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## A Group-Theoretical Approach to the Multiperipheral Model.

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**Summary.** — The convergence of the multiperipheral series is investigated in the case of the scattering of two spinless particles, when the square of the spacelike momentum transfer  $W$  is fixed as a parameter. A « coupling scheme » is introduced in which the scattering amplitude is described as a pseudobilinear functional on some Hilbert spaces which contain elements describing an incoming and an outgoing particle. So in this formalism to every multiperipheral graph is associated a pseudobilinear functional. The properties of the series are studied by means of the « crossed » partial-wave analysis. Using Fredholm techniques and analyticity properties it is shown that the graph series sum, in correspondence to physical values of the coupling constant  $g$ , is described by an unbounded functional which cannot be expressed only by means of the unitary irreducible representations of the principal series of  $O_{2,1}$ . In this formalism this means that the amplitude contains some Regge-pole contributions. The Regge asymptotic behaviour of the solution and a new formulation of the Froissart bound are pointed out. We then discuss the usefulness of the method for the ladder-approximation case.

### 1. — Introduction.

In this work we sum the ladder graph series, which describes the scattering of two spinless particles, when the square of the four-momentum transfer  $Q^2 = W$  is fixed as a parameter and  $W < 0$ . Our investigation is essentially directed to develop some mathematical techniques which allow a rigorous and natural treatment of multiperipheral models.

The detailed attention that we have devoted to the case of the ladder series (with an unstable exchanged particle) may be justified by observing that, even if this dynamical model is not a perfect tool, it is a useful one for discovering properties that more realistic amplitudes might possess. Nevertheless we think that our considerations may be useful in connection with more general theories. In this direction, recently much attention has been devoted to some multiperipheral models, suggested by unitarity, in which the nonkinematical aspects of Regge theories can be explored. The basic idea of multi-Regge<sup>(1)</sup> and multi-Veneziano<sup>(2)</sup> models is an extension of the AFS ideas<sup>(3)</sup>: the amplitude is obtained through a multiperipheral exchange of some objects which may be « Reggeons » or « families of Reggeons ».

The study of the ladder series convergence was performed by one of us in a previous work<sup>(4)</sup>: in that case the ladder graphs were considered at fixed total four-momentum. In the space of the off-shell wave functions a suitable Hilbert-space structure was introduced, in such a way that the iterated integral operator became bounded in this space and the off-shell convergence implied the on-shell one. These results provide a more satisfactory justification for the employment of integral equations of Bethe-Salpeter type and the formalism allows us to write down an equation involving bounded operators in Hilbert space without any modification of symmetry properties. This method is thus particularly useful when a group-theoretical approach is considered. Until now these group-theoretical techniques have been successfully used in order to find some « kinematical properties » of the Regge trajectories<sup>(5)</sup>, however also dynamical problems have been investigated by means of group-

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<sup>(1)</sup> G. F. CHEW and C. DETAR: *Multiperipheral dynamics at zero momentum transfer*, Berkeley preprint UCRL-18618 (1968); G. F. CHEW, M. L. GOLDBERGER and F. LOW: *Phys. Rev. Lett.*, **22**, 208 (1969); L. CANESCHI and A. PIGNOTTI: *Self-consistent Regge trajectories in a multiperipheral model*, University of California, Santa Barbara preprint (1968); L. CANESCHI and A. PIGNOTTI: *Coupled-channel model of self-consistent Regge trajectories*, University of California, Santa Barbara preprint (1969); I. G. HALLIDAY: *Self-consistent Regge singularities*, I, preprint ICTP/67/36 (1968); I. G. HALLIDAY: *Self-consistent Regge singularities*, II, preprint ICTP/68/2 (1968); I. G. HALLIDAY and L. M. SAUNDERS: *The unitarity equation of high energy*, preprint ICTP/67/34 (1968); P. D. THING: *Certain implications for high-energy total cross-sections from the multiperipheral model*, Berkeley preprint UCRL-18670 (1968); G. F. CHEW and W. R. FRAZER: *A model of the Pomeron pole cut relationship*, Berkeley preprint UCRL-18681 (1968); G. F. CHEW: *Phys. Rev. Lett.*, **22**, 364 (1969).

<sup>(2)</sup> K. BARDAKCI and H. RUEGG: *Reggeized resonance model for arbitrary production processes*, Berkeley preprint (1968).

<sup>(3)</sup> D. AMATI, S. FUBINI and A. STANGHELLINI: *Nuovo Cimento*, **26**, 896 (1962).

<sup>(4)</sup> G. MATTIOLI: *Nuovo Cimento*, **56 A**, 172 (1968).

<sup>(5)</sup> M. TOLLER: *Nuovo Cimento*, **53 A**, 671 (1968); G. COSENZA, A. SCIARRINO and M. TOLLER: *Nuovo Cimento*, **57 A**, 253 (1968); *On the group-theoretical approach to the conspiracy problem for arbitrary masses*, II, Ref. TH. 1002 CERN (1969).

theoretical techniques, especially in the framework of the B-S equation <sup>(6)</sup>. We hope that the more refined techniques we shall develop in this work will permit physicists to obtain more powerful results.

The «crossed partial-wave analysis» <sup>(7,8)</sup> allows us to study the much simpler convergence of the projected series and the original amplitude is derived by means of an inversion formula. This procedure leads to clarify the structure of the amplitude in terms of the parameters of the Lorentz group irreducible representations, as complex angular momentum and signature, and provides in this way an interesting dynamical justification of the Regge-pole assumptions.

In order to derive these results, the mathematical tools are given by the theory of analytic and meromorphic operator-valued functions and by Fredholm spectral theory in Hilbert spaces <sup>(9)</sup>. Group-theoretical methods must be employed with particular caution, since many difficulties arise in the application of the theory of group representations of noncompact groups. These difficulties, whose origin is due to the fact that one has to deal with infinite-dimensional representations, can be tackled by introducing a more general formalism suggested by TOLLER <sup>(10)</sup>. Another difficulty, which is related to the fact that the formalism of functions defined on the group is not sufficiently general for the treatment of off-shell functions <sup>(11)</sup>, can be solved by using the theory of the harmonic analysis on homogeneous spaces <sup>(12)</sup>; however, in the solution of the special problem we are considering, we can ourselves to apply these techniques in a simpler way by enlarging some results of a previous paper of SERTORIO and TOLLER <sup>(13)</sup>.

In Sect. 2 we introduce the general formalism: it consists of defining the scattering amplitude as a pseudobilinear functional defined on Hilbert spaces whose elements describe an incoming and an outgoing particle of the reaction; we call them «pseudostates». A pseudobilinear functional is then associated

<sup>(6)</sup> M. KEMMER and A. SALAM: *Proc. Roy. Soc., A* **230**, 266 (1955); R. E. CUTKOSKY: *Phys. Rev.*, **96**, 1135 (1954); *Journ. Math. Phys.*, **1**, 429 (1960); *Rev. Mod. Phys.*, **33**, 446 (1961); *Phys. Rev.*, **131**, 1888 (1963); *Proc. Rochester Conf. on Theoretical Physics* (1967); G. DOMOKOS and P. SURANYI: *Nucl. Phys.*, **54**, 529 (1964); G. DOMOKOS: *Phys. Rev.*, **159**, 1387 (1967); D. Z. FREEDMAN and J. M. WANG: *Phys. Rev.*, **153**, 1596 (1967).

<sup>(7)</sup> M. TOLLER: *Nuovo Cimento*, **37**, 631 (1965).

<sup>(8)</sup> M. TOLLER: *Nuovo Cimento*, **54 A**, 295 (1968).

<sup>(9)</sup> N. DUNFORD and J. T. SCHWARTZ: *Linear Operators*, Part I, II (New York, 1964); See also some results of: G. COSENZA, L. SERTORIO and M. TOLLER: *Nuovo Cimento*, **35**, 913 (1965), Appendices A, B.

<sup>(10)</sup> M. TOLLER: *Nuovo Cimento*, **54 A**, 295 (1968).

<sup>(11)</sup> M. TOLLER: *Nuovo Cimento*, **37**, 631 (1965), Sect. 6.

<sup>(12)</sup> I. M. GEL'FAND, M. I. GRAEV and N. YA. VILENKIN: *Generalized Functions*, vol. 5 (New York, 1966); I. M. GEL'FAND and M. I. GRAEV: *Am. Math. Soc. Transl.*, vol. **37** (1964).

<sup>(13)</sup> L. SERTORIO and M. TOLLER: *Nuovo Cimento*, **33**, 413 (1964).

to each iterated graph and it is expressed by a multiple integral containing the wave functions of the four interacting particles.

In Sect. 3 we formulate the problem of iterated graph convergence which in our formalism means convergence of the series of pseudobilinear functionals.

Sect. 4 is concerned with the crossed partial-wave analysis of the functionals: projection formulae are derived for graphs satisfying some conditions which are verified in our specific problem.

In Sect. 5 and 6 we study the convergence of the series of the projected functionals, and we will find out that it is a sum of matrix elements of bounded operators. On that basis we will be allowed to study the convergence, in the uniform topology, of the corresponding operator series. A set of values of the coupling constant  $g$  is found, where the convergence of the projected series is proved. In this domain the sum of the operator series is obtained by means of the resolvent of a projected off-shell integral operator, which is an analytic operator-valued function of  $g$  and  $l$  (in a strip of the  $l$ -plane) of the Hilbert-Schmidt type. The nonprojected amplitude is obtained through an inversion formula derived from group theory. The Fredholm theorems and the analyticity properties of the projected pseudostates enable us to perform the analytic continuation in  $g$  of the projected series outside the uniform-convergence domain. The nonprojected amplitude is obtained by shifting the integration path in the inversion formula (this is a generalization of the inverse Laplace transform). Only singularities of pole type are present in our model.

In the present formalism the Regge-pole behaviour of the solution is derived in the high-energy limit and also a new formulation of the Foissart bound is pointed out.

## 2. - The scattering amplitude as a pseudobilinear functional on the pseudostates.

The formalism introduced in ref. (8) allows us to describe the six reactions connected by the substitution rules, by means of a single multilinear functional defined on the topological product of some Hilbert spaces whose elements contain the states of a single particle and of the corresponding antiparticle (14).

