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A Sum Rule for the Pion Electromagnetic Form Factor.

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Summary. — A method for obtaining a sum rule for the pion electromagnetic form factor is proposed. Numerical evaluations give $F_\pi(k^2) \simeq F_1^v(k^2)$. Some results about residues of the pion trajectory at $t = m_\pi^2$ for every k^2 are also obtained in the evasive and in the conspiratorial case.

In this note we wish to show how it is possible to derive a sum rule for the pion electromagnetic form factor involving electroproduction amplitudes, using the gauge invariance principle and the hypothesis, first suggested by FUBINI (1), that the absorptive part of an amplitude determines it completely.

As a by-product, under the hypothesis that the pion is a Regge particle and dominates the asymptotic behaviour of the (—) amplitudes of the electroproduction process we are able to determine completely, when the pion « evades », the residues of the pion trajectory at $t = m_\pi^2$ ($m_\pi^2 \simeq 0.02$ GeV) as a function of the mass of the virtual photon k^2, and to find analogous relations also when the pion « conspires ». This is an extension of Ball and Jacob result (2) obtained in the photoproduction case.

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(1) S. FUBINI: *Nuovo Cimento*, **52** A, 224 (1967).

(2) J. S. BALL and M. JACOB: *Photoproduction and Regge behaviour*, CERN preprint TH. 825.

Let us decompose the electroproduction amplitude T^a [3] for the process

$$\gamma(k) + \mathcal{N}(p_1) \to \mathcal{N}'(p_2) + \pi^a(q) \tag{1}$$

(where $\mathcal{N}$ and $\mathcal{N}'$ are nucleons and γ is a real [virtual] photon with lightlike [spacelike] momentum k) in the usual way

$$T^a = \delta_{3a} T^{(+)} + \tfrac{1}{2}[\tau_a, \tau_3] T^{(-)} + \tau_a T^{(0)}$$

and consider the $T^{(-)}$ amplitude for which we write the following decomposition:

$$T^{(-)} = \sum_{i=1}^{8} I_i A_i^{(-)}, \tag{2}$$

where the I_i are the eight covariants

$$I_i = \bar{u}(p_2)\gamma_5 M_i u(p_1),$$

and

$$\left\{\begin{array}{ll} M_1 = \gamma\cdot k\gamma\cdot\varepsilon - \gamma\cdot\varepsilon\gamma\cdot k, & M_5 = \gamma\cdot\varepsilon, \\ M_2 = (2p_1\cdot\varepsilon + k\cdot\varepsilon), & M_6 = \gamma\cdot k(2p_1\cdot\varepsilon + k\cdot\varepsilon), \\ M_3 = (2p_2\cdot\varepsilon - k\cdot\varepsilon), & M_7 = \gamma\cdot k(2p_2\cdot\varepsilon - k\cdot\varepsilon), \\ M_4 = (2q\cdot\varepsilon - k\cdot\varepsilon), & M_8 = \gamma\cdot k(2q\cdot\varepsilon - k\cdot\varepsilon). \end{array}\right. \tag{3}$$

As shown by GERSTEIN [4] or more generally by DE ALFARO, FUBINI, FURLAN, ROSSETTI [5] the amplitudes $A_i^{(-)}$, above defined, are free of kinematical singularities. If we now impose gauge invariance on $T^{(-)}$, we get the two equations

$$(s - M^2)A_2^{(-)}(s, t, k^2) - (\bar{s} - M^2)A_3^{(-)}(s, t, k^2) - (t - m_\pi^2)A_4^{(-)}(s, t, k^2) = 0, \tag{4.1}$$

$$A_5^{(-)}(s, t, k^2) + (s - M^2)A_6^{(-)}(s, t, k^2) - (\bar{s} - M^2)A_7^{(-)}(s, t, k^2) - (t - m_\pi^2)A_8^{(-)}(s, t, k^2) = 0, \tag{4.2}$$

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(3) We define

$$_{\text{out}}\langle\mathcal{N}', \pi^a|\mathcal{N}\gamma\rangle_{\text{in}} = S_{fi} = \delta_{fi} + i(2\pi)^4\delta^{(4)}(P_f - P_i)\frac{1}{(2\pi)^6}\left[\frac{M^2}{4\omega_k\omega_q E_1 E_2}\right]^{\frac{1}{2}} T^a_{fi},$$

where ω_q, ω_k, E_1 and E_2 are the energies associated with the four particles involved in the process (1) and M is the nucleon mass.

