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B. Touschek : COVARIANT STATISTICAL MECHANICS.

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Covariant Statistical Mechanics.

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Summary. — A covariant formulation of thermodynamics is derived from statistical mechanics.

1. — The relativistic transformation properties of thermodynamical quantities have been discussed even before the advent of the special theory of relativity. In 1904 HASENÖEHL⁽¹⁾ derived a formula for the spectral intensity of the radiation emitted from a moving black body. By 1919 the problem seemed to be solved and clearly understood: a chapter in Pauli's⁽²⁾ *Relativitätstheorie* is dedicated to this question. The classical opinion can be memorized by stating that according to it the ideal gas equation (in which the pressure is considered to be a scalar) is form-invariant, so that the temperature has to transform like a volume: it is smaller in the moving frame than in the rest frame of the system.

This opinion was challenged in 1963 by OTT⁽³⁾, who held that the temperature should increase in the transition from the rest system to a moving frame. There followed an extensive discussion which is admirably documented in Møller's⁽⁴⁾ article with the significant title: *Relativistic Thermodynamics*.

⁽¹⁾ F. HASENÖEHL: *Wien. Ber.* **113**, 1039 (1904), *Ann. d. Phys.* **15**, 344 (1905); also K. V. MOSENGEIL: *Ann. d. Phys.* **22**, 867 (1907).

⁽²⁾ W. PAULI: *Relativitätstheorie* (Enz. Math. Wiss.) p. 697.

⁽³⁾ H. OTT: *Zeits. f. Phys.* **175**, 70 (1963); compare also H. ARZELIÈS: *Nuovo Cimento* **25**, 792 (1965); A. GAMBÀ: *Nuovo Cimento* **27**, 1792 (1965); T. W. B. KIBBLE: *Nuovo Cimento* **41 B**, 73 (1966).

⁽⁴⁾ C. MØLLER: *Kgl. Danske Videnskab. Selskab.* **36**, 1 (1967): this paper gives a critical summing up of the discussion of ref. (3).

A Strange Incident in the History of Physics. Ott's argument can be summarized by stating that the entropy should be an invariant and that the heat should transform like the time component of a four-vector. It then follows from $dS = dQ/T$ that, since the heat is increased by being transformed out of the rest system, the temperature should do the same.

In his contribution to the general discussion which followed the publication of Ott's paper, ROHRLICH⁽⁵⁾ observed that the transformation properties of a physical quantity cannot be determined unless one is prepared to specify in detail the way in which this quantity has to be measured. The Lorentz contraction of a length is the characteristic of a measurement in which care is taken that the terminal points are observed at the same instant of time. Other methods of measurement (as for example the interpretation of a photograph taken of a moving object) will not show Lorentz contraction. It is a remarkable feature of the «strange incident» that in all its literature no serious attempt seems to have been made to define the way in which the temperature should be measured from a moving frame. If one then remembers that no one seems to have any doubts about how to calculate for example the energy distribution in a moving black body or in an ideal gas one is driven to the conclusion that the whole controversy is one about the form rather than about the substance of the problem of making thermodynamics compatible with relativity. What is called temperature in the old texts is no longer temperature in the new ones and what *should* be called temperature remains a matter of convenience.

In his book on *The Relativistic Gas* SYNGE⁽⁶⁾ introduced a vector reciprocal temperature and by this means arrived at a completely satisfactory description of a relativistic gas of particles with point interactions. His tacit conclusion was not that the temperature *is* the reciprocal of a relativistic four-vector, but that this four-vector is a convenient tool for handling the problems of relativistic statistical mechanics. Synge's four-vector is timelike and parallel to the velocity four-vector which describes the motion of the relativistic gas as a whole. It can therefore be used to define also an invariant temperature, which in the rest system of the gas coincides with the conventional temperature. The example shows how easy it is to enlarge the discussion on the transformation properties of the temperature.

The point of view that the choice of the definition of the temperature for moving reference frames may be a matter of convenience has also recently been presented by MØLLER in his contribution to the *Colloquium on Relativistic Thermodynamics and Statistical Mechanics* which was held at Brussels⁽⁷⁾. This point of view does of course not exclude that the convenience for

⁽⁵⁾ F. ROHRLICH: *Nuovo Cimento* **45** B, 76 (1966).

