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"MASS FORMULA" FOR $M = 0$ AND $M = 1$ TOLLER FAMILIES

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A method to derive the so-called mass formulae, based on analyticity alone, is presented. The simpler $M = 0$ case is discussed in some detail. The behaviour of the residue functions is also studied; using the factorization theorem the Toller pole is reconstructed. The mass formula for $M = 1$ is given.

It is well known that Regge poles occur in families with integral spacing at zero energy. This circumstance raises some very interesting problems. In fact if the daughter trajectories are approximately linear and parallel to the parent for positive t, they could give rise to particle or resonances with well defined quantum numbers. It seems therefore important to obtain some general information about the slopes of the daughter trajectories. Some work in this direction has been done by Domokos [1] and Domokos and Suranyi [2]. For small violations of SL(2, C), using the Bethe-Salpeter equation and a perturbation expansion, they obtained, for families of poles with Toller quantum number $M = 0$, the expression

$$\alpha_n(t) = \alpha - n + [a_1 + a_2(\alpha - n)(\alpha - n + 1)]t + O(t^2)$$

where the index n enumerates the trajectories, and, here and in the following, α is the $t = 0$ intercept of the parent Regge trajectory. In the Bethe-Salpeter model the constants a_1 and a_2 can be calculated for any given kernel.

In this letter we will give some results and the general idea of a method, based only on analyticity, for studying the n-dependence of the residue functions and of the derivatives of the trajectories $\alpha_n(t)$ at $t = 0$.

For $M = 0$ the most general n-dependence of the derivatives, compatible with analyticity, is in agreement with the Domokos mass formula:

$$\alpha_n'(0) = a_1 + a_2(\alpha - n)(\alpha - n + 1) . \tag{1}$$

We have shown explicitely that one arrives at this result starting from both the unequal-unequal (UU) and the equal-unequal (EU) mass problems.

We have also determined the n-dependence of the residue functions, both in the UU and EU case: once this is done, assuming the factorization theorem, we obtained the n-dependence of the residue functions in the equal (EE) mass case: the results are in agreement with those found using the O(4) symmetry. We have thus formally reconstructed the $M = 0$ Toller pole using analyticity and factorization.

Using essentially the same method we have studied the same problem for the more interesting case of families with $M = 1$, where the parity doubling phenomenon begins to appear. We have found the n-dependence ‡

$$\alpha_n^{\pm\prime}(0) = c_1 + [c_2 \pm c_3](\alpha - n)(\alpha - n + 1) . \tag{2}$$

Let us briefly illustrate here, in the simpler case of the scattering of spinless particles (i.e. for class I poles), the method used. A more complete description of our work will be published elsewhere.

The contribution of a family of Regge poles to the scattering amplitude is given by

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‡ The ± sign refers to the natural parity. As before $\alpha = \alpha^+(0) = \alpha^-(0)$.

$$f(s,t) = \sum_{n=0}^{\infty} \frac{2\alpha_n + 1}{\sin \pi\alpha_n} [1 + \tau_n \exp(-i\pi\alpha_n)]\ \mathcal{P}_{\alpha_n}(-\cos\theta_t)\beta_n(t) \tag{3}$$

where

$$\mathcal{P}_\alpha(z) = -\frac{\mathrm{tg}\,\pi\alpha}{\pi}\mathcal{P}_{-\alpha-1}(z) = \frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\alpha+1)} z^\alpha F(-\tfrac{1}{2}\alpha, \tfrac{1}{2}-\tfrac{1}{2}\alpha, \tfrac{1}{2}-\alpha; z^{-2}) = \frac{2^\alpha\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\alpha+1)} z^\alpha \sum_{k=0}^{\infty} a_k(\alpha) z^{-2k}\ .$$

In the UU case, which we discuss here, one has

$$\cos\theta_t = \frac{s}{2q_{\text{in}}q_{\text{out}}} + \frac{t(t - \sum_i m_i^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{4t\,q_{\text{in}}q_{\text{out}}}$$

where q_{in} and q_{out} are the initial and final momenta in the crossed t-channel.

Expanding the right hand side of eq. (3) in a power series, after some rearrangement, we get

$$f(s,t) = \sum_{m=0}^{\infty} \left[\frac{B(t)}{t}\right]^m \left(\frac{s}{s_0}\right)^{-m} \sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} b^m_{n;k}(t) \left(\frac{s}{s_0}\right)^{\alpha_n(t)+n}$$

where s_0 is a scale factor,

$$B(t) = \frac{t(t - \sum_i m_i^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{2s_0}\ ,$$

$$N(m) = \begin{cases} \frac{1}{2}m & \text{if} \quad (-1)^m = 1 \\ \frac{1}{2}(m-1) & \text{if} \quad (-1)^m = -1 \end{cases}$$

and, at $t = 0$, which is the point of interest here,

$$b^m_{n;k}(0) = g_m(\alpha)\frac{[2(\alpha-n)+1]}{2^{n+2k}k!} \frac{(-1)^n\gamma_n(0)}{\Gamma(\frac{1}{2}-\alpha+n+k)(m-n-2k)!}$$

where

$$g_m(\alpha) = \pi\frac{\frac{1}{2}[1+\tau\exp(-i\pi\alpha)]}{\sin 2\pi\alpha}\frac{2^{\alpha+1}}{\Gamma(\alpha+1-m)}$$

and $\gamma_n(0)$ are the reduced residue functions, from which, for $n \geqslant 1$, the appropriate singularities [3] at $t = 0$ have been removed.