We consider the multilinear functional:

$$\begin{aligned}
 (2.1) \quad & \theta(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) = \theta(f^{(1)} \oplus Jf^{(1c)}, f^{(2)} \oplus Jf^{(2c)}, f^{(3)} \oplus Jf^{(3c)}, f^{(4)} \oplus Jf^{(4c)}) = \\
 & = ([f^{(1c)} \otimes f^{(2c)}], S - I[f^{(3)} \otimes f^{(4)}]) + ([f^{(3c)} \otimes f^{(4c)}], S - I[f^{(1)} \otimes f^{(2)}]) + \\
 & + ([f^{(1c)} \otimes f^{(3c)}], S - I[f^{(2)} \otimes f^{(4)}]) + ([f^{(2c)} \otimes f^{(4c)}], S - I[f^{(1)} \otimes f^{(3)}]) + \\
 & \quad + ([f^{(3c)} \otimes f^{(2c)}], S - I[f^{(1)} \otimes f^{(4)}]) + ([f^{(1c)} \otimes f^{(4c)}], S - I[f^{(3)} \otimes f^{(2)}])
 \end{aligned}$$

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(14) We follow the definitions and the developments used in Sect. 4 and 5 of ref. (8). All references about the representations of the Lorentz group can be found there.

defined on the topological product

$$(2.2) \quad \tilde{\mathcal{H}}^{(1)} \times \tilde{\mathcal{H}}^{(2)} \times \tilde{\mathcal{H}}^{(3)} \times \tilde{\mathcal{H}}^{(4)}$$

of the Hilbert spaces:

$$(2.3) \quad \mathcal{H}^{(i)} = \mathcal{H}^{(i)} \oplus \overline{\mathcal{H}}^{(i)} \quad (i = 1, 2, 3, 4),$$

whose elements are of the type

$$(2.4) \quad \tilde{f}^{(i)} = f^{(i)} \oplus Jf^{(i)}, \quad f^{(i)} \in \mathcal{H}^{(i)}, \quad f^{(i)} \in \mathcal{H}^{(i)},$$

where we call  $\mathcal{H}^{(i)}$  the Hilbert spaces of single-particle states and  $\overline{\mathcal{H}}^{(i)}$  the Hilbert spaces of antiparticle states. These spaces are connected by the *TCP* transformation. Furthermore we call  $J$  the antilinear canonical mapping which transforms the Hilbert space  $\mathcal{H}$  in its adjoint  $\overline{\mathcal{H}}$ . Let now  $T(P^{(1)}, P^{(2)}, P^{(3)}, P^{(4)})$  be the scattering amplitude, defined in the usual way; using (2.1) we may write

$$(2.5) \quad \theta(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) = \frac{(2\pi)^4}{i} \iiint \delta^4\left(\sum_{i=1}^4 P^{(i)}\right) T(P^{(1)}, P^{(2)}, P^{(3)}, P^{(4)}) \cdot \\ \cdot \tilde{f}^{(1)}(P^{(1)}) \tilde{f}^{(2)}(P^{(2)}) \tilde{f}^{(3)}(P^{(3)}) \tilde{f}^{(4)}(P^{(4)}) \prod_{i=1}^4 \delta(P^{2(i)} - M_{(i)}^2) d^4P^{(i)},$$

where  $M_{(i)}$  is the mass of the particle of type  $(i)$ .  $\tilde{f}^{(i)}(P^{(i)})$  is the wave function associated to the vector  $\tilde{f}^{(i)}$ : if  $P_0^{(i)} > 0$  it describes the single-particle state  $f^{(i)}$ , if  $P_0^{(i)} < 0$  the corresponding antiparticle state  $f^{(i)}$  (15). In the spaces  $\mathcal{H}^{(i)}$  the scalar product is given by

$$(2.6) \quad (\tilde{f}^{(i)}, \tilde{f}^{(i)}) = (f^{(i)}, f^{(i)}) + (f^{(i)}, f^{(i)}) = \int \tilde{f}^{(i)*}(P) \tilde{f}^{(i)}(P) \delta(P^2 - M_{(i)}^2) d^4P$$

and therefore  $\tilde{\mathcal{H}}^{(i)}$  is realized by function  $L^2$  on the two-sheet hyperboloid  $P^2 = M_{(i)}^2$ .

From Lorentz invariance it turns out that

$$(2.7) \quad T(LP^{(1)}, LP^{(2)}, LP^{(3)}, LP^{(4)}) = T(P^{(1)}, P^{(2)}, P^{(3)}, P^{(4)}) \quad \text{if } L \in O_{3,1}.$$

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(15) In order to obtain from the functional (2.5) the matrix element of *S-I* for one of the six processes we are considering we have to determine its arguments in the following way: *a*) if  $(i)$  is an incoming particle, put  $\tilde{f}^{(i)}(P) = f^{(i)}(P)\theta(P_0)$  where  $f^{(i)}(P)$  is the corresponding wave function ( $\theta$  is the step function); *b*) if  $(i)$  is an outgoing particle, put  $\tilde{f}^{(i)}(P) = f^{(i)}(-P)\theta(-P_0)$  where  $f^{(i)}(P)$  also represents the wave function. As a pure mathematical abstraction we consider also functions  $f^{(i)}(P)$  different from 0 for both positive and negative values of  $P_0$ .

It is convenient for our purposes to introduce a new set of variables as follows. We choose on the orbit  $Q^2 = W$  the four-vector

$$(2.8) \quad Q_W = \begin{cases} (0, 0, 0, \sqrt{-W}) & \text{if } W < 0, \\ (\sqrt{W}, 0, 0, 0) & \text{if } W > 0. \end{cases}$$

If  $L_Q, L_{Q'}$ , have the properties

$$(2.9) \quad L_Q Q_W = Q, \quad L_{Q'} Q_W = Q',$$

we can put

$$(2.10) \quad \begin{cases} P^{(1)} = L_Q \left( \frac{Q_W}{2} + P \right) = \frac{Q}{2} + L_Q P, \\ P^{(2)} = L_{Q'} \left( -\frac{Q_W}{2} - P \right) = -\frac{Q'}{2} - L_{Q'} P', \\ P^{(3)} = L_Q \left( \frac{Q_W}{2} - P \right) = \frac{Q}{2} - L_Q P, \\ P^{(4)} = L_{Q'} \left( -\frac{Q_W}{2} + P' \right) = -\frac{Q'}{2} + L_{Q'} P', \end{cases}$$

in the variables defined by (2.10) the multilinear functional (2.5) is written as

$$(2.11) \quad \theta(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}) = \frac{(2\pi)^4}{i} \iiint \delta^4(Q - Q') \cdot T \left( \frac{Q}{2} + L_Q P, -\frac{Q'}{2} - L_{Q'} P', \frac{Q}{2} - L_Q P, -\frac{Q'}{2} + L_{Q'} P' \right) \cdot \tilde{f}^{(1)} \left( \frac{Q}{2} + L_Q P \right) \tilde{f}^{(2)} \left( -\frac{Q'}{2} - L_{Q'} P' \right) \tilde{f}^{(3)} \left( \frac{Q}{2} - L_Q P \right) \tilde{f}^{(4)} \left( -\frac{Q'}{2} + L_{Q'} P' \right) \cdot \delta \left[ \left( \frac{Q}{2} + L_Q P \right)^2 - M_1^2 \right] \delta \left[ \left( -\frac{Q'}{2} - L_{Q'} P' \right)^2 - M_2^2 \right] \cdot \delta \left[ \left( \frac{Q}{2} - L_Q P \right)^2 - M_3^2 \right] \delta \left[ \left( -\frac{Q'}{2} + L_{Q'} P' \right)^2 - M_4^2 \right] d^4 P d^4 P' d^4 Q d^4 Q',$$

where we used the Lorentz invariance of the measure:  $d^4 P = d^4 L_Q P$ .

Let us write the particular process we want to consider, as follows:

$$(2.12) \quad (1) + (2) \rightarrow (3c) + (4c).$$

In order to describe this process at fixed momentum transfer it is convenient,

following Sect 4 of ref. <sup>(8)</sup>, to introduce the new Hilbert spaces <sup>(16)</sup>:

$$(2.13) \quad \begin{cases} \mathcal{H}^A = \tilde{\mathcal{H}}^{(1)} \otimes \tilde{\mathcal{H}}^{(3)}, \\ \mathcal{H}^B = \tilde{\mathcal{H}}^{(2)} \otimes \tilde{\mathcal{H}}^{(4)}, \end{cases}$$

whose elements we indicate with  $f^A$ ,  $f^B$  respectively. Of course only elements of the form  $f^A = \tilde{f}^{(1)} \otimes \tilde{f}^{(3)}$ ,  $f^B = J\tilde{f}^{(2)} \otimes J\tilde{f}^{(4)}$  are connected with quantities of physical meaning; as pure mathematical abstraction we shall use also finite and infinite sums of elements of this kind. A pseudobilinear functional  $\Phi(f^B, f^A)$  can be defined over the dense subspaces  $\mathcal{L}^A \subset \mathcal{H}^A$ ,  $\mathcal{L}^B \subset \mathcal{H}^B$  formed by elements of factorized type and by their finite sums by the relation:

$$(2.14) \quad \Phi(f^B, f^A) = \Phi(J\tilde{f}^{(2)} \otimes J\tilde{f}^{(4)}, f^{(1)} \otimes f^{(3)}) = \theta(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)}).$$

We call the elements of  $\mathcal{L}^A$ ,  $\mathcal{L}^B$  the pseudostates of the reaction. In the variables defined by (2.10) the wave functions of the factorized pseudostates are written as follows:

$$(2.15) \quad \begin{cases} f_Q^A(P) = \tilde{f}^{(1)} \left( \frac{Q}{2} + L_Q P \right) \tilde{f}^{(3)} \left( \frac{Q}{2} - L_Q P \right), \\ f_{Q'}^B(P') = \tilde{f}^{*(2)} \left( -\frac{Q'}{2} - L_{Q'} P' \right) \tilde{f}^{*(4)} \left( -\frac{Q'}{2} + L_{Q'} P' \right). \end{cases}$$

If we integrate in (2.11) over  $d^4 Q'$  by means of the  $\delta$ -function and take into account of the Lorentz invariance (2.7), the relation (2.14) becomes:

$$(2.16) \quad \begin{aligned} \Phi(f^B, f^A) &= \frac{(2\pi)^4}{i} \iiint \int f_Q^{*B}(P) \cdot \\ &\cdot T \left( \frac{Q_W}{2} + P, -\frac{Q_W}{2} - P', \frac{Q_W}{2} - P, -\frac{Q_W}{2} + P' \right) f_Q^A(P) \cdot \\ &\cdot \delta \left[ \left( \frac{Q_W}{2} + P \right)^2 - M_1^2 \right] \delta \left[ \left( \frac{Q_W}{2} - P' \right)^2 - M_2^2 \right] \cdot \\ &\cdot \delta \left[ \left( \frac{Q_W}{2} - P \right)^2 - M_3^2 \right] \delta \left[ \left( \frac{Q_W}{2} + P' \right)^2 - M_4^2 \right] d^4 P d^4 P' d^4 Q. \end{aligned}$$

We now decompose the Hilbert spaces  $\mathcal{H}^A$  and  $\mathcal{H}^B$ , as a direct integral, in the

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<sup>(16)</sup> See also F. T. HADJIOANNOU: *Nuovo Cimento*, **44** A, 185 (1966).



following way:

$$(2.17) \quad \begin{cases} \mathcal{H}^A = \int d^4Q \mathcal{H}_Q^A, \\ \mathcal{H}^B = \int d^4Q \mathcal{H}_Q^B, \end{cases}$$

then we have

$$(2.18) \quad \begin{cases} \|f^A\|^2 = \int d^4Q \|f_Q^A\|^2, \\ \|f^B\|^2 = \int d^4Q \|f_Q^B\|^2, \end{cases}$$

where  $f_Q^A, f_Q^B$  are elements of the spaces  $\mathcal{H}_Q^A, \mathcal{H}_Q^B$  with the norms

$$(2.19) \quad \begin{cases} \|f_Q^A\|^2 = \int |f_Q^A(P)|^2 \delta \left[ \left( \frac{Q_W}{2} + P \right)^2 - M_1^2 \right] \delta \left[ \left( \frac{Q_W}{2} - P \right)^2 - M_3^2 \right] d^4P, \\ \|f_Q^B\|^2 = \int |f_Q^B(P)|^2 \delta \left[ \left( \frac{Q_W}{2} - P' \right)^2 - M_4^2 \right] \delta \left[ \left( \frac{Q_W}{2} + P' \right)^2 - M_2^2 \right] d^4P'. \end{cases}$$

