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Linear Theory of Motion in Electron Storage Rings

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The (single-particle) motion of electrons or positrons in storage rings is studied, giving special attention to the effects determining the beam size in general magnetic structures.

1. The role of the beams in storage rings is twofold, each stored particle being at a time projectile and target. Number densities are of great importance in connection with the practical observation of high-energy events from beam–beam collisions; in particular, the luminosity L has to be known[1] in order to get the cross section σ from the measured rate $\dot{n} = L\sigma$.

The case of electron storage rings (both e^-e^- and e^+e^-) must be given special attention due to radiation phenomena strongly influencing the single-beam geometrical properties.

Usually a two-step approach is followed in studying the particle motion: first, the single-particle motion is described (*1*) and the effect of synchrotron radiation, or other small perturbations, on the beam size is studied (*2*), (*3*); next, intensity dependent effects [such as space-charge (*4*) or resistive wall instabilities (*5*)] are considered. Both steps have been given wide attention in the literature in the past years, but we had frequent occasions to regret how much the results concerning the first step are scattered and how difficult it is to correlate them.

In particular, the single particle motion in electron storage rings is a delicate matter calling for an unified treatment. We will attempt to give this unified treatment in this paper; even restricting the subject to the single particle motion in the linear approximation the description of the various effects determining for instance the beam size will not be easy especially when considering the two transverse betatron modes of oscillation to be strongly coupled.

[1] The luminosity for head-on colliding bunched beams is defined as

$$L = fk \int_s \rho^+\rho^- \, dS$$

where f is the revolution frequency, k the rf harmonic number, ρ^+ and ρ^- the transverse densities of the two beams.

To put some order in the problems to be considered in the following paragraphs, we find it convenient to indicate the scale of importance of the terms to be included in the single particle equations of motion:

(i) first a reference trajectory (RT) will be defined; the actual motion will develop around it and the particles will have parameters (positions, momenta) so slightly different from those corresponding to the reference trajectory that Taylor expansion truncated to the linear terms can be considered as fairly accurate.

(ii) displaced orbits of particles having energy constant in time will be considered next; this step will be called as usual "betatron oscillations."

(iii) phase oscillations and energy losses originate from weaker forces than betatron oscillations so that they can be included as third step.

(iv) eventually the radiation from displaced particles and the radiation fluctuations (giving rise to dampings and finite s_{top} effects, respectively) will be considered.

2. We take as starting point, defining "zero-order" motion (that is, an RT) the equation for a nonradiating particle in a magnetic field constant in time

$$\ddot{\mathbf{r}} = \dot{\mathbf{r}} \times \boldsymbol{\omega}(\mathbf{r}), \tag{1}$$

where

$$\boldsymbol{\omega}(\mathbf{r}) = (ec/E)\, \mathbf{H}(\mathbf{r})$$

is the cyclotron frequency vector of an electron of total energy E in the magnetic field $\mathbf{H}(\mathbf{r})$.

The RT will be chosen in the following as a particular solution $\mathbf{r}_s(t)$ of (1) associated with a certain energy E_s; a better characterization of the RT will be given later in terms of the frequency of the main radio-frequency field.

Equation (1), together with the value of E_s, embodies the "machine parameters" usually specified to describe a ring: magnetic structure, radii, quadrupole strenghts, etc.: all of these informations are collected into the single vector function $\boldsymbol{\omega}(\mathbf{r})$.

Now we introduce further terms into the equations of motion; namely, radio-frequency fields $\boldsymbol{\epsilon}$; and terms associated with random events producing a sudden change of the vector momentum of the electron, such as the emission of synchrotron light or scattering and bremsstrahlung on the residual gas.

As a consequence (1) changes into

$$\frac{d}{dt}(E\dot{\mathbf{r}}) = ec^2\boldsymbol{\epsilon} + ec\dot{\mathbf{r}} \times \mathbf{H} + c^2\Sigma_j\Sigma_i\mathbf{q}_i^{(j)}\,\delta(t - t_i), \tag{2}$$

where $\mathbf{q}_i^{(j)}$ is the variation in the electron momentum due to an event of type "j" at the time t_i.