⁽⁶⁾ J. L. SYNGE: *The Relativistic Gas* (Amsterdam 1957).

⁽⁷⁾ *Colloquium on Relativistic Thermodynamics and Statistical Mechanics*, Brussels 9-10 May 1968.

a particular choice of the definition of temperature may be so great as to make it preferable to all other choices; this is the line which I intend to defend in this paper.

The main reason for such a choice is that a «one-parameter temperature» is not sufficient to ensure equilibrium if the systems envisaged in a relativistic situation are allowed to exchange momentum as well as energy. This can be seen by considering two identical «clouds» of particles in relative motion to one another. For the establishment of an equilibrium between these two clouds it is not sufficient to require that their temperatures (measured in their respective rest frames) be the same, but one has also to insist that their relative velocity be zero, since otherwise the energy corresponding to their relative motion can be converted into thermal energy of the «compound cloud». It is therefore seen that what was one parameter (the temperature) in the non-relativistic theory becomes four parameters (the rest-frame temperature + + the three components of the velocity of relative motion) in a relativistic situation. Synge's procedure presents a natural way of grouping these four parameters together:

$$(1) \quad \beta^\mu = u^\mu / T(0) \quad (\mu = 0, 1, 2, 3).$$

Here β^μ is the Synge vector, u^μ is the velocity of the centre of mass of the system ($(u, u) = 1$) and $T(0)$ is the temperature of the system measured in its rest frame.

In the following Section I shall give what I believe to be a generalization of Synge's procedure and show that in this way it is possible to establish a complete framework of relativistic thermodynamics as well as relativistic statistical mechanics for an equilibrium situation.

2. – Statistical mechanics deals with systems, which are defined to possess the following properties:

Systems can assume states, which are labelled by an index α (a «discrete» notation is used here, since the generalization to systems with a continuous spectrum is trivial).

To every state of the system there corresponds an energy-momentum 4-vector P_μ^α . ($\mu = 0, 1, 2, 3$; the metric here used is $g_{00} = 1$, $g_{ii} = -1$, $i = 1, 2, 3$, $g_{\mu\nu} = 0$ for $\mu \neq \nu$.)

The energy-momentum 4-vector of the state of a system is timelike. The mass of the state can therefore be defined as

$$(2) \quad (P^\alpha, P^\alpha) = m^{\alpha^2} > 0$$

and it is assumed that $P_0^\alpha > 0$.

$A \gg 1$ systems can be combined to form an ensemble. A relativistic ensemble allows for the exchange of energy and momentum between its systems. The interaction responsible for this exchange of energy and momentum between the systems is assumed to be so weak as not to change the spectrum of the single systems.

The ensemble of A systems is itself a closed system and as such can be used to study the thermodynamical properties of closed systems in general (which by definition should be independent of the detailed mechanism of energy-momentum exchange within the system).

The total energy-momentum vector of the ensemble will be called $P_\mu A$, P_μ being the average energy-momentum of the constituent systems. Since the ensemble is closed P_μ will be a constant. Because of (2) it will be a time-like vector.

A « complexion » of the ensemble is defined by specifying a set of numbers a_α equal to the number of systems in the state α . The a_α are therefore subject to the conditions

$$(3) \quad \sum a_\alpha = A \quad \sum a_\alpha P_\mu^\alpha = AP_\mu.$$

The terminology is Fowler's⁽⁸⁾.

A state of the ensemble is defined by specifying the state of each individual component system. A configuration contains many states, precisely

$$(4) \quad W = A! / \prod a_\alpha!.$$

The equilibrium complexion (defined in terms of the numbers \bar{a}_α) contains the maximum number of states compatible with the conditions (3). Variation of $\log W$ subject to the conditions (3) gives

$$(5) \quad \bar{a}_\alpha = AZ^{-1} \exp[-(\beta, P^\alpha)],$$

where—at this stage— β is a 4-parameter, which has to be determined from the second condition (3) and Z —the sum over states or partition function—is given by

$$(6) \quad Z = \sum_\alpha \exp[-(\beta, P^\alpha)].$$

With this choice the first condition of (3) is automatically satisfied. If the definition of the systems is such that the summation over their states can be considered as an invariant operation, Z will be an invariant provided that β

⁽⁸⁾ R. H. FOWLER: *Statistical Mechanics* (Cambridge 1936).

transforms like a four-vector. The expression (6) will converge provided that β is timelike with $\beta_0 > 0$ and that the level density does not increase too rapidly with energy.