(4) I. S. GERSTEIN: *Phys. Rev.*, **161**, 1631 (1967).

(5) V. DE ALFARO, S. FUBINI, G. FURLAN and C. ROSSETTI: *Superconvergence and current algebra*, preprint Istituto di Fisica dell'Università di Torino.

where s, $\bar{s}$, t are the Mandelstam variables

$$s = (p_1 + k)^2 \qquad \bar{s} = (p_2 - k)^2 \qquad t = (q - k)^2 .$$

Introducing now $\nu = (s - \bar{s})/4M$ we may rewrite eq. (4.1) in the form

$$(5) \qquad -2M\nu\big(A_2^{(-)}(\nu, t, k^2) + A_3^{(-)}(\nu, t, k^2)\big) + \\ + \tfrac{1}{2}(t - m_\pi^2 - k^2)\big(A_2^{(-)}(\nu, t, k^2) - A_3^{(-)}(\nu, t, k^2)\big) + (t - m_\pi^2)A_4^{(-)}(\nu, t, k^2) = 0 .$$

Let us perform in this equation the limit $t \to m_\pi^2$. Remembering that $M_4 = (2q - k)\cdot\varepsilon$, one would get in $A_4^{(-)}$ the Born diagram in Fig. 1 so the residue of $A_4^{(-)}$ at $t = m_\pi^2$, is $-egF_\pi(k^2)$ (6).

Therefore we have

$$(6) \qquad 2M\nu\big(A_2^{(-)}(\nu, t = m_\pi^2, k^2) + A_3^{(-)}(\nu, t = m_\pi^2, k^2)\big) + \\ + \frac{k^2}{2}\big(A_2^{(-)}(\nu, t = m_\pi^2, k^2) - A_3^{(-)}(\nu, t = m_\pi^2, k^2)\big) = -egF_\pi(k^2) .$$

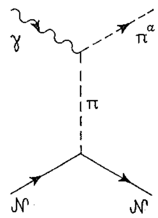

Fig. 1.

Following FUBINI (1), we shall assume that, expanding an amplitude, say $A(\nu, t, k^2)$, for which we can write a fixed t and k^2 dispersion relation with N subtractions, as a power series in the ν variable (Khuri representation (7)):

$$A(\nu, t, k^2) = \sum_{n=0}^{\infty} A_n(t, k^2)\nu^n ,$$

the analytic function of z

$$A(z, t, k^2) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\operatorname{Im} A(\nu, t, k^2)}{\nu^{1+z}}\, d\nu$$

will interpolate the expansion coefficients A_n not only for the integer values of $z > N$ for which the integral is convergent, but also for every integer n with $0 \leqslant n \leqslant N$.

We now assume that our amplitudes verify the above-mentioned analyticity properties. Equation (6) may be rewritten

$$(7) \qquad \nu D(\nu, m_\pi^2, k^2) + C(\nu, m_\pi^2, k^2) = -egF_\pi(k^2) ,$$

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(6) $F_\pi(0) = 1$, $g^2/4\pi = 14.5$, $e^2/4\pi = 1/137$.