From (2.17) we have the following decomposition of (2.16):

$$(2.20) \quad \Phi(f^B, f^A) = \int d^4Q \Phi_Q(f_Q^B, f_Q^A),$$

where we define the  $Q$ -fixed functional as

$$(2.21) \quad \begin{aligned} \Phi_Q(f_Q^B, f_Q^A) &= \frac{(2\pi)^4}{i} \iint f_Q^{B*}(P') T(P', W, P) f_Q^A(P) \delta \left[ \left( \frac{Q_W}{2} + P' \right)^2 - M_2^2 \right] \\ &\cdot \delta \left[ \left( \frac{Q_W}{2} - P' \right)^2 - M_4^2 \right] \delta \left[ \left( \frac{Q_W}{2} - P \right)^2 - M_3^2 \right] \delta \left[ \left( \frac{Q_W}{2} + P \right)^2 - M_1^2 \right] d^4P d^4P' \end{aligned}$$

and we have put

$$T(P', W, P) = T \left( \frac{Q_W}{2} + P, -\frac{Q_W}{2} - P', \frac{Q_W}{2} - P, -\frac{Q_W}{2} + P' \right).$$

We now assume for simplicity

$$(2.22) \quad \begin{cases} M_1 = M_2 = M_a, \\ M_3 = M_4 = M_b, \end{cases}$$

and the reaction we consider is

$$(2.23) \quad a + \bar{a} \rightarrow b + \bar{b}.$$

Let us define the following sets:

$$(2.24) \quad \begin{cases} \Gamma^+(M_a, M_b) : Q^2 \geq (M_a + M_b)^2, \\ \Gamma^-(M_a, M_b) : Q^2 \leq (M_a - M_b)^2, \\ \Gamma(M_a, M_b) = \Gamma^+ \cup \Gamma^- (\Gamma^+ \cap \Gamma^- = \emptyset), \end{cases}$$

then we rewrite (2.20) as

$$(2.25) \quad \Phi(f^B, f^A) = \int_{\Gamma} d^4Q \Phi_q(f_q^B, f_q^A) = \int_{\Gamma^+} d^4Q \Phi_q(f_q^B, f_q^A) + \int_{\Gamma^-} d^4Q \Phi_q(f_q^A, f_q^B).$$

The first member of the second equality in (2.25) is the  $S$ - $T$  matrix element between the initial and final states of the reaction

$$(2.26) \quad a + b \rightarrow a + b \quad (\bar{a} + \bar{b} \rightarrow \bar{a} + \bar{b}).$$

$W$  is the square of the c.m. energy. The second member (which describes the scattering in the crossed channels) is a pseudobilinear functional on the « pseudostates » of the crossed reactions of (2.26):

$$(2.27) \quad \begin{cases} a + \bar{a} \rightarrow b + \bar{b} & (b + \bar{b} \rightarrow a + \bar{a}), \\ a + \bar{b} \rightarrow a + \bar{b} & (\bar{a} + b \rightarrow \bar{a} + b). \end{cases}$$

$W$  in this case is the square of the four-momentum transfer.

The aim of our investigation is the study of the region  $W < 0$  (which corresponds to the reactions (2.27)). We start by considering the  $Q$ -fixed functional defined by (2.20); successively we will return to the  $W$ -fixed one by means of an integration on the orbit  $Q^2 = W$ . Finally we will study the limit of  $T(P', W, P)$  for  $S = (P - P')^2 \rightarrow \infty$ , which corresponds to the high-energy behaviour of the first reaction in (2.27).

### 3. - Perturbative expansion of the pseudobilinear functional.

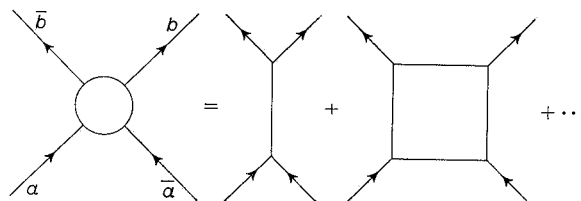
In the perturbative models that we consider the scattering amplitude  $T(P, Q_W, P')$  is obtained by iteration of a given amplitude:

$$(3.1) \quad \begin{array}{ccc} -\frac{Q}{2} + L_q P' & \searrow & \frac{Q}{2} - L_q P \\ & \circlearrowleft G & \\ -\frac{Q}{2} - L_q P' & \nearrow & \frac{Q}{2} + L_q P \end{array}$$

we call  $G(P, Q_W, P')$  the amplitude associated to this bubble.



following series:

(3.6) 

and in this case we have

(3.7) 
$$G(P, Q_w, P') = \frac{-g}{(P - P')^2 - \mu^2 + i\gamma},$$

where  $g$  is the square of the coupling constant  $g_0$  and  $\gamma > 0$ , as we assume the exchanged scalar particle of mass  $\mu$  to be unstable. In the theory given below we refer to an unspecified amplitude  $G(P', Q_w, P)$  and some conditions for its applicability are derived. These conditions are verified in the ladder-approximation case: we will show this in some detail in Appendix A.

Our purpose is now to look for a method which allows us to calculate the sum of the series defined by (3.5). In order to solve this problem it is essential to formulate it in a well-defined mathematical framework. In this connection it will be extremely useful to be able to relate our series to an operator series of the Neumann type. For this purpose it is important to observe that every bounded pseudobilinear functional can be written, from a general theorem <sup>(18)</sup>, as a matrix element of a bounded linear operator. This fact suggests to establish under which hypothesis the generical  $n$ -functional  $\Phi_{\alpha_n}(f_Q^B, f_Q^A)$  is bounded and then to study the corresponding operator series. We shall prove the uniform convergence of this series for a set of values of the coupling constant  $g$  and in this domain the uniform convergence of the series of the bounded functionals follows too. The graph series sum we obtain, can be extended outside this set, by means of an analytic continuation in  $g$ , using the results of Fredholm theory and of spectral analysis: this is the very important consequence of the employed formalism, because in the context of the Regge-pole theory, physical amplitudes correspond to higher values of the coupling constant, such that the series of the bounded functionals converges in a weak sense towards an unbounded functional. In order to develop our program we introduce the crossed partial-wave projection method. This decomposition is a very powerful mathematical tool as well as it clarifies the physical structure of the amplitude in terms of usual Regge parameters and it allows a new formulation of the Froissart bound in the present formalism.

<sup>(18)</sup> M. A. NAIMARK: *Normed Rings*, Chap. I.5, Sect. 3 (Gröningen, 1964).

We consider this procedure in the next Section but it is useful before to introduce a system of appropriate variables. We express a generical four-vector by two Lorentz-invariant variables and by two co-ordinates on the one- and two-sheet hyperboloids. Note that these hyperboloids are equivalent to the coset spaces  $O_{2,1}/O_{1,1}$  and  $O_{2,1}/O_2$  respectively <sup>(19)</sup>. To do this we firstly define the following transformation: given a four-vector  $P = (P_0, P_1, P_2, P_3)$  we put

$$(3.8) \quad \begin{cases} P_0 = pz, & p = \sqrt{|P_0^2 - P_1^2 - P_2^2|} \geq 0, \\ P_1 = p\sqrt{z^2 - \varrho} \cos \varphi, & \varrho = \frac{P_0^2 - P_1^2 - P_2^2}{p^2} = \pm 1, \\ P_2 = p\sqrt{z^2 - \varrho} \sin \varphi, & z^2 \geq \varrho, \quad 0 \leq \varphi < 2\pi, \\ P_3 = P_3. \end{cases}$$

We have for the Lorentz-invariant measure

$$(3.9) \quad d^4P = dP_0 dP_1 dP_2 dP_3 = p^2 dp d\hat{p} dP_3,$$

where  $dp = dz d\varphi$  is the invariant measure on the hyperboloid  $p^2 = \varrho$ . We now introduce the new system of variables

$$(3.10) \quad \begin{cases} \left(\frac{Q_W - P}{2}\right)^2 = u, \\ \left(\frac{Q_W + P}{2}\right)^2 = v, \\ z = z, \\ \varphi = \varphi. \end{cases}$$

Then we obtain

$$(3.11) \quad \begin{cases} p = p(u, v) = \sqrt{\frac{|(u-v)^2 + W^2 - 2W(u+v)|}{-4W}}, \\ \varrho = \varrho(u, v) = \frac{(u-v)^2 + W^2 - 2W(u+v)}{|(u-v)^2 + W^2 - 2W(u+v)|}, \\ P_3 = P_3(u, v) = \frac{u-v}{2\sqrt{-W}}, \end{cases}$$

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<sup>(19)</sup> See ref. <sup>(12)</sup>, Chapt 6.

and for the measure (3.9)

$$(3.12) \quad d^4P = \frac{p(u, v)}{4\sqrt{-W}} du dv d\hat{p};$$

on the mass-shell we have

$$(3.13) \quad \begin{cases} u = \bar{u} = M_a^2, \\ v = \bar{v} = M_b^2. \end{cases}$$

From (2.21) we write in the new variables

$$(3.14) \quad \begin{aligned} \Phi_q(f_q^B, f_q^A) &= \\ &= \frac{(2\pi)^4}{i} \iiint \iiint f_q^{B*}(u, v, \hat{p}) M(u, v, \hat{p} \cdot \hat{p}', u', v') f_q^A(u, v, \hat{p}) \frac{p(u, v)}{4\sqrt{-W}} \frac{p(u', v')}{4\sqrt{-W}} \\ &\cdot \delta(u - \bar{u}) \delta(v - \bar{v}) \delta(u' - \bar{u}') \delta(v' - \bar{v}') du dv d\hat{p} du' dv' d\hat{p}' = \\ &= \iint f_q^{B*}(\hat{p}) M_w(\hat{p} \cdot \hat{p}') f_q^A(\hat{p}) \frac{\bar{p}^2}{-16W} d\hat{p} d\hat{p}' \end{aligned}$$

and we have put

$$(3.15) \quad \begin{cases} M_w(\hat{p} \cdot \hat{p}') = M(\bar{u}, \bar{v}, \hat{p} \cdot \hat{p}', \bar{u}', \bar{v}'), \\ \bar{p} = p(\bar{u}, \bar{v}), \\ f_q^A(\hat{p}) = f_q^A(\bar{u}, \bar{v}, \hat{p}), \quad f_q^B(\hat{p}) = f_q^B(\bar{u}, \bar{v}, \hat{p}). \end{cases}$$

It is interesting to observe that the square of the c.m. energy is given by

$$(3.16) \quad S = (P - P')_{\text{on shell}}^2 = 2\bar{p}(1 - \hat{p} \cdot \hat{p}'),$$

we also write

$$(3.17) \quad \begin{cases} G(P, Q_w, P') = G_w(u, v, \hat{p} \cdot \hat{p}', u', v'), \\ K(P, Q_w, P') = \frac{1}{i(2\pi)^4} \frac{G_w(u, v, \hat{p} \cdot \hat{p}', v', u')}{(u - M_b^2 + i\varepsilon)(v - M_a^2 + i\varepsilon)}. \end{cases}$$

In short we put

$$(3.18) \quad \begin{cases} x = (u, v), \quad dx = \frac{p(u, v)}{4\sqrt{-W}} du dv, \quad \varrho(u, v) = \varrho(x), \\ \Delta(x) = (u - M_b^2 + i\varepsilon)(v - M_a^2 + i\varepsilon). \end{cases}$$