That β^μ can be chosen to be a four-vector follows from the second condition (3), which because of (5) and (6) can be written as

$$(7) \quad P_\mu = -\frac{\partial}{\partial \beta^\mu} \log Z.$$

That β is a four-vector and parallel to P as shown in eq. (1) can be seen in the following way. In the rest frame of the ensemble—which is indicated by (0)—one has $P_i(0) = 0$, so that in this frame eq. (7) becomes

$$(7') \quad P_0(0) = -(\partial/\partial \beta^0(0)) \log Z(0), \quad (\partial/\partial \beta^i(0)) \log Z(0) = 0.$$

The second of these equations requires that

$$(7'') \quad \sum_{\alpha} P_i^{\alpha}(0) \exp[-(\beta(0), P^{\alpha})] = 0.$$

If the summation over the states α is an invariant operation then—in the rest frame—one must have that to every state with $P_i^{\alpha} = K_i$ there must correspond another α' with $P_i^{\alpha'} = -K_i$. This means that (7'') can only vanish if $\beta_i(0) = 0$. In the rest frame of the ensemble β is therefore seen to be purely timelike:

$$(8) \quad \beta_{\mu}(0) = (\beta_0(0), 0, 0, 0).$$

In this case the sum over states becomes

$$(6') \quad Z(0) = \sum_{\alpha} \exp[-\beta^0(0) P_0^{\alpha}(0)]$$

and this is the partition function of the Gibbs ensemble, in which no exchange of momentum is contemplated. In the rest frame the relativistic ensemble therefore reduces to the Gibbs ensemble and only gives the additional information that the average momentum of the systems (which is not discussed in the Gibbs ensemble) must be zero. This argument shows (and this will be discussed later on) that relativistic covariance does not introduce any new parameters into the description of the equilibrium properties of a statistical system: these properties are completely determined in its rest frame. The only addition is—in the words of ROHRLICH—trivial in that it involves only the four-velocity u_{μ} defined by

$$(9) \quad P^{\mu} = P^0(0) u^{\mu},$$

which determines the Lorentz transformation, which leads away from the rest frame. Equation (1) is an example of this.

To establish a system of thermodynamical equations one has to find an expression for the entropy. This definition should be compatible with $dS = (dQ)_{\text{rev}}/T$ as well as with Boltzmann's equation $S = k \log W$. A large part of the discussion of the relativistic transformation properties of thermodynamic quantities was expended on the transformation properties of the heat transfer: heat and work should sum up to the four-vector of the total energy, but this—in the absence of separate conservation laws—does not imply that both heat and work have to transform like a relativistic four-vector. It will be seen, however, that a good argument can be given for assuming that they do.

To separate heat from work one can consider the total relativistic ensemble as an example of a closed system. A change of its energy-momentum vector is then given by

$$(10) \quad A \delta P_{\mu} = \sum_{\alpha} (\bar{a}_{\alpha} \delta P_{\mu}^{\alpha} + P_{\mu}^{\alpha} \delta \bar{a}_{\alpha}).$$

In the rest system the first term in the expression for the change of total energy is always considered as work, and the second as heat. It is therefore convenient to adopt the same interpretation also in an arbitrary frame of reference and as well for the spacelike as for the timelike components of the energy-momentum vector. One can then define a « four-heat » by

$$(11) \quad A(\delta Q_{\mu})_{\text{rev}} = \sum P_{\alpha}^{\mu} \delta \bar{a}_{\alpha},$$

where the reversibility of the change indicated by the subscript is achieved by letting the $\delta \bar{a}_{\alpha}$ correspond to a change in which subject to the boundary conditions W remains maximum. Equation (11) expresses that work is the change of energy-momentum which results from changing the energies and momenta of every component of a system without changing their statistical arrangement, whereas heat is the change of energy-momentum of the whole system resulting from a change in the statistical arrangement of its components.