(7) N. N. KHURI: *Phys. Rev.*, **132**, 914 (1963).

where

$$D(\nu, m_\pi^2, k^2) = 2M\big(A_2^{(-)}(\nu, m_\pi^2, k^2) + A_3^{(-)}(\nu, m_\pi^2, k^2)\big)$$

and

$$C(\nu, m_\pi^2, k^2) = \frac{k^2}{2}\big(A_2^{(-)}(\nu, m_\pi^2, k^2) - A_3^{(-)}(\nu, m_\pi^2, k^2)\big) .$$

D and C, as linear combinations of $A_2^{(-)}$ and $A_3^{(-)}$, will enjoy the same analyticity properties as $A_3^{(-)}$ and $A_2^{(-)}$ so that expanding D and C in power series of ν we have the relations

$$D_{n-1}(t = m_\pi^2, k^2) = - C_n(t = m_\pi^2, k^2) \qquad \text{for } n \geqslant 1 ,$$

i.e. in our hypothesis

$$D(z-1, t = m_\pi^2, k^2) = - C(z, t = m_\pi^2, k^2) ,$$

which at $z = 0$ becomes

$$D(-1, t = m_\pi^2, k^2) = - C(0, t = m_\pi^2, k^2) .$$

Therefore, as $C_0(t = m_\pi^2, k^2) = - egF_\pi(k^2)$, we obtain, taking into account crossing symmetry

$$\lim_{z\to-1} \frac{4M}{\pi} \int_0^\infty \frac{\operatorname{Im}\big(A_2^{(-)}(\nu, t = m_\pi^2, k^2) + A_3^{(-)}(\nu, t = m_\pi^2, k^2)\big)}{\nu^{1+z}}\, d\nu = egF_\pi(k^2) . \tag{8}$$

At first sight this sum rule looks different from the one written in the photoproduction case by MUKUNDA and RHADA [8]; they in fact obtained an equation which, in our formalism, is

$$\lim_{z\to-1} \frac{4M}{\pi} \int_0^\infty \frac{\operatorname{Im}\big(A_2^{(-)}(\nu, t, k^2 = 0) + A_3^{(-)}(\nu, t, k^2 = 0)\big)}{\nu^{1+z}}\, d\nu = 0 . \tag{9}$$

However, on the basis of Regge arguments it is possible to show that eq. (8) and eq. (9) for $k^2 = 0$ in the limit $t \to m_\pi^2$ actually agree. The point is simply that, while in eq. (8) one has to integrate a function in which the limit $t \to m_\pi^2$ is already performed; in the case of eq. (9) one has to integrate the

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([8]) N. MUKUNDA and T. RHADA: *Nuovo Cimento*, **44** A, 726 (1966).

same function still dependent on t and interchanging the integration with the limit is not allowed. We shall show here the agreement in the evasive case (no pion conspirator); it will be seen in the following that the same result still holds if the pion conspires with its parity doublet partner.

In fact assuming that the pion trajectory dominates the (—) amplitudes, on the basis of Ball and Jacob results, one can very easily show that in the evasive case

$$A_2^{(-)}(\nu, t, k^2) + A_3^{(-)}(\nu, t, k^2) \xrightarrow[\nu\to\infty]{} \sqrt{2}\, M\beta_2(t, k^2)\alpha_\pi(t) s^{\alpha_\pi(t)-1} \cdot \frac{1+\exp[-i\pi\alpha_\pi(t)]}{\sin \pi\alpha_\pi(t)}, \tag{10}$$

Remembering that $\alpha_\pi(m_\pi^2) = 0$, it is immediately seen that

$$\mathrm{Im}\left(A_2^{(-)}(\nu, t = m_\pi^2, k^2) + A_3^{(-)}(\nu, t = m_\pi^2, k^2)\right) \xrightarrow[\nu\to\infty]{} 0\,, \tag{11}$$

so that the Reggeized pion does not contribute to eq. (8). This is not the case for eq. (9), in fact let us perform the limit $t \to m_\pi^2$ in this equation:

$$\lim_{t\to m_\pi^2} \lim_{z\to -1} \frac{4M}{\pi} \int_0^\infty \frac{\mathrm{Im}\,(A_2^{(-)} + A_3^{(-)})}{\nu^{1+z}}\, \mathrm{d}\nu = 0\,.$$