It is convenient now to define the new functions

$$(3.19) \quad A_W(x, \hat{p} \cdot \hat{p}') = G_W(x, \hat{p} \cdot \hat{p}', \bar{x}'),$$

$$(3.20) \quad B_W(\hat{p} \cdot \hat{p}', x') = K_W(\bar{x}, \hat{p} \cdot \hat{p}', x'),$$

so that finally we may rewrite (3.3) as

$$(3.21) \quad \Phi_{\mathfrak{q}_n}(f_{\mathfrak{q}}^B, f_{\mathfrak{q}}^A) = \int \dots \int f_{\mathfrak{q}}^{B*}(\hat{p}_1) B_W(\hat{p}_1 \cdot \hat{p}_2, x_2) K_W(x_2, \hat{p}_2 \cdot \hat{p}_3, x_3) \dots \\ \dots K_W(x_{n-1}, \hat{p}_{n-1} \cdot \hat{p}_n, x_n) A_W(x_n, \hat{p}_n \cdot \hat{p}_{n+1}) f_{\mathfrak{q}}^A(\hat{p}_{n+1}) \frac{\bar{p}^3}{-16W} \prod_{i=2}^n dx_i \prod_{i=1}^{n+1} d\hat{p}_i.$$

#### 4. - The crossed partial-wave analysis.

In this Section we perform the decomposition of the series of the pseudo-bilinear functionals (3.5) by means of the irreducible representations of the three-dimensional Lorentz group  $O_{2,1}$ . In order to do this we consider the unitary representations of  $O_{2,1}$  acting on Hilbert spaces formed by functions  $L^2$  on the coset spaces  $O_{2,1}/O_2$  and  $O_{2,1}/O_{1,1}$ . We consider the group  $O_{2,1}$  with the reflection operation  $t$  included, defined by

$$(4.1) \quad t\hat{p} = -\hat{p}.$$

The representations  $\mathcal{D}^e(g)$  we are considering act in the following way <sup>(20)</sup>:

$$(4.2) \quad \mathcal{D}^e(g)f^e(\hat{p}) = f^e(\hat{p}_g), \quad \hat{p}_g = g^{-1}\hat{p}, \quad g \in O_{2,1}.$$

The infinitesimal operators of the Lie group are  $L_0, L_1, L_2$  (with the analogue angular-momentum commutation relation) and the Casimir operators are  $L^2 = L_0^2 - L_1^2 - L_2^2$  and  $T$  ( $T = \mathcal{D}^e(t)$ ).

We consider the decomposition of the representations (4.2) under the group-chain  $O_{2,1} \supset O_2$  and we call  $D_m^{e\ell\tau}(\hat{p})$  the eigenfunctions of the operators  $L^2, L_0, T$ . They satisfy the following set of equations:

$$(4.3) \quad \left( \frac{\partial}{\partial z} (z^2 - \varrho) \frac{\partial}{\partial z} + \frac{1}{\varrho z^2 - 1} \frac{\partial^2}{\partial \varphi^2} \right) D_m^{e\ell\tau}(z, \varphi) = l(l+1) D_m^{e\ell\tau}(z, \varphi),$$

$$(4.4) \quad -i \frac{\partial}{\partial \varphi} D_m^{e\ell\tau}(z, \varphi) = m D_m^{e\ell\tau}(z, \varphi),$$

$$(4.5) \quad D_m^{e\ell\tau}(-z, \varphi \pm \pi) = \tau D_m^{e\ell\tau}(z, \varphi).$$

<sup>(20)</sup> See ref. (12).

These functions are explicitly given by

$$(4.6) \quad D_m^{1\tau}(z, \varphi) = C_{i\tau m}^1 [\theta(z-1)P_{lm}(z) + \tau e^{im\pi} \theta(-z-1)P_{lm}(-z)] e^{im\varphi},$$

$$(4.7) \quad D_m^{-1\tau}(z, \varphi) = C_{i\tau m}^{-1} [P_{lm}(iz) + \tau e^{im\pi} P_{lm}(-iz)] e^{im\varphi},$$

where  $P_{lm}(z)$  are the Legendre functions of the first kind <sup>(21)</sup>. If  $\text{Re } l = -\frac{1}{2}$  or  $\text{Im } l = 0$  the  $D_m^{e^{i\tau}}(\hat{p})$  form a basis for the irreducible representations of the principal and complementary series of  $O_{2,1}$  respectively <sup>(22)</sup> ( $l$  not integer); if  $l$  is any other complex number the functions (4.6) and (4.7) still define non-unitary irreducible representations of the group.

If we write the norms defined by (2.19) in the variables introduced in the previous Section we obtain

$$(4.8) \quad \begin{cases} \|f_{\mathfrak{Q}}^A\|^2 = \int_{p^2=1} |f_{\mathfrak{Q}}^A(\bar{u}, \bar{v}, \hat{p})|^2 \frac{\bar{p}}{4\sqrt{-W}} d\hat{p}, \\ \|f_{\mathfrak{Q}}^B\|^2 = \int_{p^2=1} |f_{\mathfrak{Q}}^B(\bar{u}, \bar{v}, \hat{p})|^2 \frac{\bar{p}}{4\sqrt{-W}} d\hat{p}, \end{cases}$$

therefore the Hilbert spaces  $\mathcal{H}_{\mathfrak{Q}}^A$  and  $\mathcal{H}_{\mathfrak{Q}}^B$  are realized by functions  $L^2$  on the two-sheet hyperboloid.

In order to avoid convergence difficulties we assume  $f_{\mathfrak{Q}}^A, f_{\mathfrak{Q}}^B \in L^1 \cap L^2$  which is a dense set in  $L^2$ . Then we have the projection formula

$$(4.9) \quad f_{\mathfrak{Q}m}^{A1\tau} = \int_{p^2=1} f_{\mathfrak{Q}}^A(\hat{p}) D_m^{1\tau}(\hat{p}) d\hat{p}.$$

This transformation can be defined in the strip  $-1 < \text{Re } l < 0$ . From the Plancherel formula for the group  $O_{2,1}$  (see Appendix A) the following completeness relation can be written

$$(4.10) \quad \|f_{\mathfrak{Q}}^A\|^2 = \int_{p^2=1} |f_{\mathfrak{Q}}^A(\hat{p})|^2 \frac{\bar{p}}{4\sqrt{-W}} d\hat{p} = \frac{1}{2} \frac{\bar{p}}{4\sqrt{-W}} \sum_{\tilde{\mathfrak{Q}}} \int_{m=-\infty}^{m=+\infty} |f_{\mathfrak{Q}m}^{A1\tau}|^2 \eta(l) dl,$$

where

$$(4.11) \quad \begin{cases} \tilde{\mathfrak{Q}} & \text{is the set } \text{Re } l = -\frac{1}{2}, \\ \eta(l) & = (2l+1) \text{ctg } l\pi. \end{cases}$$

<sup>(21)</sup> A. ERDELYI, W. MAGNUS, F. OBERHETTINGER and F. G. TRICOMI: *Higher Transcendental Functions*, vol. 1, Chap. 3 (New York, 1953).

<sup>(22)</sup> V. BARGMANN: *Ann. Math.*, **48**, 568 (1947); I. M. SHIROKOV: *Sov. Phys. JETP*, **6**, 919 (1959).



Formula (4.10) can be extended to an arbitrary  $L^2$  function. For brevity we write

$$\frac{1}{2} \sum_{\tau} \int d\tau \eta(l) = \int d\mathcal{A},$$

where  $\mathcal{A} = (l, \tau)$ ; we still have

$$(4.12) \quad \|f_{\mathcal{Q}}^A\|^2 = \frac{\bar{p}}{4\sqrt{-W}} \int_{\tilde{\Omega}} d\mathcal{A} \|f_{\mathcal{Q}}^{AA}\|^2,$$

with

$$(4.13) \quad \|f_{\mathcal{Q}}^{AA}\|^2 = \sum_m |f_{\mathcal{Q}_m}^{AA}|^2,$$

and we put

$$(4.14) \quad f_{\mathcal{Q}}^{AA} = \{f_{\mathcal{Q}_m}^{AA}\}.$$

From (4.12) and (4.13) we have obtained the direct integral decomposition<sup>(23)</sup>:

$$(4.15) \quad \mathcal{H}_{\mathcal{Q}}^A = \frac{\bar{p}}{4\sqrt{-W}} \int_{\tilde{\Omega}}^{\oplus} d\mathcal{A} \mathcal{H}_{\mathcal{Q}}^{AA}.$$

For  $f_{\mathcal{Q}}^B$  formulae analogue to (4.9), (4.10), (4.11), (4.12), (4.13), (4.14) and (4.15) hold too.  $\mathcal{H}_{\mathcal{Q}}^{AA}$  and  $\mathcal{H}_{\mathcal{Q}}^{BA}$  are the Hilbert spaces of the sequences like (4.14) with the norm (4.13). Note that the measure of the direct integral (4.15) has as a support only the right line  $\text{Re } l = -\frac{1}{2}$  (principal series). For our purposes, however, it is also necessary to consider the Hilbert spaces  $\mathcal{H}_{\mathcal{Q}}^{AA}$  and  $\mathcal{H}_{\mathcal{Q}}^{BA}$  corresponding to values of  $l$  in the strip  $-1 < \text{Re } l < 0$ .

In order to perform the projection of the pseudobilinear functional series (3.5) we now evaluate the convolution of the function  $G_w(x, \hat{p} \cdot \hat{p}', x')$  with the hyperbolic harmonics  $D_m^{ql}(\hat{p})$ .

The following theorem holds<sup>(24)</sup>:

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<sup>(23)</sup> We have so performed, in two steps, the following decomposition of our Hilbert spaces:  $\mathcal{H}^A \sim \int_{\mathcal{Q}^2 < 0} d^4 Q \int_{\tilde{\Omega}} d\mathcal{A} \mathcal{H}_{\mathcal{Q}}^{AA}$ . This decomposition does not coincide with the central decomposition obtained in Sect. 7 of ref. (8) of the reducible representations of the Poincaré group acting on  $\mathcal{H}^A$  and which is written as  $\mathcal{H}^A \sim \int_{w < 0} dW \int_{\tilde{\Omega}} d\mathcal{A} \mathcal{H}_w^{AA}$ . The connection between these different decompositions is obtained observing that:  $\mathcal{H}_w^{AA} \sim \int d\mu_w(Q) \mathcal{H}_{\mathcal{Q}}^{AA}$  (similar formulae hold for  $A \rightarrow B$ ). Moreover the spaces  $\mathcal{H}_{\mathcal{Q}}^A$  and  $\mathcal{H}_{\mathcal{Q}}^B$  are not Lorentz invariant but their introduction is useful in order to simplify the mathematical problem.

<sup>(24)</sup> S. FERRARA and G. MATTIOLI: Internal report, to be published.

If  $G_W(x, \hat{p} \cdot \hat{p}', x') \in L^p$ ,  $1 \leq p < 2$  in the variable  $\hat{p} \cdot \hat{p}'$ , almost everywhere in the variables  $x, x'$ , then we have in the strip  $C_p$ ,  $-1/p < \text{Re } l < -1 + 1/p$

$$(4.16) \quad \int_{p'^2 = \varrho'} G_W(x, \hat{p} \cdot \hat{p}', x') D_m^{\varrho(w)l\tau}(\hat{p}') d\hat{p}' = G_W^{l\tau}(x, x') D_m^{\varrho(w)l\tau}(\hat{p}),$$

where the function  $G_W^{l\tau}(x, x')$  is analytic in the strip  $C_p$  and can be evaluated by the projection formula (4.21) of ref. (13).

