor does it live in the Lie algebra of the group even if $A_\mu(x)$ does. If the lattice transformations have to transform into one another objects of the same type we must introduce auxiliary fields. We can then assume for $A_\mu(x)$ the form

$$A_\mu(x) = \Big[\mathcal{A}^b_\mu(x) + i\,a\,B^b_\mu(x)\Big]\,T_b + a\,V_\mu(x), \tag{6}$$

where $\mathcal{A}^b{}_\mu$ and $B^b{}_\mu$ are real fields while $V_\mu(x)$ are complex. $\mathcal{A}_\mu(x)$ is the gauge field which obtains in the continuum limit of $A_\mu(x)$. In fact according to Eq. (5)

$$V'_\mu(x) = \frac{1}{a}\,T_r\,A'_\mu(x)$$

is finite in the limit of $a \to 0.$ so that the product $a\,V'_\mu$ vanishes. The same holds true for $B'_\mu(x)$.

Our problem is how to construct an action invariant under the transformation (5) and such that in the continuum limit become the Yang-Mills action for the field $\mathcal{A}_\mu$.

To start with we need a definition of the strengths such that under the transformation (5) they should transform homogeneously. We observe that the quantity playing the role of the covariant derivative is

$$D_\mu(x) = \frac{1}{a} + i\,g\,A_\mu(x) \tag{7}$$

whose transformation results to be

$$D_\mu(x) \to U^+(x)\,D_\mu(x)\,U(x+\mu)\,. \tag{8}$$

It is in order to obtain this transformation law that we have put $U(x+\mu)$ instead of $U(x)$ in Eq. (5).

We can now define the field strengh in analogy with the continuum definition

$$F_{\mu\nu}(x) = \frac{1}{ig}\Big[D_\mu(x)\;D_\nu(x+\mu) \;-\; D_\nu(x)\;D_\mu(x+\nu)\Big]. \tag{9}$$

$F_{\mu\nu}$ is antisymmetric in μ, ν, and transforms according to

$$F_{\mu\nu}(x) \to U^{+}(x) F_{\mu\nu}(x) U(x+\mu+\nu). \tag{10}$$

It should be noted that $F_{\mu\nu}$ also does not live in the Lie algebra of the gauge group, but in the formal continuum limit it becomes the continuum strength

$$\lim_{a \to 0} F_{\mu\nu} = \mathcal{F}^{a}_{\mu\nu} T_a = \left(\partial_\mu \mathcal{A}^{a}_{\nu} - \partial_\nu \mathcal{A}^{a}_{\mu} - g f^{a}_{bc} \mathcal{A}^{b}_{\mu} \mathcal{A}^{c}_{\nu}\right) T_a. \tag{11}$$

We assume the invariant action density on the lattice

$$S = S_G + S_F, \tag{12}$$

where

$$S_G = -\frac{1}{4} \sum_{\mu\nu} \frac{1}{C} T_r F^{\mu\nu}(x) F^{+}_{\mu\nu}(x) \tag{13}$$

$$S_F = \frac{i}{2} \bar{\psi}(x) \gamma^{\mu} D_\mu \psi(x+\mu) + \text{h.c.} \tag{14}$$

The fermion field transforms obviously according to

$$\psi(x) \to U^{+}(x) \psi(x), \tag{15}$$

and we have disreguarded the problem of the doubling as inessential to the present issue.

The formal continuum limit of S_G and S_F gives the Yang-Mills and Dirac Lagrangian densities

$$\mathcal{S}_G = -\frac{1}{4} \sum_{\mu\nu a} \mathcal{F}^{\mu\nu a}(x) \mathcal{F}^{a}_{\mu\nu}(x), \tag{16}$$

$$\mathcal{S}_F = \frac{i}{2} \bar{\psi}(x) \gamma^{\mu} \mathcal{D}_\mu(x) \psi(x) + \text{h.c.}, \qquad \mathcal{D}_\mu = \partial_\mu + i g \mathcal{A}_\mu. \tag{17}$$

But the actual limit in an explicit calculation might be different due to a dynamical growing of V_μ and/or B_μ like $1/a$. The latter evenience is most dangerous because B_μ has nonabelian interactions with $\mathcal{A}_\mu$ and ψ. This possibility can be prevented by adding to the action (12) the gauge invariant term

$$S_C = \sum_\mu T_r \left\{ \frac{1}{2\lambda a^3} - \frac{1}{\lambda a} D^{\mu}(x) D^{+}_{\mu}(x) + \frac{a}{2\lambda} \left[D^{\mu}(x) D^{+}_{\mu}(x)\right]^2 \right\}, \tag{18}$$

where λ is a parameter with the dimension of a length.

In the limit of $a \to 0$ S_C becomes

$$S_C = \lim_{a\to 0} \sum_{\mu} \frac{g^2}{\lambda a}\left[- M a^2 \left(V_\mu - V_\mu^*\right)^2 + C \sum_b a^2 B^{\mu b} B_\mu^b \right]$$

$$+ i \frac{g^3}{\lambda} T_r A_\mu \left(A_\mu - A_\mu^+\right) A_\mu^+, \qquad (19)$$

where M is the dimension of the representation of the gauge group. S_C is vanishing unless V_μ and/or B_μ grow like 1/a. In such a case the first-term of S_C provides a factor

$$\prod_{x,\mu} \delta\left[a\left(V_\mu(x) - V_\mu^*(x)\right)\right] \text{ and/or } \prod_{x,b,\mu} \delta\left[a B_\mu^b(x)\right]$$

in the measure of the partition function, while the last term vanishes with such a measure. This ensures that the auxiliary field B_μ always disappears in the continuum and V_μ,if it survives, is real. It only contributes a term

$$\lim_{a\to 0} \left(\partial_\mu aV_\nu - \partial_\nu aV_\mu\right)^2$$

to the gauge field action, and a term

$$\lim_{a\to 0} i\bar{\psi}\gamma^\mu aV_\mu \psi$$

to the Dirac action. It is perhaps worth while noticing that $\lim_{a\to 0} a V_\mu$ is gauge invariant.

As far as quark confinement is concerned we can limit ourselves to the study of the ground state with static quarks. In this situation the contribution of the field V_μ can be ignored. Alternatively we can fix the gauge on the lattice by the constraint

$$T_r \ A_\mu = 0,$$

which does not spoil the gauge-invariance of the continuum limit.

Unlike Wilson's formulation or the polar formulation there appears to be no simple way to study the strong coupling limit, and one has to resort to numerical calculations. Once one has checked that the continuum limit has the right properties, thanks to the invariance of the action one can simplify such calculations by putting everywhere $A_\mu(x) = \mathcal{A}^a_\mu(x)\, T_a$.

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