Subtracting and adding to the integrand the assumed asymptotic behaviour (eq. (10)), we may write

$$\lim_{t\to m_\pi^2} \lim_{z\to -1} \frac{4M}{\pi} \cdot \tag{12}$$

$$\cdot \int_0^\infty \frac{\left[\mathrm{Im}\left(A_2^{(-)}(\nu, t, k^2=0) + A_3^{(-)}(\nu, t, k^2=0)\right) + \theta(\nu' - \bar\nu)\sqrt{2}\, M\beta_2(t)\alpha_\pi(t)(\nu)^{\alpha_\pi(t)-1}\right]}{\nu^{1+z}}\, \mathrm{d}\nu -$$

$$- \lim_{t\to m_\pi^2} \lim_{z\to -1} \frac{4M}{\pi} \int_0^\infty \frac{\left[\theta(\nu' - \bar\nu)\sqrt{2}\, M\beta_2(t)\alpha_\pi(t)(\nu)^{\alpha_\pi(t)-1}\right]}{\nu^{1+z}}\, \mathrm{d}\nu = 0\,.$$

($\bar\nu$ is a critical energy above which $\mathrm{Im}\,(A_2^{(-)} + A_3^{(-)})$ has a Regge behaviour.) Since now the first integral defines an analytic function of t and z also for $t = m_\pi^2$ and $z = -1$, see eq. (10), we may interchange the limits $t \to m_\pi^2$ and $z \to -1$ and also perform the limit $t \to m_\pi^2$ before evaluating the integral, so we get from (12)

$$\lim_{z\to -1} \frac{4M}{\pi} \int_0^\infty \frac{\mathrm{Im}\,(A_2^{(-)} + A_3^{(-)})_{t=m_\pi^2}}{\nu^{1+z}}\, \mathrm{d}\nu + \frac{2\sqrt{2}}{\pi} M\beta_2(t = m_\pi^2) = 0\,.$$

But, as found by BALL and JACOB [2] in eliminating kinematical singularities from the invariant amplitudes A_i, one has the relation

$$\frac{2\sqrt{2}\,M}{\pi}\,\beta_2(m_\pi^2, k^2=0) = -eg\,.$$

This shows the announced agreement of eq. (9) with eq. (8).

Let us try, now, to saturate our sum rule (8). We approximate the imaginary parts of the amplitudes as:

$$\text{(13)}\qquad \operatorname{Im} A(\nu, t, k^2) = \begin{cases} \operatorname{Im} A_{\text{res}}(\nu, t, k^2) & \text{for } \nu < \bar{\nu}\,, \\ \operatorname{Im} A_{\text{high}}(\nu, t, k^2) & \text{for } \nu > \bar{\nu}\,, \end{cases}$$

where $\operatorname{Im} A_{\text{res}}$ represents the low-energy resonant contribution, $\operatorname{Im} A_{\text{high}}$ the high-energy contribution (in our hypothesis Regge contribution) and $\bar{\nu}$, defined as above, has to be determined from the available experimental data. Remembering eq. (11) and extracting the one-particle contribution, we get (9)

$$\text{(14)}\qquad \frac{4M}{\pi}\int_{\nu_0}^{\bar{\nu}} \operatorname{Im}\left(A_2^{(-)}(\nu, t=m_\pi^2, k^2) + A_3^{(-)}(\nu, t=m_\pi^2, k^2)\right)\mathrm{d}\nu = eg\left(F_\pi(k^2) - F_1^v(k^2)\right).$$

At $k^2=0$ our sum rule is identically satisfied, since from the gauge equation (6) one easily obtains for $\nu \neq 0$

$$\text{(15)}\qquad \lim_{t\to m_\pi^2} \operatorname{Im}[A_2^{(-)}(\nu, t, k^2=0) + A_3^{(-)}(\nu, t, k^2)] = 0\,.$$