If we define the following function:

$$(4.17) \quad (A_W f_{\varrho}^A)(x, \hat{p}) = \int_{p'^2 = \mathbf{1}} A_W(x, \hat{p} \cdot \hat{p}') f_{\varrho}^A(\hat{p}') d\hat{p}',$$

we have, by means of an allowed exchange of the integration order (see ref. (24)):

$$(4.18) \quad (A_W f_{\varrho}^A)^{l\tau}(x) = A_W^{l\tau}(x) f_{\varrho_m}^{Al\tau}.$$

If we put

$$(4.19) \quad (B_W K_W^{n-2} A_W f_{\varrho}^A)(\hat{p}_1) = \int \dots \int B_W(\hat{p}_1 \cdot \hat{p}_2, x_2) K_W(x_2, \hat{p}_2 \cdot \hat{p}_3, x_3) \cdot K_W(x_{n-1}, \hat{p}_{n-1} \cdot \hat{p}_n, x_n) A_W(x_n, \hat{p}_n \cdot \hat{p}_{n+1}) f_{\varrho}^A(\hat{p}_{n+1}) \prod_{i=2}^{n+1} dx_i \prod_{i=2}^{n+1} d\hat{p}_i,$$

we obtain

$$(4.20) \quad (B_W K_W^{n-2} A_W f_{\varrho}^A)^{l\tau} = B_W^{l\tau} K_W^{l\tau n-2} A_W^{l\tau} f_{\varrho_m}^{Al\tau},$$

where

$$(4.21) \quad B_W^{l\tau} K_W^{l\tau n-2} A_W^{l\tau} f_{\varrho_m}^{Al\tau} = \int \dots \int B_W^{l\tau}(x_2) K_W^{l\tau}(x_2, x_3) \dots K_W^{l\tau}(x_{n-1}, x_n) A_W^{l\tau}(x_n) f_{\varrho_m}^{Al\tau} \prod_{i=2}^n dx^i.$$

If we define the following projected functional:

$$(4.22) \quad \Phi_{\varrho_n}^A(f_{\varrho}^{BA}, f_{\varrho}^{AA}) = \sum_m f_{\varrho_m}^{B^*l\tau} (B_W K_W^{n-2} A_W f_{\varrho}^A)^{l\tau} = \sum_m \int \dots \int f_{\varrho_m}^{B^*l\tau} B_W^{l\tau}(x_2) K_W^{l\tau}(x_1, x_3) \dots K_W^{l\tau}(x_{n-1}, x_n) A_W^{l\tau}(x_n) f_{\varrho_m}^{Al\tau} \prod_{i=2}^n dx^i,$$

we may write (see (4.10))

$$(4.23) \quad \Phi_{\varrho_n}(f_{\varrho}^B, f_{\varrho}^A) = \frac{\bar{p}^3}{-16W} \int_{\bar{\Sigma}} d\Lambda \Phi_{\varrho_n}^A(f_{\varrho}^{BA}, f_{\varrho}^{AA}).$$

(This equality holds only if the integral on the right side converges.) The projected functional (4.22) can be associated to the  $n$ -th-order fixed angular momentum  $l$  and fixed momentum transfer  $Q$  graph. In the next Section we establish some sufficient conditions for the boundedness of the functionals (4.23); to do this we will start by studying the boundedness of the projected functionals (4.22).

### 5. - Off-shell wave functions and boundness of the pseudobilinear functionals.

We introduce in this Section a suitable Hilbert space of projected off-shell wave functions  $f_{\mathcal{Q}_m}^{i\tau}(u, v)$ . The norm  $\|f_{\mathcal{Q}_m}^{i\tau}\|$  in this space has the property that the off-shell integral operators we consider are bounded with respect to this norm<sup>(25)</sup>. The Hilbert-space structure is described in detail in Appendix A. In this Section we indicate in a formal way  $\|f_{\mathcal{Q}_m}^{i\tau}\|$  this norm. Let  $f_{\mathcal{Q}}(u, v, \hat{p})$  a generical off-shell function defined almost everywhere in the space of the  $(u, v)$  variables, whose on-shell restriction is  $f_{\mathcal{Q}}(M_b^2, M_a^2, \hat{p}) = f_{\mathcal{Q}}(\hat{p})$ .

We define the following quantities:

$$(5.1) \quad f_{\mathcal{Q}_m}^{i\tau}(u, v) = \int_{v^2 = \mathcal{Q}(u, v)} f_{\mathcal{Q}}(u, v, \hat{p}) D_m^{e(u, v)i\tau}(\hat{p}) d\hat{p}$$

we call  $\mathcal{H}_{\mathcal{Q}}^A$  the Hilbert spaces of projected off-shell functions which satisfy the property

$$(5.2) \quad \sum_{m=-\infty}^{+\infty} \|f_{\mathcal{Q}_m}^A\|^2 < \infty,$$

where  $\|f_{\mathcal{Q}_m}^A\|$  is the suitable norm defined in the space of  $f_{\mathcal{Q}_m}^{i\tau}(u, v)$ .

If we call  $\mathcal{H}_{\mathcal{Q}_m}^A$  the Hilbert spaces with the norm  $\|f_{\mathcal{Q}_m}^A\|$  we obtain

$$(5.3) \quad \mathcal{H}_{\mathcal{Q}}^A = \bigoplus_m \mathcal{H}_{\mathcal{Q}_m}^A.$$

Consider now the functions  $K_W^A(x, x')$ ,  $A_W^A(x)$ ,  $B_W^A(x)$  defined by (3.17), (3.19) and (3.20); they act as kernels of integral transformations on  $\mathcal{H}_{\mathcal{Q}_m}^A$ ,  $\mathcal{H}_{\mathcal{Q}_m}^{AA}$  and  $\mathcal{H}_{\mathcal{Q}_m}^{BA}$  respectively. The Hilbert-Schmidt norm<sup>(26)</sup> of these transformations can be written explicitly as an integral (see Appendix A) which introduces a further

<sup>(25)</sup> This method is strictly connected to some techniques of regularization of the relativistic B-S equation, first introduced by I. G. TAYLOR (*Suppl. Nuovo Cimento*, **1**, 857 (1963)) and more recently developed in ref. (4). We apply these last results to the projected fixed-momentum-transfer off-shell wave functions.

<sup>(26)</sup> See ref. (9), part II, p. 1009.

condition on these functions. If these integrals converge these functions are the kernels of H-S operators in the Hilbert spaces we have introduced above and we have

$$(5.4) \quad A_w^A(x) \text{ defines a H-S operator } A_w^A \text{ from } \mathcal{H}_{\mathfrak{q}_m}^{AA} \text{ in } \mathcal{H}_{\mathfrak{q}_m}^A,$$

with domain  $\mathcal{D}_{A_w^A} = \mathcal{H}_{\mathfrak{q}_m}^{AA}$  and range  $\mathcal{R}_{A_w^A} \subseteq \mathcal{H}_{\mathfrak{q}_m}^A$ ;

$$(5.5) \quad B_w^A(x) \text{ defines a H-S operator } B_w^A \text{ from } \mathcal{H}_{\mathfrak{q}_m}^A \text{ in } \mathcal{H}_{\mathfrak{q}_m}^{BA}$$

with  $\mathcal{D}_{B_w^A} = \mathcal{H}_{\mathfrak{q}_m}^A, \mathcal{R}_{B_w^A} \subseteq \mathcal{H}_{\mathfrak{q}_m}^{BA}$ ;

$$(5.6) \quad K_w^{AA}(x, x') \text{ defines an H-S operator } K_w^A \text{ in } \mathcal{H}_{\mathfrak{q}_m}^A$$

with  $\mathcal{D}_{K_w^A} \subseteq \mathcal{D}_{K_w^A} = \mathcal{H}_{\mathfrak{q}_m}^A$ .

We shall indicate in the future by  $|A_w^A|, |B_w^A|, |K_w^A|$  the H-S norms of these operators.

For our purposes it is also necessary to introduce the on-shell operator  $\tilde{A}_w^A$  which corresponds to the kernel of the first graph (the irreducible one):  $G_w(\bar{x}, \hat{p} \cdot \hat{p}', \bar{x}') = A_w(\bar{x}, \hat{p} \cdot \hat{p}')$ . The definition of our norm ensures that the H-S condition for  $\tilde{A}_w^A$  follows from (5.4).

Note that the operators  $A_w^A, B_w^A$  and  $K_w^A$  cannot be of the H-S type on the spaces  $\mathcal{H}_{\mathfrak{q}}^{AA}, \mathcal{H}_{\mathfrak{q}}^{BA}$  and  $\mathcal{H}_{\mathfrak{q}}^A$  defined by a direct sum as in (5.3): *i.e.* consider the operator  $K_w^A$  on  $\mathcal{H}_{\mathfrak{q}}^A$ . The square of its complete H-S norm is

$$(5.7) \quad \sum_{mm'} |K_w^A|^2 \delta_{mm'} = \sum_m |K_w^A|^2,$$

which is always divergent.

However the propositions (5.4), (5.5) and (5.6) are sufficient for the boundedness of the operators on  $\mathcal{H}_{\mathfrak{q}}^{AA}, \mathcal{H}_{\mathfrak{q}}^{BA}$  and  $\mathcal{H}_{\mathfrak{q}}^A$ . In fact consider for example  $K_w^A$ , we obtain

$$(5.8) \quad \|K_w^A\| = \sup_{\{f_{\mathfrak{q}}^A \in \mathcal{H}_{\mathfrak{q}}^A\}} \frac{\|K_w^A f_{\mathfrak{q}}^A\|}{\|f_{\mathfrak{q}}^A\|} = \sup_{\{f_{\mathfrak{q}}^A \in \mathcal{H}_{\mathfrak{q}}^A\}} \frac{(\sum_m \|K_w^A f_{\mathfrak{q}_m}^A\|^2)^{\frac{1}{2}}}{(\sum_m \|f_{\mathfrak{q}_m}^A\|^2)^{\frac{1}{2}}} \leq \frac{(\sum_m |K_w^A|^2 \|f_{\mathfrak{q}_m}^A\|^2)^{\frac{1}{2}}}{(\sum_m \|f_{\mathfrak{q}_m}^A\|^2)^{\frac{1}{2}}} = |K_w^A|.$$

Thus we have that the H-S norm of  $K_w^A$  restricted on  $\mathcal{H}_{\mathfrak{q}_m}^A$  exceeds its norm on  $\mathcal{H}_{\mathfrak{q}}^A$ ,

If the H-S norms  $|A_w^A|, |B_w^A|, |K_w^A|$  converge we obtain, considering the expression (4.18), (4.20) and (4.22):

$$(5.9) \quad \begin{cases} |\Phi_{\mathfrak{q}_1}^A(f_{\mathfrak{q}}^{BA}, f_{\mathfrak{q}}^{AA})| \leq |\tilde{A}_w^A| \|f_{\mathfrak{q}}^{BA}\| \|f_{\mathfrak{q}}^{AA}\|, \\ |\Phi_{\mathfrak{q}_n}^A(f_{\mathfrak{q}}^{BA}, f_{\mathfrak{q}}^{AA})| \leq |B_w^A| |K_w^A|^{n-2} |A_w^A| \|f_{\mathfrak{q}}^{BA}\| \|f_{\mathfrak{q}}^{AA}\|. \end{cases}$$

From (5.9) we have the boundedness of the projected functionals defined by (4.22), in fact

$$(5.10) \quad \begin{cases} \|\Phi_{e_1}^A\| \leq |\tilde{A}_W^A|, \\ \|\Phi_{e_n}^A\| \leq |B_W^A| |K_W^A|^{n-2} |A_W^A|. \end{cases}$$

From (5.9) it turns out that the projected functionals may be also written as

$$(5.11) \quad \Phi_{e_n}^A(f_{e_n}^{BA}, f_{e_n}^{AA}) = g^n (f_{e_n}^{BA} T_{W_n}^A f_{e_n}^{AA}),$$

where the operators  $T_{W_n}^A$  defined as

$$(5.12) \quad \begin{cases} gT_{W_1}^A = \tilde{A}_W \\ g^n T_{W_n}^A = B_W^A K_W^{An-1} A_W^A, \end{cases}$$

are bounded.