The reversibility of the changes of the \bar{a}_{α} in (11) can be formulated more explicitly. One notes that Z because of (6) can be considered as a function of the parameters $x^{\alpha} = (\beta, P^{\alpha})$ so that after putting $f = \log Z$ one can write for (5)

$$(5') \quad \bar{a}_{\alpha} = -A(\partial f / \partial x^{\alpha}).$$

A reversible change in the occupation numbers a_{α} can now be defined by

$$(12) \quad \delta \bar{a}_{\alpha} = -A \sum_{\beta} (\partial^2 f / \partial x^{\alpha} \partial x^{\beta}) \delta x^{\beta}$$

and it is seen after inserting (12) into (11) that in general $(\delta Q_{\mu})_{\text{rev}}$ will not represent a Pfaffian in the variables x^{β} . Equation (11), however has an integrating factor—precisely β^{μ} —and one has

$$(13) \quad dG = (\beta, \delta Q_{\text{rev}}) = - \sum_{\alpha, \beta} (\partial^2 f / \partial x^{\alpha} \partial x^{\beta}) x^{\alpha} \delta x^{\beta}.$$

Using eqs. (7) and (5') and disposing of a constant of integration one gets

$$(14) \quad G = (\beta, P) + \log Z,$$

an invariant expression, which in the rest frame reduces to $G = \beta_0(0)P_0(0) + \log Z$, *i.e.* apart from the constant k (which in the following we choose to be 1) the conventional expression for the average entropy of a system in the ensemble, the total entropy being $S = (AP_0/T) + A \log Z$. $S = AG$ holds generally in every frame of reference. One easily verifies by the use of the (mutilated) Stirling formula $\log n! = n(\log n - 1)$ that, because of (3), (5) and (6) one has

$$(15) \quad S = \log W,$$

where W —defined by eq. (4)—is evaluated for the most probable complexion compatible with the conditions (3). It is therefore seen that the relativistically invariant form (14) has indeed all the properties of an entropy: it is the Pfaffian of the heat transfer and satisfies Boltzmann's relation (15).

The importance of the entropy results from the fact that its knowledge as a function of the appropriate variables completely determines the thermodynamical behaviour of the system. In the particular case of gas theory the appropriate variables are: N the number of particles, V the volume of the container and $E = P_0(0)$ the total energy of the gas. N is obviously a relativistic invariant, the last two quantities can be interpreted as forming part of time-like relativistic four-vectors, namely

$$(16) \quad D_{\mu}(0) = (1/V, 0, 0, 0), \quad P_{\mu}(0) = (E, 0, 0, 0).$$

By definition, the timelike vectors of energy and density are parallel, which can be expressed by writing

$$(17) \quad (D, P) = D_0(0)P^0(0) = E/V.$$

Instead of D it is sometimes convenient to use a vector V defined by

$$(18) \quad V_{\mu} = D_{\mu}(D, D)^{-1},$$

In particular one has $V_{\mu}(0) = (V, 0, 0, 0)$.

Any specific gas theory will define $G = G(E, V, N)$. The present formalism then allows one to extract the relativistically invariant contents of such a theory by replacing E by $P = (P, P)^\frac{1}{2}$ and V by $V = (V, V)^\frac{1}{2}$. The paradoxical result of this is that also a non covariant theory of the gas can give rise to a covariant thermodynamical behaviour! This should not be too great a surprise since thermodynamics picks just a very small number of degrees of freedom from the very big reservoir of such degrees represented by the gas as a whole. It is a matter of convenience to choose this limited number of degrees of freedom in such a way that the resulting thermodynamics is covariant, the only limitation being that the new formalism should reduce to the old in the rest frame.

In the rest frame the equations which determine thermodynamical equilibrium are

$$(19) \quad 1/T(0) = (\partial G/\partial E)_{V,N}, \quad p(0)/T(0) = (\partial G/\partial V)_{E,N} \quad \text{and} \quad i(0)/T(0) = (\partial G/\partial N)_{E,V},$$

the first of which defines the absolute temperature of the system, the second its pressure and the third its chemical potential i . After the substitution which renders G manifestly invariant the first of the equations (19) becomes

$$(20) \quad (\partial G/\partial P_\mu) = (\partial G/\partial P)(\partial P/\partial P_\mu) = (\partial G/\partial P)P^\mu/P = w^\mu/T(0) = \beta^\mu,$$

w^μ being the four-velocity of the system. This verifies eq. (1).