At $k^2 \neq 0$ we can transform eq. (14) into a sum rule for the difference between the mean-square radius of the pion $\langle r_\pi^2\rangle$ and of the nucleon $\langle r_{\mathcal{N}}^2\rangle$ (we mean the Dirac radius of the nucleon), that is

$$\frac{4M}{\pi}\frac{\partial}{\partial k^2}\left[\int_{\nu_0}^{\bar{\nu}} \operatorname{Im}\left(A_2^{(-)}(\nu, t=m_\pi^2, k^2) + A_3^{(-)}(\nu, t=m_\pi^2, k^2)\right)\mathrm{d}\nu\right]_{k^2=0} = \frac{eg}{6}\left(\langle r_\pi^2\rangle - \langle r_{\mathcal{N}}^2\rangle\right).$$

We have evaluated the integral using Zagury's treatment of the electro-production which only includes the $\mathcal{N}^*(1236)$ contribution (10), obtaining

$$\langle r_\pi^2\rangle - \langle r_{\mathcal{N}}^2\rangle = -6.8\cdot 10^{-2}\ (\text{fm})^2 \qquad \text{and} \qquad \sqrt{\langle r_\pi^2\rangle} = 0.76\ \text{fm}\,.$$

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(9) $F_1^v(k^2)$ is the isovector nucleon form factor: $F_1^v = F_1^p - F_1^n$ with the normalization $F_1^v(0) = 1$ and $\nu_0 = (1/2M)[(m_\pi^2 + 2m_\pi M) - \frac{1}{2}k^2]$.

(10) R. ZAGURY: *Phys. Rev.*, **145**, 1112 (1966).

This result agrees with the widespread belief that the pion form factor is about equal to the Dirac proton form factor, which is also supported by some experimental evidence [11].

In order to find the relations between the residues of the pion trajectory and $F_\pi(k^2)$, let us consider separately the evasive and the conspiratorial case.

Let us start with the evasive case. Following again Ball and Jacob [2], for the contribution of the pion trajectory to the amplitudes A_2+A_3, A_2-A_3, A_4, we find the expressions:

(16.1) $$A_2^{(-)}+A_3^{(-)} \xrightarrow[s\to\infty]{} \sqrt{2}\,M\beta_2^{(e)}(t,k^2)\alpha_\pi(t)s^{\alpha_\pi(t)-1}L^\pi(t)\,,$$

(16.2) $$A_2^{(-)}-A_3^{(-)} \xrightarrow[s\to\infty]{} -2M\left\{\frac{3t+m_\pi^2-k^2}{[(t-m_\pi^2+k^2)^2-4tk^2]}\cdot\left[-\sqrt{2}\beta_2^{(e)}(t,k^2)\alpha_\pi(t)+\frac{4}{t}\beta_5(t,k^2)\right]-\frac{1}{t}\beta_7(t,k^2)\right\}s^{\alpha_\pi(t)}L^\pi(t)\,,$$

(16.3) $$A_4^{(-)} \xrightarrow[s\to\infty]{} 2M\left\{\frac{t+m_\pi^2-k^2}{[(t-m_\pi^2+k^2)^2-4tk^2]}\cdot\left[-\sqrt{2}\beta_2^{(e)}(t,k^2)\alpha_\pi(t)+\frac{4}{t}\beta_5(t,k^2)\right]-\frac{1}{t}\beta_7(t,k^2)\right\}s^{\alpha_\pi(t)}L^\pi(t)\,,$$

where $L^\pi(t)=\big(1+\exp[-i\pi\alpha_\pi(t)]\big)/\sin\pi\alpha_\pi(t)$, $\alpha_\pi(t=m_\pi^2)=0$ and $\beta_2^{(e)}$, β_5, β_7 are the residues of the pion trajectory appearing in $\overline{F}_2$, $\overline{F}_5$, $\overline{F}_7$ which are definite parity helicity amplitudes [2] free of kinematical singularities; with the superscript (e) in $\beta_2^{(e)}$ we wish to emphasize that β_2 is different in the evasive or in the conspiratorial case.