If also the following conditions are verified:

$$(5.13) \quad \sup_{\{e_p\}} |A_W^A| = ga_W,$$

$$(5.14) \quad \sup_{\{e_p\}} |B_W^A| = gb_W,$$

$$(5.15) \quad \sup_{\{e_p\}} |K_W^A| = gc_W,$$

the pseudobilinear functionals  $\Phi_{e_n}(f_{e_n}^B, f_{e_n}^A)$  are uniformly bounded in  $\mathcal{A}$  in the strip  $C_p$  and we have:

$$(5.16) \quad \begin{cases} \|\Phi_{e_1}^A\| \leq ga_W, \\ \|\Phi_{e_n}^A\| \leq g^n a_W b_W c_W^{n-2}, \end{cases} \quad n = 2, 3, \dots$$

From (5.16) it follows that also the nonprojected  $n$ -th functional given by (3.21) is bounded, in fact <sup>(27)</sup>

$$(5.17) \quad \begin{cases} \|\Phi_{e_1}\| = \text{ess. max}_{\{\Omega\}} \|\Phi_{e_1}^A\| \leq ga_W, \\ \|\Phi_{e_n}\| = \text{ess. max}_{\{\Omega\}} \|\Phi_{e_n}^A\| \leq g^n a_W b_W c_W^{n-2}, \end{cases} \quad n = 2, 3, \dots$$

---

<sup>(27)</sup> We indicated by  $\text{ess. max } f(x)$  the least number  $C$  with the property that the set of values of  $x$  such that  $f(x) > C$  has vanishing measure. In general, if the functional  $\Phi$  is given by a direct integral  $\Phi = \int_{\oplus} d\mathcal{A} \Phi^A$ ,  $\|\Phi\| = \text{ess. max } \|\Phi^A\|$ .

From (5.17) it turns out that also the nonprojected functionals can be written as matrix elements of bounded operators as

$$(5.18) \quad \Phi_{\mathfrak{q}_n}(f_{\mathfrak{q}}^B, f_{\mathfrak{q}}^A) = g^n(f_{\mathfrak{q}}^B, T_{W_n} f_{\mathfrak{q}}^A),$$

where the bounded operators  $T_{W_n}$  are given by

$$(5.19) \quad \begin{cases} gT_{W_1} = \tilde{A}_W, \\ gT_{W_n} = B_W K_W^{n-2} A_W. \end{cases}$$

Note that the functionals  $\Phi_{\mathfrak{q}_n}$ , which are defined only if their arguments  $f_{\mathfrak{q}}^A, f_{\mathfrak{q}}^B$  can be derived (by means of a direct integral decomposition) from elements belonging to the dense spaces  $\mathcal{L}^A$  and  $\mathcal{L}^B$ , can be extended, by means of formula (5.18), to generic elements  $f_{\mathfrak{q}}^B, f_{\mathfrak{q}}^A$  of the spaces  $\mathcal{H}_{\mathfrak{q}}^B, \mathcal{H}_{\mathfrak{q}}^A$ .

## 6. – The convergence of the series of bounded functionals.

The uniform convergence of the series of the bounded pseudobilinear functionals (3.21) follows from (5.17) in the open set  $g < 1/c_W$ . The sum  $\Phi_{\mathfrak{q}} = \sum_{n=1}^{\infty} \Phi_{\mathfrak{q}_n}$  is a bounded functional on  $\mathcal{H}_{\mathfrak{q}}^B \times \mathcal{H}_{\mathfrak{q}}^A$  and for its norm we have the inequality

$$(6.1) \quad \|\Phi_{\mathfrak{q}}\| \leq g a_W \left( 1 + g b_W + g^2 \frac{b_W c_W}{1 - g c_W} \right).$$

The functional  $\Phi_{\mathfrak{q}}$  is so generated by a bounded operator  $T_W$  which we call the  $W$ -fixed crossed scattering operator.

For larger values of the coupling constant in general the series of the functionals does not converge in norm towards a bounded functional. However we can still define an unbounded and not everywhere defined functional putting

$$(6.2) \quad \Phi_{\mathfrak{q}}(f_{\mathfrak{q}}^B, f_{\mathfrak{q}}^A) = \sum_{n=1}^{\infty} \Phi_{\mathfrak{q}_n}(f_{\mathfrak{q}}^B, f_{\mathfrak{q}}^A) = \sum_{n=1}^{\infty} g^n(f_{\mathfrak{q}}^B, T_{W_n} f_{\mathfrak{q}}^A).$$

This functional is defined only for pairs of the arguments such that the series on the right-hand side converges (weak convergence). If this series converges we can compute it for small values of  $g$ , such that the series converges in the uniform topology, and then continue it analytically to the interesting value of  $g$ ; this is just the sum if it converges. It will be pointed out that this technique can be applied only if the weak convergence is restricted to the spaces  $\mathcal{L}^B$  and

$\mathcal{L}^A$ . However this is sufficient and in fact these spaces just contain the elements of physical meaning (of factorized type).

The analytic continuation, in  $g$  can be performed using the crossed partial-wave projection, therefore we start by considering the series of the projected functionals

$$(6.3) \quad \sum_{n=1}^{\infty} \Phi_{\varrho_n}^A(f_{\varrho}^{BA}, f_{\varrho}^{AA}) = \sum_{n=1}^{\infty} g^n (f_{\varrho}^{BA}, T_W^A f_{\varrho}^{AA}).$$

If  $g < 1/c_W$ ,  $l \in C_p$  we can write the sum of (6.3) as

$$(6.4) \quad \varphi_{\varrho}^A(f_{\varrho}^{BA}, f_{\varrho}^{AA}) = (f_{\varrho}^{BA}, T_W^A f_{\varrho}^{AA}),$$

where  $T_W^A$  is a bounded operator given by

$$(6.5) \quad T_W^A = \tilde{A}_W^A + B_W^A A_W^A + \sum_{n=3}^{\infty} B_W^A K_W^{A^{n-2}} A_W^A,$$

and the right-hand side converges in the uniform topology (see (5.13), (5.14) and (5.15)). In the open set  $g < 1/c_W$ ,  $l \in C_p$  the series (6.5) converges uniformly with respect to  $g$  and  $l$  <sup>(28)</sup>. If we call  $R_W^A$  the resolvent <sup>(29)</sup> of  $K_W^A$  defined by

$$(6.6) \quad R_W^A = K_W^A (I - K_W^A)^{-1},$$

we write (6.5) as

$$(6.7) \quad T_W^A = \tilde{A}_W^A + B_W^A A_W^A + B_W^A R_W^A A_W^A$$

and the norm of the operator  $T_W^A$  satisfies the inequality

$$(6.8) \quad \|T_W^A\| \leq g a_W \left( 1 + g b_W + g^2 \frac{b_W c_W}{1 - g c_W} \right).$$

From the Fredholm theory we have, under our hypothesis, that <sup>(30)</sup>: the resolvent operator  $R_W^A$  is analytic for  $l \in C_p$ ,  $g < 1/c_W$ , meromorphic for  $l \in C_p$ ,  $g > 1/c_W$  and it is an integral operator whose kernel we indicate by  $R_W^A(x, x')$ .

The operator  $T_W^A(g)$  (we have explicitated its dependence on  $g$ ), which is

<sup>(28)</sup> We have in fact:

$$\|(K_W^{A^{n+1}} + \dots + K_W^{A^{n+m}})\| \leq (|K_W^A|^{n+1} + \dots + |K_W^A|^{n+m}) \leq (g c_W)^{n+1} (1 + g c_W + \dots + (g c_W)^{m-1}).$$

<sup>(29)</sup> See ref. <sup>(9)</sup> and also: F. RIESZ and B. NAGY: *Leçons d'analyse fonctionnelle* (Paris, 1965).

<sup>(30)</sup> See ref. <sup>(29)</sup>.

a multiple of the unit operator on  $\mathcal{H}_q^{AA}$ , is explicitly written as

$$(6.9) \quad T_W^{lr}(g) = A_W^{lr}(\bar{x}) + \int dx A_W^{lr}(x) B_W^{lr}(x) + \iint dx dx' B_W^{lr}(x) R_W^{lr}(x, x') A_W^{lr}(x').$$

The function (6.9) is analytic in the open set  $l \in C_p$ ,  $g < 1/c_W$ , meromorphic for  $l \in C_p$ ,  $g > 1/c_W$ . Of course we have

$$\|T_W^A(g)\| = |T_W^A(g)|.$$

The sum (6.4) of the series (6.3) is given by

$$(6.10) \quad \Phi_q^A(f_q^{BA}, f_q^{AA}) = T_W^A(g)(f_q^{BA}, f_q^{AA}).$$

From the above properties and using the completeness relation (4.23) we derive for  $g < 1/c_W$ :

$$(6.11) \quad \begin{aligned} \Phi_q(f_q^B, f_q^A) &= \sum_{n=1}^{\infty} \Phi_{q_n}(f_q^B, f_q^A) = \frac{\bar{p}^2}{-16W} \sum_{n=1}^{\infty} \int_{\bar{Q}} d\Lambda \Phi_{q_n}^A(f_q^{BA}, f_q^{AA}) = \\ &= \frac{\bar{p}^2}{-16W} \int_{\bar{Q}} d\Lambda \sum_{n=1}^{\infty} \Phi_{q_n}^A(f_q^{BA}, f_q^{AA}) = \frac{\bar{p}^2}{-16W} \int_{\bar{Q}} d\Lambda T_W^A(g)(f_q^{BA}, f_q^{AA}). \end{aligned}$$

For more generality we now consider the  $W$ -fixed pseudobilinear functional

$$(6.12) \quad \Phi_W(f_W^B, f_W^A) = \int d\mu_W(Q) \Phi_q(f_q^B, f_q^A); \quad d\mu_W(Q) = \delta(Q^2 - W) d^4Q.$$

$f_W^A, f_W^B$  are elements of the Lorentz invariant Hilbert spaces <sup>(31)</sup>:

$$(6.13) \quad \mathcal{H}_W^A = \int^{\oplus} d\mu_W(Q) \mathcal{H}_q^A,$$

with the norm

$$(6.14) \quad \|f_W^A\|^2 = \int d\mu_W(Q) \|f_q^A\|^2$$

and similar formulae for  $f_W^B$ .

For  $g < 1/c_W$  the  $W$ -fixed functional is bounded and it is

$$(6.15) \quad \|\Phi_W\| = \text{ess. max}_{(q^2=W)} \|\Phi_q\| \leq g a_W \left( 1 + g b_W + g^2 \frac{b_W c_W}{1 - g c_W} \right).$$

<sup>(31)</sup> See footnote <sup>(23)</sup>.