If the amplitude has to be analytic at $t = 0$ we must require that, for $t \to 0$

$$\sum_{k=0}^{N(m)} \sum_{n=0}^{m-2k} b^m_{n;k}(t)\left(\frac{s}{s_0}\right)^{\alpha_n(t)+n} = \mathrm{O}(t^m)\ .$$

These are the fundamental equations in our approach. From these conditions we can extract the relations between the residues of the parent and daughter trajectories and, differentiating with respect to t, the n-dependence of the derivatives $\alpha'_n(0)$, i.e. the mass formula. In fact at $t = 0$ the system (4) turns out to be

$$\sum_{n=0}^{m} \frac{[2(\alpha-n)+1]}{2^n} \frac{(-1)^n}{2^{2\alpha+1}\sqrt{\pi}} \frac{1}{(m-n)!} \frac{2^{m+n}\Gamma(m-\alpha)}{\Gamma(m+n-2\alpha)}\gamma_n(0) = 0$$

whose solutions can be shown to be

$$\gamma_n(0) = -(2\alpha+1)\frac{(-1)^n}{n!}\frac{\Gamma(n-1-2\alpha)}{\Gamma(-2\alpha)}\gamma_0(0)\ . \tag{5}$$

Once these relations are known we can deduce the mass formula. In fact from the system (4), differentiating with respect to t, we obtain, for $m \geqslant 2$

$$\sum_{k=0}^{N(m)}\sum_{n=0}^{m-2k} b^m_{n;k}(0)\alpha'_n(0) = 0\ .$$

From this system we can express the slopes of all the daughter trajectories in terms of those of the parent and of the first daughter in the form

$$\alpha'_n(0) = \frac{\alpha'_1(0)n(1+2\alpha-n)}{2\alpha} + \frac{\alpha'_0(0)(n-1)(n-2\alpha)}{2\alpha}$$

which is readily seen to be equivalent to eq. (1).

At this point it is intersting to investigate the EU problem, as a first step toward the proof of the consistency of the whole scheme. Assuming that the equal mass particles are particle-antiparticle systems, only the daughters with $P = G(-1)^I$ contribute: for a class I family these are the even one.

Using essentially the same methods described above we get

$$\gamma^{\mathrm{EU}}_{2n}(0) = (2\alpha+1)\frac{(-1)^{n+1}}{2n!}\frac{\Gamma(n-\frac{1}{2}-\alpha)}{\Gamma(\frac{1}{2}-\alpha)}\gamma^{\mathrm{EU}}_0(0)$$

and

$$\alpha'_{2n}(0) = \frac{\alpha'_2(0)n(1+2\alpha-2n)}{2\alpha-1} + \frac{\alpha'_0(0)(n-1)(1+2n-2\alpha)}{2\alpha-1}$$

which, as expected, is again of the form of eq. (1).

Having determined the most singular part of the residue functions in the EU and UU case, using the factorization theorem [4] we can obtain, at $t = 0$, the daughter residue in the EE case, where, as is well known [5], analyticity itself does not give any information. In this way we obtain

$$\gamma^{\mathrm{EE}}_{2n}(0) = \frac{(2n)!}{(n!)^2 2^{2n}}\frac{\Gamma(\alpha-n+1)}{\Gamma(\alpha+1)}\frac{\Gamma(\frac{3}{2}+\alpha)}{\Gamma(\alpha+\frac{3}{2}-n)}\gamma^{\mathrm{EE}}_0(0)\ . \tag{7}$$

From the O(4) symmetry [5], in the spinless equal mass case, one has

$$\gamma^{\mathrm{EE}}_{2n}(0) = \frac{R}{2\pi^2}\frac{(\alpha+1)^2}{[2(\alpha-2n)+1]}\,|\,d^{\alpha,0}_{\alpha-2n;0;0}(\tfrac{1}{2}\pi)\,|^2$$

where R is the Toller pole residue and

$$d^{n,0}_{J,0,0}(\tfrac{1}{2}\pi) = \left[\frac{(2J+1)\Gamma(n-J+1)}{(n+1)\Gamma(n+J+2)}\right]^{\frac{1}{2}}(2\mathrm{i})^J(-1)^{\frac{1}{2}(n-J)}\frac{\Gamma(1+\frac{1}{2}(n+J))}{\Gamma(1+\frac{1}{2}(n-J))}\ .$$

It is easily seen that the residues given by the O(4) symmetry in the equal mass case coincide with those, given above, derived from the usual Regge pole theory.

We have thus reconstructed the contribution of the Toller pole to the scattering amplitude, and it is interesting to note that this has been done, via factorization, through the study of the unequal mass problem, where the extension of the group-theoretical approach is not trivial.

In conclusion, the main results of our work have been to establish the mass formula, eqs. (1) and (2), both for $M = 0$ and $M = 1$, starting from general analyticity properties, without any additional dynamical assumption. We have also shown, for the simpler $M = 0$ case, that the group-theoretical results on the residue functions can be obtained from the analyticity requirements and the factorization theorem.

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