Gauge invariance gives the known relation between β_5 and β_7:

(17) $$\beta_5(t,k^2)=-\tfrac{1}{4}(t-m_\pi^2+k^2)\beta_7(t,k^2)\,.$$

From eq. (16.1) we deduce also at $t=m_\pi^2$ the relation

$$\lim_{s\to\infty} s(A_2^{(-)}+A_3^{(-)})_{t=m_\pi^2}=\frac{2\sqrt{2}}{\pi}\beta_2^{(e)}(t=m_\pi^2,k^2)M\,,$$

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(11) For experimental data on $\langle r_N^2\rangle$ see: C. DE VRIES, R. HOFSTADTER, A. JOHANSSON and R. HERMAN: *Phys. Rev.*, **134**, B 848 (1964): they give for $\sqrt{\langle r_N^2\rangle}$ the two values 0.81 and 0.77 fm.

For the ones on $\langle r_\pi^2\rangle$ see K. BERKELMANN, *et al.*: *Measurement of the pion form factor*, Internal Report of the Laboratory of Nuclear Studies, Cornell University, Ithaca, N.Y. He has found that within the errors the pion and proton charge form factors are identical and that the root-mean-square radius of the pion is $\sqrt{\langle r_\pi^2\rangle}=(0.80\pm0.10)$ fm.

which gives a sum rule simply by noticing that for $t \leqslant m_\pi^2$, $A_2+A_3 \xrightarrow[s\to\infty]{} s^{-1-\varepsilon}$ with $\varepsilon \geqslant 0$, so that we can assume unsubtracted dispersion relations for A_2+A_3. We get

$$(18)\qquad \frac{4M}{\pi}\int_0^\infty \mathrm{Im}\left(A_2^{(-)}(\nu, t=m_\pi^2, k^2)+A_3^{(-)}(\nu, t=m_\pi^2, k^2)\right)\mathrm{d}\nu = \\ = -\frac{2\sqrt{2}}{\pi}M\beta_2^{(e)}(t=m_\pi^2, k^2)\,.$$

Comparing equations (8) and (18), we have

$$(19)\qquad -\frac{2\sqrt{2}}{\pi}\beta_2^{(e)}(t=m_\pi^2, k^2)M = egF_\pi(k^2)\,.$$

Let us remember that for every ν

$$\lim_{t\to m_\pi^2}(t-m_\pi^2)A_4(\nu, t, k^2) = -egF_\pi(k^2)\,,$$

but from eq. (16.3) one gets

$$(20)\qquad \lim_{t\to m_\pi^2}(t-m_\pi^2)A_4(\nu, t, k^2)\xrightarrow[\nu\to\infty]{}\frac{8M}{\pi\alpha_\pi'(m_\pi^2)}\frac{1}{k^2-4m_\pi^2}\beta_7(t=m_\pi^2, k^2)\,,$$

where $\alpha_\pi'(m_\pi^2) = \mathrm{d}\alpha_\pi/\mathrm{d}t|_{t=m_\pi^2}$, so we conclude

$$(21)\qquad \frac{8M}{4m_\pi^2-k^2}\frac{1}{\pi\alpha_\pi'(m_\pi^2)}\beta_7(t=m_\pi^2, k^2) = egF_\pi(k^2)$$

and then from eq. (19)

$$(22)\qquad \beta_7(t=m_\pi^2, k^2) = -\frac{\sqrt{2}}{4}(4m_\pi^2-k^2)\alpha_\pi'(m_\pi^2)\beta_2(t=m_\pi^2, k^2)\,.$$

Equation (22) was also obtained for $k^2=0$ by BALL and JACOB as a necessary condition in order not to have kinematical singularity at $k^2=0$ in the amplitudes $A_i^{(-)}$.