Then we have <sup>(32)</sup>

$$(6.16) \quad \Phi_w(f_w^B, f_w^A) = (f_w^B, T_w(g)f_w^A).$$

From the Fubini theorem we can write

$$(6.17) \quad \begin{aligned} \Phi_w(f_w^B, f_w^A) &= \frac{\bar{p}^2}{-16W} \int \tilde{d}\mu_w(Q) \int_{\tilde{\Omega}} d\Lambda T_w^A(g)(f_w^{BA}, f_w^{AA}) = \\ &= \int_{\tilde{\Omega}} d\Lambda T_w^A(g) \int \tilde{d}\mu_w(Q)(f_w^{BA}, f_w^{AA}) \frac{\bar{p}^2}{-16W} = \int_{\tilde{\Omega}} d\Lambda T_w^A(g) \Psi_w^A, \end{aligned}$$

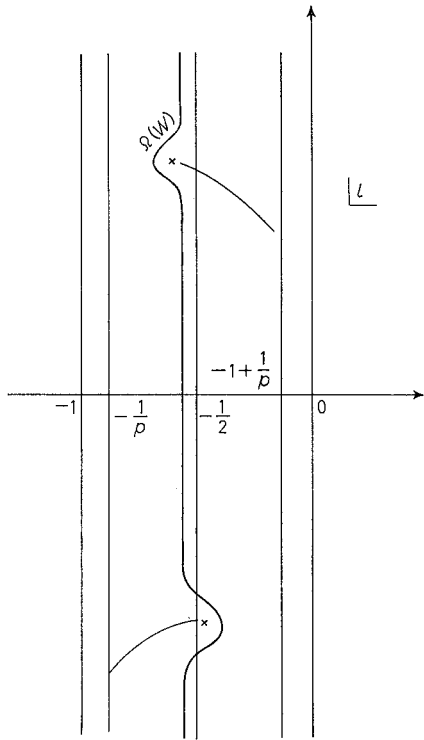


Fig. 1.

where the quantities  $\Psi_w^A = \int \tilde{d}\mu_w(Q) \cdot (f_w^{BA}, f_w^{AA}) (\bar{p}^2 / -16W)$  are the projections of elements belonging to some normed spaces <sup>(33)</sup>. It is shown in ref. <sup>(8)</sup> that the functions  $\Psi_w^A$  are analytic in the strip  $-2 < \text{Re } l \leq 1$  if  $f^A, f^B \in \mathcal{L}^A, \mathcal{L}^B$  respectively.

These remarks enable us to consider the series of the functionals  $\sum_{n=1}^{\infty} \Phi_w(f_w^B, f_w^A)$  for  $g > 1/c_w$ . For these values of the coupling constant the series does not uniformly converge, the functional (6.12) is not bounded and the crossing scattering operator does not exist. However the quantity (6.12) has a physical meaning if its arguments  $f_w^B, f_w^A$  can be derived from pseudostates  $f^B, f^A$  of factorized type <sup>(34)</sup>. Therefore we may evaluate  $\Phi_w(f_w^B, f_w^A)$  for  $g > 1/c_w$  performing an analytic continuation in  $g$  using the formula (6.17). This continuation is possible

<sup>(32)</sup> See ref. <sup>(18)</sup>.

<sup>(33)</sup> This result is found in Sect. 8 and 9 of ref. <sup>(8)</sup>.

<sup>(34)</sup> More explicitly it turns out, from the unitarity of the  $S$ -matrix that, for  $f^B$  and  $f^A$  of factorized type,

$$|\Phi(f^B, f^A)| = \left| \int dW \Phi_w(f_w^B, f_w^A) \right| = |\theta(\tilde{f}^{(1)}, \tilde{f}^{(2)}, \tilde{f}^{(3)}, \tilde{f}^{(4)})| \leq 2 \|\tilde{f}^{(1)}\| \|\tilde{f}^{(2)}\| \|\tilde{f}^{(3)}\| \|\tilde{f}^{(4)}\|$$

and therefore the quantity  $\Phi_w(f_w^B, f_w^A)$  is finite (almost everywhere for fixed  $W$ ).

from the analyticity properties of  $\Psi_W^A$  and of  $T_W(g)$  that we have derived.

Note that if a Regge pole has a real part larger than one the analytic continuation is not in general possible and the functional (6.12) becomes divergent even in correspondence with physically significant (factorized) pseudostates. This is a formulation of the Froissart bound in the present formalism.

The analytic continuation of (6.17) is obtained by a modification of the integration path as in the Figure, when the poles of  $T_W^A(g)$  cross the line  $\text{Re } l = -\frac{1}{2}$ :

$$(6.18) \quad \Phi_W(f_W^B, f_W^A) = \int_{\Omega(W)} d\Lambda T_W^A(g) \Psi_W^A, \quad \Omega(W) \subset C_2.$$

Note that if  $f^A$  and  $f^B$  do not belong to  $\mathcal{L}^A$  and  $\mathcal{L}^B$  the function  $\Psi_W^A$  is no longer analytic and the integration path cannot be modified, so that the functional  $\Phi_W$  need not to be defined for arguments of this kind.

In order to show the Regge behaviour of the  $W$ -fixed scattering amplitude we compare the usual expression

$$(6.19) \quad \Phi_W(f_W^B, f_W^A) = \iiint d\mu_W(Q) d\hat{p} d\hat{p}' f_0^{*B}(\hat{p}) T_W(\hat{p} \cdot \hat{p}') f_0^A(\hat{p}') \frac{\bar{P}^2}{-16W},$$

with the formulae (6.17) and we obtain

$$(6.20) \quad T_W(z) = \int_{\Omega(W)} d\Lambda T_W^A(g) D_0^A(z) = \sum_{\tau} \int_{\Omega(W)} T_W^{\tau}(g) D_0^{\tau}(z) (2l+1) \text{ctg } \varrho \pi d\varrho = \\ = \int_{\Omega(W)} [T_W^{l,1}(g) \pm T_W^{l,-1}(g)] Q_{-l-1}(\pm z) (2l+1) dl, \quad \pm z \geq 1, \quad (z = p \cdot p'),$$

where the last step derives from some well-known properties of the Legendre functions.

We observe that  $S$  (see (3.16)) is linearly related to  $z$  and

$$(6.21) \quad Q_{-l-1}(z) \xrightarrow{z \rightarrow \infty} \frac{\Gamma(-l)}{\Gamma(-l + \frac{1}{2})} (2z)^l$$

then, in the high-energy limit, the main contribution to (6.20) comes from the pole  $l_0(W)$  of  $T_W^{\tau}(g)$  with the greatest real part:

$$(6.22) \quad T_W(z) \underset{z \rightarrow \infty}{\sim} \beta'(W) z^{l_0(W)}$$

and this is the Regge asymptotic behaviour.

\* \* \*

We thank Prof. M. TOLLER and Dr. G. COSENZA for many stimulating discussions and helpful correspondence.

## APPENDIX A

In this Appendix we explain in some detail the Hilbert-space structure which we introduce in the space of the projected off-shell wave functions. So we can write down in an explicit way the conditions on the H-S norms of our operator. Finally we discuss in short how these conditions are verified. Convergence difficulties arise from the on-shell singularities of the kernel  $K_{\mathbb{W}}^A(x, x')$ : in fact in the usual  $L^2$ -norm the integral operator  $K_{\mathbb{W}}^A$  is never bounded. In order to avoid these difficulties we define the scalar product in the space of the projected off-shell functions  $f_{\mathbf{Q}_m}^A(u, v)$ , in such a way that the singularities are subtracted.

Starting from a function  $f_{\mathbf{Q}_m}^A(u, v)$  defined almost everywhere in the  $(u, v)$  variables and defined on shell let us introduce the new quantities

$$(A.1) \quad f_{\mathbf{Q}_m}^{A(1)} = f_{\mathbf{Q}_m}^A(\bar{u}, \bar{v}),$$

$$(A.2) \quad f_{\mathbf{Q}_m}^{A(2)}(u) = \frac{f_{\mathbf{Q}_m}^A(u, \bar{v}) - f_{\mathbf{Q}_m}^A(\bar{u}, \bar{v})\alpha(u)}{u - \bar{u}} [a(u)]^2,$$

$$(A.3) \quad f_{\mathbf{Q}_m}^{A(3)}(v) = \frac{f_{\mathbf{Q}_m}^A(\bar{u}, v) - f_{\mathbf{Q}_m}^A(\bar{u}, \bar{v})\beta(v)}{v - \bar{v}} [b(v)]^2,$$

$$(A.4) \quad f_{\mathbf{Q}_m}^{A(4)}(u, v) = \frac{f_{\mathbf{Q}_m}^A(u, v) - f_{\mathbf{Q}_m}^A(u, \bar{v})\beta(v) - f_{\mathbf{Q}_m}^A(\bar{u}, v)\alpha(u) + f_{\mathbf{Q}_m}^A(\bar{u}, \bar{v})\alpha(u)\beta(v)}{(u - \bar{u})(v - \bar{v})} [a(u)b(v)]^2,$$

where  $\alpha(u)$  is a suitable real function of  $u$ , belonging to  $C^1$ , whose support is a compact set  $\Delta u = (u_1, u_2)$  which contains the point  $u = \bar{u}$  where  $\alpha(\bar{u}) = 1$ ;  $\beta(v)$  has similar properties.

Inverting these definitions we obtain

$$(A.5) \quad f_{\mathbf{Q}_m}^A(u, v) = f_{\mathbf{Q}_m}^{A(1)}\alpha(u)\beta(v) + f_{\mathbf{Q}_m}^{A(2)}(u) \frac{(u - \bar{u})\beta(v)}{[a(u)]^2} + \\ + f_{\mathbf{Q}_m}^{A(3)}(v) \frac{(v - \bar{v})\alpha(u)}{[b(v)]^2} + f_{\mathbf{Q}_m}^{A(4)}(u, v) \frac{(u - \bar{u})(v - \bar{v})}{[a(u)b(v)]^2}.$$

In order to simplify the formalism let us call  $r_2 = u$ ,  $r_3 = v$ ,  $r_4 = (u, v)$ ; of course  $r_1$  is not a true variable but its introduction is convenient for uniformity of notation. In this way the four quantities associated to every off-

shell function  $f_{\mathfrak{Q}_m}^A(u, v)$  may be written in a more compact form  $\{f_{\mathfrak{Q}_m}^{A(i)}(r_i)\}$ . Let us consider now the set  $\mathcal{H}_{\mathfrak{Q}_m}^A$  of the functions  $f_{\mathfrak{Q}_m}^A(u, v)$  which satisfy the condition

$$(A.6) \quad \sum_{i=1}^4 \int |f_{\mathfrak{Q}_m}^{A(i)}(r_i)|^2 dr_i < \infty.$$

$\mathcal{H}_{\mathfrak{Q}_m}^A$  becomes a Hilbert space by introducing the following scalar product:

$$(A.7) \quad (f_{\mathfrak{Q}_m}^A, g_{\mathfrak{Q}_m}^A) = \sum_{i=1}^4 \int f_{\mathfrak{Q}_m}^{*A(i)}(r_i) g_{\mathfrak{Q}_m}^{A(i)}(r_i) dr_i;$$

from (A.6) and (A.7) we observe that  $f_{\mathfrak{Q}_m}^A \in \mathcal{H}_{\mathfrak{Q}_m}^A$  if and only if the quantities  $f_{\mathfrak{Q}_m}^{A(i)}$  belong respectively to the Hilbert spaces  $\mathcal{H}_{\mathfrak{Q}_m}^{A(i)}$  formed by the functions  $f_{\mathfrak{Q}_m}^{A(i)}(r_i)$  which are  $L^2$  in their variables. It is clear that the previous definitions establish an isomorphism between the spaces  $\mathcal{H}_{\mathfrak{Q}_m}^A$  and  $\mathcal{H}_{\mathfrak{Q}_m}^A = \bigoplus_i \mathcal{H}_{\mathfrak{Q}_m}^{A(i)}$ . By means of this isomorphism we associate to the operators  $K_W^A, A_W^A, B_W^A$  the following operators:

$$(A.8) \quad \begin{cases} \hat{K}_W^A = UK_W^A U^{-1}, & \text{from } \hat{\mathcal{H}}_{\mathfrak{Q}_m}^A \text{ in } \mathcal{H}_{\mathfrak{Q}_m}^A, \\ \hat{A}_W^A = UA_W^A, & \text{from } \hat{\mathcal{H}}_{\mathfrak{Q}_m}^{AA} \text{ in } \hat{\mathcal{H}}_{\mathfrak{Q}_m}^A, \\ \hat{B}_W^A = B_W^A U^{-1}, & \text{from } \hat{\mathcal{H}}_{\mathfrak{Q}_m}^A \text{ in } \mathcal{H}_{\mathfrak{Q}_m}^{BA}, \end{cases}$$

where  $U$  is an isometric operator with inverse. Formulae (A.8) enable us to study the H-S norms of the corresponding operators  $\hat{K}_W^A, \hat{A}_W^A, \hat{B}_W^A$ . This is much simpler because in this representation the operators  $\hat{K}_W^A, \hat{A}_W^A, \hat{B}_W^A$  are written as matrices of nonsingular integral operators.