A function $P_j(\mathbf{q}, t)\, d\mathbf{q}$, will describe the rate of events of type "j" which produce a change in the electron momentum of a quantity between $\mathbf{q}$ and $\mathbf{q} + d\mathbf{q}$; we can define average values as follows

$$\mathbf{b}^{(j)}(t) = \int \mathbf{q} P_j(\mathbf{q}, t)\, dq \tag{3}$$

and rewrite (2) as

$$\frac{d}{dt}(E\dot{\mathbf{r}}) = ec^2\boldsymbol{\epsilon} + ec\dot{\mathbf{r}} \times \mathbf{H} + c^2\mathbf{R} + c^2\mathbf{G} \tag{4}$$

where

$$\mathbf{R}(t) = \Sigma_j\, \mathbf{b}^{(j)}(t) \tag{5}$$

and

$$\mathbf{G}(t) = \sum_j G^{(j)}(t) = \sum_j \left\{\sum_\alpha \mathbf{q}_\alpha^{(j)}\, \delta(t - t_\alpha) - \mathbf{b}^{(j)}(t)\right\}. \tag{6}$$

$\mathbf{R}(t)$ is the average effect of random forces; $\mathbf{G}(t)$ is the term responsible for fluctuations.

We want to study the motion defined by Eq. (4) in the case of small displacements from RT; that is we assume that the influence of the added terms will limit the actual motion to a small volume around the RT, so that Taylor expansions can be confidently used.

Putting

$$\mathbf{r} = \mathbf{r}_s + \delta\mathbf{r}, \qquad E = E_s(1 + p),$$

and linearizing Eq. (4) with respect to $\delta\mathbf{r}$ and p, we get

$$\delta\ddot{\mathbf{r}} + \dot{\mathbf{r}}_s\dot{p} = -p\ddot{\mathbf{r}}_s + \delta\dot{\mathbf{r}} \times \boldsymbol{\omega}(\mathbf{r}_s) + \dot{\mathbf{r}}_s \times [(\delta\mathbf{r} \cdot \nabla)\, \boldsymbol{\omega}(\mathbf{r})]_{r=r_s} + \boldsymbol{\psi}\dot{\mathbf{r}}_s^2, \tag{7}$$

where

$$\boldsymbol{\psi} = \left\{\frac{ec^2}{E_s}\boldsymbol{\epsilon} + \frac{c^2}{E_s}(\mathbf{R} + \mathbf{G})\right\}\frac{1 - p}{\dot{\mathbf{r}}_s^2}. \tag{8}$$

According to our definition the velocity of the reference particle $\mathbf{v}_s = \dot{\mathbf{r}}_s$ is constant in modulus so that the arc length on RT is given simply by $s = v_s t$.

It will be convenient in the following to use instead of the derivatives with respect to time, as in (7), those with respect to the arc length, s, on RT, which will be denoted by primes.

To establish the relationship between these derivatives we first note that, calling dl the arc length on the actual trajectory, one has, to first order in $\delta\mathbf{r}$,

$$dl = ds\{1 + \mathbf{r}_s' \cdot \delta\mathbf{r}'\},$$

so that

$$\frac{d}{dt} = v\frac{d}{dl} = v\{1 + \mathbf{r}'_s \cdot \delta\mathbf{r}'\}^{-1}\frac{d}{ds}\,. \tag{9}$$

The quantity v appearing in (9) is the velocity on the actual trajectory. It is interesting to note that from the definition of p it follows that

$$\frac{v - v_s}{v_s} \simeq \frac{c^2}{v_s^2}\left(\frac{m_0c^2}{E_s}\right)^2 p$$

and therefore $(v - v_s)/v_s$ is much smaller, for ultrarelativistic particles, than the first-order quantity p. Hence in the following we will assume

$$v = v_s$$

in accordance with the well known fact that, in electron circular accelerators, the angular frequency very nearly behaves as the inverse radius. The approximation $v_s \simeq c$ will also be used whenever possible.

Having decided to use the derivatives with respect to s in the description of the motion, it is natural to use as frame of reference a frame defined on RT in the usual way by means of the orthonormal tangent, normal and binormal vectors $\boldsymbol{\alpha}(s)$, $\boldsymbol{\beta}(s)$, $\boldsymbol{\gamma}(s)$, such that

$$\begin{aligned} \dot{\mathbf{r}}_s &= v_s\boldsymbol{\alpha}, & \boldsymbol{\beta}' &= -K(s)\,\boldsymbol{\alpha} + H(s)\,\boldsymbol{\gamma}, \\ \boldsymbol{\alpha}' &= K(s)\,\boldsymbol{\beta}, & \boldsymbol{\gamma}' &= -H(s)\,\boldsymbol{\beta}, \end{aligned} \tag{10}$$

$K(s)$ and $H(s)$ being, respectively, the curvature and the torsion of RT.