Equations (17), (19) and (22) fix completely the three residues $\beta_2^{(e)}$, β_5 and β_7 at $t=m_\pi^2$ for any k^2, *i.e.*

$$(23)\qquad \begin{cases} \beta_2^{(e)}(t=m_\pi^2, k^2) = -\dfrac{\pi eg}{2\sqrt{2}M}F_\pi(k^2)\,, \\[2ex] \beta_5\;(t=m_\pi^2, k^2) = \dfrac{\pi eg}{32M}(k^2-4m_\pi^2)k^2\alpha_\pi'(m_\pi^2)F_\pi(k^2)\,, \\[2ex] \beta_7\;(t=m_\pi^2, k^2) = -\dfrac{\pi eg}{8M}(k^2-4m_\pi^2)\alpha_\pi'(m_\pi^2)F_\pi(k^2)\,. \end{cases}$$

Equations (23) together with (16.1)-(16.3) determine completely the Regge behaviour of $A_2^{(-)}$, $A_3^{(-)}$, $A_4^{(-)}$ in the near forward direction, also in the electron-production case.

In the conspiratorial case, we have to consider in the expressions of our $A_i^{(-)}$ also the amplitudes $\bar{F}_1$ and $\bar{F}_3$, to which the conspiring scalar trajectory contributes, so we get for $A_2^{(-)}+A_3^{(-)}$

$$(24)\qquad A_2^{(-)}+A_3^{(-)} \xrightarrow[s\to\infty]{} \frac{M}{2}\left\{\frac{t-m_\pi^2+k^2}{M(t-4M^2)}\beta_1(t,k^2)\alpha^c(t)\frac{1+\exp[-i\pi\alpha^c(t)]}{\sin\pi\alpha^c(t)}s^{\alpha^c(t)-1}+\right.$$
$$+\frac{M}{t}(t-m_\pi^2+k^2)\beta_3(t,k^2)\alpha^c(t)\frac{1+\exp[-i\pi\alpha^c(t)]}{\sin\pi\alpha^c(t)}s^{\alpha^c(t)-1}+$$
$$\left.+\frac{8}{t}\beta_2^{(c)}(t,k^2)\alpha_\pi(t)\frac{1+\exp[-i\pi\alpha_\pi(t)]}{\sin\pi\alpha_\pi(t)}s^{\alpha_\pi(t)-1}\right\},$$

with

$$\beta_3(t=0,k^2)=\frac{8}{M(m_\pi^2-k^2)}\beta_2^{(c)}(t=0,k^2)\qquad\text{and}\qquad\alpha^c(0)=\alpha_\pi(0)\,.$$

Just looking at eq. (24) one can see easily that, as announced, the agreement between eq. (8) and eq. (9) still holds. Note that gauge invariance gives again eq. (17) and no constraints on β_1, $\beta_2^{(c)}$ and β_3; but we have for $\beta_2^{(c)}$ at $k^2=0$ the analogue of eq. (22), *i.e.*

$$(25)\qquad \beta_7(m_\pi^2,0)=-4\alpha_\pi'(m_\pi^2)\beta_2^{(c)}\,,$$

furthermore, also in this case, eq. (21) is valid.

Following the previous procedure in deriving eq. (19), it is immediately seen that eq. (18) would hold in the form

$$(26)\qquad \frac{4M}{\pi}\int_0^\infty \operatorname{Im}(A_2^{(-)}+A_3^{(-)})_{t=m_\pi^2}\,d\nu=-\frac{8M}{m_\pi^2\pi}\beta_2^{(c)}(m_\pi^2,k^2)$$

if $\alpha^c(t=m_\pi^2)<0$, if $\alpha^c(t=m_\pi^2)=0$ we would obtain

$$(27)\qquad \frac{4M}{\pi}\int_0^\infty \operatorname{Im}(A_2^{(-)}+A_3^{(-)})_{t=m_\pi^2}\,d\nu=-\frac{8M}{m_\pi^2\pi}\beta_2^{(c)}(m_\pi^2,k^2)+$$
$$+\left[-\frac{1}{\pi}\frac{k^2}{(m_\pi^2-4M^2)}\beta_1(m_\pi^2,k^2)-\frac{M^2}{m_\pi^2\pi}k^2\beta_3(m_\pi^2,k^2)\right]$$

if on the contrary $\alpha^c(t=m_\pi^2)>0$, we cannot obtain in this way any relation of this kind.