For example the equation

$$(A.9) \quad g_{\mathfrak{Q}_m}^A(u, v) = \int K_W^A(u, v, u', v') f_{\mathfrak{Q}_m}^A(u', v') du' dv'$$

becomes

$$(A.10) \quad g_{\mathfrak{Q}_m}^{A(i)}(r_i) = \sum_{j=1}^4 \int K_W^{A(i,j)}(r_i, r'_j) f_{\mathfrak{Q}_m}^{A(j)}(r'_j) dr'_j, \quad i = 1, 2, 3, 4,$$

and the H-S norm of the operator  $K_W^A$  is given by

$$(A.11) \quad |K_W^A| = |\hat{K}_W^A| = \left( \sum_{i,j=1}^4 \iint dr_i dr'_j |K_W^{A(i,j)}(r_i, r'_j)|^2 \right)^{\frac{1}{2}}.$$

The norms of the operator  $A_W^A, B_W^A$  are written in a similar way.

Let us consider now the case of the ladder approximation. Then the function  $G_W(u, v, \hat{p} \cdot \hat{p}', u', v')$  has the form (see (3.7))

$$(A.12) \quad G_W(u, v, \hat{p} \cdot \hat{p}', u', v') = \frac{-g}{\frac{1}{2}(u+v) + \frac{1}{2}(u'+v') - (1/2W)(u-v)(u'-v') - 2p(u, v)p(u', v') \hat{p} \cdot \hat{p}' - \mu^2 - W/2 + i\gamma}.$$

In general a convenient situation, in order to study the integral conditions (A.11), is obtained by transforming the kernel of the integral operator  $K_W^4$  so that it assumes a symmetric asymptotic behaviour with respect to the two group of variables  $u, v$  and  $u', v'$ . This can be achieved by means of the following transformation on the function  $G_W(u, v, \hat{p} \cdot \hat{p}', u', v')$ :

$$(A.13) \quad G(u, v, \hat{p} \cdot \hat{p}', u', v') = \Gamma(u, v) G(u, v, \hat{p} \cdot \hat{p}', u', v') \frac{1}{\Gamma(u', v')},$$

where

$$\Gamma(u, v) = \frac{\sqrt{p(u, v)}}{a(u)b(v)}, \quad a(u) = (\sqrt{(u-\bar{u})^2 + 1})^{\frac{1}{2}}, \\ b(v) = (\sqrt{(v-\bar{v})^2 + 1})^{\frac{1}{2}}.$$

It is immediate to observe that the  $n$ -th functional (see (3.21)) is unchanged and the required symmetry is now verified.

If we apply the theorem on the convolution with the hyperbolic harmonics given in Sect. 4 (see (4.16)) to the function (4.12) we obtain for its Laplace transform <sup>(35)</sup>

$$(A.14) \quad G_W^{lr}(u, v, u', v') = \frac{g}{2p(u, v)p(u', v')} V_W^{lr}(u, v, u', v'),$$

where

$$(A.15) \quad \left\{ \begin{array}{l} V_W^{lr}(u, v, u', v') = \frac{-2\pi^2}{\sin l\pi} [P_i(-z) - \tau P_i(z)], \quad \text{if } \varrho(u, v) = \varrho(u', v') = 1, \\ V_W^{lr}(u, v, u', v') = \frac{2\pi^2 i}{\sqrt{\sin l\pi}} \tau P_i(-iz), \quad \text{if } \varrho(u, v) = -\varrho(u', v') = 1, \\ V_W^{lr}(u, v, u', v') = \frac{-4\pi^2 i}{\sqrt{\sin l\pi}} P_i(-iz), \quad \text{if } \varrho(u, v) = -\varrho(u', v') = -1, \\ V_W^{lr}(u, v, u', v') = \frac{-2\pi^2 i}{\sin l\pi} [P_i(z) + \tau P_i(-z)], \\ \hspace{15em} \text{if } \varrho(u, v) = \varrho(u', v') = -1, \end{array} \right.$$

and

$$z = z_W(u, v, u', v') = \\ = \frac{\frac{1}{2}(u+v) + \frac{1}{2}(u'+v') - (1/2W)(u-v)(u'-v') - 2p(u, v)p(u', v') - \frac{1}{2}W - \mu^2}{2p(u, v)p(u', v')} + \\ + i \frac{\gamma}{2p(u, v)p(u', v')}.$$

The functions  $V_W^{lr}(u, v, u', v')$  are analytic in every strip  $C_\delta$ :  $-1 + \delta < \text{Re } l < -\delta$  ( $0 < \delta < \frac{1}{2}$ ). In ref. <sup>(36)</sup> it is shown that the norm defined by (A.11) is convergent for the function (A.14), therefore in the ladder approximation the (5.4), (5.5)

<sup>(35)</sup> See Sect. 4 of ref. (13).

<sup>(36)</sup> S. FERRARA: Thesis, Rome (1968).

and (5.6) are verified. Moreover it is shown that in the strip  $C_\delta$  the following condition is verified:

$$(A.16) \quad |\widehat{K}_W^A| < g c_W \quad (\text{and similar expressions hold for the operators } \widehat{A}_W^A \text{ and } \widehat{B}_W^A)$$

and so the conditions (5.13), (5.14) and (5.15) hold too. Then the results we have pointed out in Sect. 6 are valid in our oversimplified model: ladder graph series converges and its sum verifies the Regge behaviour and the Froisart bound.

### APPENDIX B

We now derive the completeness relation (4.10) by means of the Plancherel formula for the group  $O_{2,1}$  <sup>(37)</sup>.

The Haar measure  $dg$  on  $O_{2,1}$  and the invariant measure on  $O_{2,1}/O_2$  (two-sheet hyperboloid) are so related

$$(B.1) \quad \int_{|z| \geq 1} \int_0^{2\pi} f(z, \varphi) dz d\varphi = \frac{1}{2\pi} \int f(g) dg .$$

From the Appendix B of ref. <sup>(38)</sup> we have

$$(B.2) \quad P_{lm}(\cosh \zeta) = \frac{\Gamma(l+1)}{\Gamma(l-m+1)} d_{m0}^l(\zeta) ,$$

then it turns out that

$$(B.3) \quad P_{lm}(\cosh \zeta) \exp[im\varphi] = \mathcal{D}_{m0}^l(g) ,$$

where  $\mathcal{D}_{m0}^l(g)$  are the matrix elements of the analytic irreducible representations of  $SU_{1,1}$ . Remembering now the explicit expression for the hyperbolic harmonics (4.6) and choosing

$$C_{lm}^1 = \frac{\Gamma(l-m+1)}{\Gamma(l+1)} ,$$

we have

$$(B.4) \quad \int_{p^2=1} f(\widehat{p}) D_m^{l\tau}(\widehat{p}) d\widehat{p} = \frac{1}{2\pi} \int_{(SU_{1,1})} f(g) \mathcal{D}_{m0}^{l\tau}(g) dg = f_m^{l\tau} ,$$

<sup>(37)</sup> See Appendix B of ref. <sup>(8)</sup>, formula (B.11).

<sup>(38)</sup> This formula is easily obtained comparing formula (B.9) of Appendix B of ref. <sup>(7)</sup> and formula 3 of ref. <sup>(21)</sup>, Subsect. 3'2.

where  $t$  is the reflection and  $(SU_{1,1}, t)$  is the group  $SU_{1,1}$  with reflection. Therefore we obtain

$$(B.5) \quad \int_{p^2=1} |f(\hat{p})|^2 d\hat{p} = \frac{1}{2\pi} \int_{(SU_{1,1}, t)} |f(g)|^2 dg = \frac{1}{4\pi} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \sum_m |f_m^{lr}|^2 (2l+1) \operatorname{ctg} l\pi dl.$$

#### RIASSUNTO

Si investiga la convergenza della serie multiperiferica nel caso di diffusione di due particelle senza spin, quando il quadrato del momento trasferito di tipo spazio è fissato come parametro. Si introduce uno schema di accoppiamento in cui l'ampiezza di diffusione viene descritta come un funzionale pseudobilineare su certi spazi di Hilbert che contengono elementi descriventi una particella entrante ed una uscente. In tal modo in questo formalismo ad ogni grafico multiperiferico rimane associato un funzionale pseudobilineare. Le proprietà della serie sono studiate per mezzo dell'analisi in onde parziali incrociate. Facendo uso delle tecniche di Fredholm e della proprietà di analiticità viene mostrato che la somma della serie dei grafici, in corrispondenza a valori fisici della costante di accoppiamento  $g$ , è descritta da un funzionale non limitato che non può essere espresso solamente tramite le rappresentazioni unitarie irriducibili della serie principale di  $O_{2,1}$ . In questo formalismo ciò significa che l'ampiezza contiene un contributo di poli di Regge. Si ricavano l'andamento asintotico di Regge della soluzione e una nuova formulazione del limite di Froissart. Viene discussa l'utilità del metodo nel caso dell'approssimazione di grafici a scala.

#### Подход теории групп к многопериферической модели.

**Резюме (\*).** — Исследуется сходимость многопериферических рядов в случае рассеяния двух бесспиновых частиц, когда квадрат пространственно-подобного передаваемого импульса  $W$  является фиксированным, как параметр. Вводится «схема связи», в которой амплитуда рассеяния описывается, как псевдобилинейный функционал на некоторых гильбертовых пространствах, которые содержат элементы, описывающие падающие и уходящие-частицы. Таким образом, в этом формализме каждому многопериферическому графику сопоставляется псевдобилинейный функционал. Исследуются свойства этих рядов с помощью «перекрестного» анализа парциальных волн. Используя технику Фредгольма и свойства аналитичности, показывается, что графическая сумма ряда, в соответствии с физическими значениями константы связи  $g$ , описывается неограниченным функционалом, который не может быть выражен только посредством унитарных неприводимых представлений главного ряда  $O_{2,1}$ . В этом формализме это означает, что амплитуда содержит вклады некоторых полюсов Редже. Отмечаются асимптотическое поведение Редже полученных решений и новая формулировка границы Фройссарта. Обсуждается ценность этого метода для случая лестничного приближения.

(\*) Переведено редакцией.