The second equation of (19) gives

$$(21) \quad (\partial G/\partial V_\mu) = p(0)V^\mu/T(0)V = p(0)w^\mu/T(0) = p(0)\beta^\mu$$

and it is therefore seen that the pressure can be defined as a relativistic invariant p by putting

$$(22) \quad \beta^\mu p = (\partial G/\partial V_\mu), \quad p = p(0).$$

The last equation of (19) leads to the introduction of a chemical 4-potential i_μ with $i_\mu(0) = (1, 0, 0, 0)$ and

$$(23) \quad (\beta, i) = \partial G/\partial N$$

and one has of course $i^\mu = i^0(0)w^\mu$.

The physical importance of eqs. (20), (22) and (23) rests upon the fact that they permit one to formulate the conditions for the equilibrium between two systems. This results from the fact that the entropy $G(12)$ of the compound system ($G(12) = G(1) + G(2)$) must have its maximum when the two systems are in equilibrium. One can, for example, consider the transfer of an infinitesimal four-momentum δP_μ from system (1) to system (2). The resulting change in

entropy of the total system must be zero, so that $0 = \delta P_\mu (-\partial G(1)/\partial P_\mu + \partial G(2)/\partial P_\mu)$ and therefore because of (20)

$$(24) \quad \beta^\mu(1) = \beta^\mu(2) .$$

(24) is a necessary condition for equilibrium, which is therefore only possible if the two systems have a common rest frame.

The significance of eq. (22) is the following. The infinitesimal movement of an imagined wall separating the two systems should not change the entropy of the compound. This gives

$$(25) \quad p(1) = p(2)$$

as a further condition of equilibrium. Finally the compound entropy should remain unchanged if δN particles are transferred from system 1 to system 2. This gives

$$(26) \quad i_\mu(1) = i_\mu(2)$$

as a consequence of eq. (23).

Equations (24) and (26) contain as a common element the necessity of

$$(27) \quad u_\mu(1) = u_\mu(2)$$

for the equilibrium of two systems.

It is customary in gas theory to consider the limit $V \rightarrow \infty$ with N/V finite. In this limit G and P , but not β , p and i , become infinite. The infinity of the total four-momentum as well as of the entropy is not shared by the energy-momentum tensor $\theta_{\mu\nu}$ and the entropy flux vector g_μ , which can be defined as

$$(28) \quad \theta_{\mu\nu} = P_\mu D_\nu \quad \text{and} \quad g_\mu = G D_\mu .$$

Since D and P are parallel vectors the energy-momentum tensor is automatically symmetrical.

It has already been remarked with reference to the definition of the partition function, that the summation over the states of the system has to be considered as an invariant operation. The basic element in this procedure of summation is the quantity $d^3 \varrho_0(p)$ which specifies the number of states available to a particle confined in a volume V . This number is given by

$$(29) \quad d^3 \varrho_0(p) = V d^3 p / h^3 ,$$

where $d^3 p$ is the element of momentum space of this particle. It is quite easy by means of the use of the 4-vector V to replace (29) by a manifestly covariant expression and cover at the same time the case of a particle defined by a

mass spectrum, so that $s(m)dm$ is the number of mass states in the interval dm . Indeed

$$(30) \quad d^3 \varrho(p) = \frac{2V^\mu}{h^3} \int dm s(m) p_\mu \delta(p^2 - m^2) \theta(p) d^4 p,$$

in which δ is the Dirac delta-function and $\theta(p) = 1$ inside and on the positive light cone but zero otherwise, is manifestly covariant and reduces to (29) in the special case $s(m) = \delta(m - m_0)$.

Equation (30) therefore shows how the 4-vector V_μ can be used to determine the sum over states in a manifestly covariant way and eq. (28) indicates how this rather artificial object can be finally eliminated from the theory.