But in the finite-energy sum-rule philosophy ([12]), we can write for every $\alpha^c(m_\pi^2)$

$$(28)\qquad \frac{4M}{\pi}\int\limits_0^{\bar{\nu}} \mathrm{Im}\,(A_2^{(-)}(\nu, t=m_\pi^2, k^2) + A_3^{(-)}(\nu, t=m_\pi^2, k^2)) +$$

$$+\frac{1}{\pi}\frac{k^2}{(m_\pi^2-4M^2)}\beta_1(m_\pi^2, k^2)\,(2M\bar{\nu})^{\alpha^c(m_\pi^2)} + \frac{1}{\pi}\frac{M^2}{m_\pi^2}k^2\beta_3(m_\pi^2, k^2)(2M\bar{\nu})^{\alpha^c(m_\pi^2)} =$$

$$= -\frac{1}{\pi}\frac{8M}{m_\pi^2}\beta_2^{(c)}(m, k^2)\,.$$

Let us point out that in saturating the sum rule (eq. (8)) one should take into account in this case the presence of the pion conspirator, so that eq. (14) should read

$$(29)\qquad \frac{4M}{\pi}\int\limits_0^{\bar{\nu}} \mathrm{Im}\,(A_2^{(-)}(\nu, t=m_\pi^2, k^2) + A_3^{(-)}(\nu, t=m_\pi^2, k^2)) +$$

$$+\frac{1}{\pi}\frac{k^2}{(m_\pi^2-4M^2)}\beta_1(m_\pi^2, k^2)(2M\bar{\nu})^{\alpha^c(m_\pi^2)} + \frac{1}{\pi}\frac{M^2}{m_\pi^2}k^2\beta_3(m_\pi^2, k^2)(2M\bar{\nu})^{\alpha^c(m_\pi^2)} = egF_\pi(k^2)$$

for every $\alpha^c(m_\pi^2)$ except for the value $\alpha^c(m_\pi^2)=0$, since, as already noted, in this case one cannot interchange the limit $t\to m_\pi^2$ and the integration in ν, so we shall obtain again eq. (14).

Now comparing eq. (28) with eq. (29) if $\alpha^c(m_\pi^2)\neq 0$, we obtain the relation

$$(30)\qquad egF_\pi(k^2) = -\frac{8}{\pi}\frac{M}{m_\pi^2}\beta_2^{(c)}(m_\pi^2, k^2)\,,$$

while if $\alpha^c(m_\pi)=0$, we have to compare eq. (28) with eq. (14), obtaining

$$(31)\qquad egF_\pi(k^2) = -\frac{8M}{\pi m_\pi^2}\beta_2^{(c)}(m_\pi^2, k^2) - \frac{k^2}{\pi(m_\pi^2-4M^2)}\beta_1(m_\pi^2, k^2) - \frac{M^2}{m_\pi^2\pi}k^2\beta_3(m_\pi^2, k^2)\,.$$

* * *

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([12]) R. DOLEN, D. HORN and C. SCHMID: *Finite energy sum rules and their application to πN charge exchange*, preprint Calt 68-143.

RIASSUNTO

Si propone un metodo per ottenere una regola di somma per il fattore di forma elettromagnetico del pione. Numericamente si ottiene $F_\pi(k^2) \simeq F_1^v(k^2)$. Si ricavano inoltre alcuni risultati sui residui della traiettoria del pione a $t = m_\pi^2$ e per ogni k^2, nel caso di evasione e nel caso di cospirazione.

Правило сумм для электромагнитного форм-фактора пионов.

Резюме (*). — Предлагается метод получения правила сумм для электромагнитного форм-фактора пионов. Численные вычисления дают. $F_\pi(k^2) \simeq F_1^v(k^2)$. В «конспиративном» и «неуловимом» случаях также получены некоторые результаты относительно вычета для траектории пиона при $t = m_\pi^2$ для каждого k^2.

(*) *Переведено редакцией.*