To conclude this Section I show two examples which demonstrate the way in which the canonical distribution as well as the law of black-body radiation can be represented in the covariant formalism. In either case the key quantity is the flux of particles with momentum P contained in the four-momentum interval $d^4 P$. This quantity will be denoted by $d^4 J_\mu(P)$, the zeroth component indicating the density of particles in $d^4 P$ the i -th component the flux of particles in the direction indicated by the index i . For the canonical distribution we consider a gas of particles with a unique mass m . One then finds

$$(31) \quad d^4 J_\mu(P) = a \delta(P^2 - m^2) \theta(P) P_\mu \exp[-(\beta, P)] d^4 P.$$

This is covariant, provided that a is invariant, a is a function of the total particle density $\bar{d}_\mu = D_\mu N$, which in contrast to D_μ is not an artificial quantity. One has

$$(32) \quad a = (\bar{d}, \bar{d})^{\frac{1}{2}} / 2\pi m^3 g(\sigma) \quad \text{with} \quad g(\sigma) = \int dx x^2 \exp[-\sigma \sqrt{1+x^2}]$$

and σ is given as $\sigma = m(\beta, \beta)^{\frac{1}{2}}$. $g(\sigma)$ is the partition function first introduced by JÜTTNER⁽⁹⁾ in his description of the ideal gas of relativistic particles.

The formula corresponding to (31) in the case of black-body radiation is even simpler:

$$(33) \quad d^4 J_\mu(p) = (\pi \hbar c)^{-3} \delta(P^2) \theta(P) (\exp[(\beta, P)] - 1)^{-1} P_\mu d^4 P.$$

It is seen that the density vector D has completely disappeared from this expression. Indeed any covariant result with a physical meaning will contain D only if there is a conservation law and then in the form $\bar{d} = ND$. The vector \bar{d}_μ is physically meaningful in that it represents the total current-density vector of the « charge » N .

(9) F. JÜTTNER: *Ann. Phys.* **34**, 856 (1911).

Equations (31) and (33) represent what may be termed «microscopic» statements about the two systems considered: the ideal gas and the black body. Generally to every state α of the system one can attribute a J_μ^α defined by

$$(34) \quad J_\mu^\alpha = \bar{a}_\alpha P_\mu^\alpha / A(V, P^\alpha)$$

representing the four-vector formed from the density and current of systems in the state α . Equation (34) is the tool with the help of which the unphysical quantities V and D can be eliminated.

3. - The relation between the mean value of the energy-momentum four-vector and the reciprocal temperature 4-vector β given in eq. (7) may be generalized to determine the expectation value $\langle P_{\mu_1}, P_{\mu_2}, \dots, P_{\mu_n} \rangle$ of the symmetrical tensor of rank n formed from the energy-momentum vector. Indeed it follows from eq. (5) that one can write

$$(35) \quad \langle P_{\mu_1}, P_{\mu_2}, \dots, P_{\mu_n} \rangle Z = (-1)^n \frac{\partial}{\partial \beta^{\mu_1}} \dots \frac{\partial}{\partial \beta^{\mu_n}} Z,$$

which represents a very puzzling analogue to the diagonalization of energy and momentum for a situation that can be described by a Schrödinger function $\psi(x)$, namely

$$(35') \quad P_{\mu_1}, P_{\mu_2}, \dots, P_{\mu_n} \psi(x) = \left(\frac{-i}{\hbar} \right)^n \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_n}} \psi(x).$$

The importance of eq. (35) rests upon the fact that it serves to develop the theory of fluctuations in a relativistic ensemble.

We shall discuss these fluctuations for what may be described as simple systems. These systems are defined as having a partition function of the form

$$(36) \quad Z = (\beta, V) z(\beta),$$

where, as indicated in the notation, z only depends on $\beta = (\beta, \beta)^\dagger$. An example of a system which is not simple is an ideal gas of N particles (without its container), for which the partition function has the form $Z = (\beta, V)^N z_N(\beta)$. A single particle in a gas of distinguishable particles is a simple system. This is still true if the particle has many mass states m_n with $n = 0, 1, \dots$. Simple systems are simple models of a thermometer. The observation of their average four-momentum in an environment which is in equilibrium at a reciprocal four-temperature β_μ allows one to determine this temperature by applying eq. (7). The attribute of simplicity will usually only refer to a limited range of temperature. A drop of water is simple at low temperatures but will cease to be simple at high temperatures.

For particles with a constant mass m it follows from eq. (35) that one must have

$$(37) \quad \langle (P, P) \rangle Z = \square Z = m^2 Z,$$

where we have put $\square = (\partial/\partial\beta_\mu)(\partial/\partial\beta^\mu)$ as an abbreviation. Since a single particle is a simple system, it Z must be of the form (36) and eq. (37) therefore becomes

$$(38) \quad z'' + 5z'/\beta - m^2 z = 0$$

(the primes ' representing differentiation with respect to β). This is the Jüttner equation. Its solution can be defined by the requirement that in the nonrelativistic limit (*i.e.* for $\beta m = \sigma \rightarrow \infty$) one should obtain for Z the classical expression

$$(39) \quad Z = \frac{V}{h^3} \left(\frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \exp[-\beta m].$$

Expression (39) is essentially nonrelativistic, but it has been adapted for use in a relativistic theory by addition of the factor $\exp[-\beta m]$ ($c=1$). This is necessary since, though nonrelativistic thermodynamics leaves one the freedom of choosing the zero of the energy scale, relativistic mechanics does not. From (38) and the limiting condition (39) one finds

$$(40) \quad z = A\sigma^{-2} H_2^{(1)}(i\sigma) \text{ with } A = (1-i)\sqrt{2}\pi^2 m^4/h^3.$$

This inserted in (36) gives the Jüttner sum over states for a perfect gas of nonidentical particles.

Equation (40) may be used to arrive at an invariant as opposed to a covariant formulation of thermodynamics. Indeed, for a system with a mass spectrum $m = m_n$ ($m = 0, 1, 2, \dots$) it is easily seen that because of (36) one has

$$(41) \quad z = \sum_n z_n.$$

It then follows from (37), that

$$(42) \quad \langle m^{2k} \rangle z = \sum_n m_n^{2k} z_n \quad \text{for } k = 0, 1, \dots$$

This is the invariant form of eq. (35). The moments of the invariant m^2 are defined in terms of the invariant temperature $\beta = (\beta, \beta)^{\frac{1}{2}}$. The exponential $\exp[-(\beta, P)]$ which defines the partition function in the covariant formulation is replaced by the Jüttner function (40).

Equation (35) can be used to derive a general law for the momentum fluctuations of a simple system. Using (35) to determine $\langle P^i P_i \rangle$ and $\langle P_0 \rangle$ for

a system described by the partition function (36) one sees that in the rest frame ($\beta^i = 0$) one has

$$(43) \quad \langle (\mathbf{P}, \mathbf{P}) \rangle_0 = 3kT(\langle P_0 \rangle_0 + kT),$$

in which (\mathbf{P}, \mathbf{P}) indicates the scalar product of the three-dimensional momentum components and $\langle \rangle_0$ indicates the average taken in the rest frame. Since in this frame $\langle P^i \rangle_0 = 0$, the left-hand side of this equation represents the momentum fluctuations of the system in an environment which has no drift velocity and which is kept at the constant temperature T . Equation (43) is a consequence of relativistic mechanics since the right-hand side requires a fixed zero for the total energy. For a macroscopic particle one has $\langle P_0 \rangle_0 \sim m \ll kT$. In this case $\langle (\mathbf{P}, \mathbf{P}) \rangle_0 = 3mkT$ and (43) therefore reduces to the well-known equipartition result in the case of Brownian motion. For an extreme relativistic gas ($kT \gg m$) one has $\langle (\mathbf{P}, \mathbf{P}) \rangle_0 = 12(kT)^2$, which also follows from Jüttner's formula.

The relativistic nature of (43) can be understood by remembering that in a nonrelativistic theory it is in principle possible to exchange energy without exchanging momentum. This is no longer the case in a relativistic theory and (43) defines just the way in which the exchange of energy—necessary for the maintenance of an equilibrium—is accompanied by fluctuations in the momentum of the particles.

* * *

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RIASSUNTO

Dalla discussione della meccanica statistica di sistemi capaci di scambiare fra di loro energia ed impulso, si deriva una forma covariante della termodinamica.

Ковариантная статистическая механика.

Резюме (*). — Из статистической механики выводится ковариантная формулировка термодинамики.

(*) *Переведено редакцией.*