Any vector $\mathbf{u}$ can then be written as

$$\mathbf{u} = u_1\boldsymbol{\alpha} + u_2\boldsymbol{\beta} + u_3\boldsymbol{\gamma},$$

and in particular we chose

$$\delta\mathbf{r} = x(s)\boldsymbol{\beta} + z(s)\boldsymbol{\gamma}. \tag{11}$$

It can be useful to write down explicit formulas for $\delta\mathbf{r}'$, $\delta\mathbf{r}''$:

$$\begin{aligned} \delta\mathbf{r}' &= -Kx\boldsymbol{\alpha} + (x' - Hz)\boldsymbol{\beta} + (z' + Hx)\boldsymbol{\gamma}, \\ \delta\mathbf{r}'' &= (-K'x - 2Kx' + KHz)\boldsymbol{\alpha} + (x'' - K^2x - 2z'H - zH' - H^2x)\boldsymbol{\beta} \\ &\quad + (z'' - zH^2 + 2Hx' + H'x)\boldsymbol{\gamma}. \end{aligned}$$

These formulas, together with (9), can be used to rewrite Eq. (7) in terms of derivatives with respect to s. To further simplify Eq. (7) we also use the definition (1) of RT, from which it follows that

$$\begin{aligned} \omega_1(\mathbf{r}_s) &= -v_s H(s), \qquad \omega_2(\mathbf{r}_s) = 0, \\ \omega_3(\mathbf{r}_s) &= -v_s K(s), \end{aligned} \tag{12}$$

and the fact that, on account of Maxwell equations and (10), the field gradient must be such that, on RT,

$$\begin{aligned} (\delta\mathbf{r} \cdot \nabla)\omega_2 &= (L + M)\, v_s x - K^2 n v_s z, \\ (\delta\mathbf{r} \cdot \nabla)\omega_3 &= -K^2 n v_s x + (L - M)\, v_s z, \end{aligned}$$

where

$$-K^2 n v_s = \frac{\partial\omega_2}{\partial z} = \frac{\partial\omega_3}{\partial x},$$

$$L v_s = -\frac{1}{2}\,\omega_1', \qquad M v_s = \frac{\partial\omega_2}{\partial x} + \frac{1}{2}\,\omega_1'.$$

Then the equations of motion (7) can be put in the form

$$\begin{aligned} p' &= \psi_1(1 - Kx), \\ x'' + K^2(1 - n)x &= -Kp + Hz' + H'z - (L - M)\, z + \psi_2, \\ z'' + K^2 n z &= -Hx' - H'x + (L + M)\, x + \psi_3. \end{aligned} \tag{13}$$

The left-hand side of the last two equations (13) has now a quite familiar form; ordinary focusing forces appear expressed by the field index n. Fringe fields can be properly accounted for since derivatives of the magnetic field, such as H' and L, have been retained. However it is often a very good approximation to consider machine parameters as stepwise varying functions of s, thus simplifying (13) by elimination of the terms describing fringing fields. In this case a lattice is precisely specified and (13), apart from the ψ_i terms that will be specified later, become constant coefficient equations in every cell of the lattice. However, edge focusing or short magnetic quadrupoles can be easily and accurately introduced in this lattice structure by the addition of zero-length cells in which the parameters have a δ-type behavior.

Rewriting (13) in the case of step machine parameters, one has

$$\begin{aligned} &p' = \psi_1(1 - Kx), \\ &x'' + K^2(1 - n)\, x - Hz' - Mz = -Kp + \psi_2, \\ &z'' + K^2 n z + Hz' - Mx = \psi_3. \end{aligned} \tag{14}$$

3. Our problem is now to find the solution of Eqs. (14). As we will see to find the solution we will use the fact that for electron synchrotrons or electron–positron storage rings the radiation reaction force is usually much smaller than magnetic forces (or the centrifugal force in a typical magnet). This fact we will refer to in the following as condition "a".

Condition "a" allows the use of perturbation techniques to solve (14). Hence the first thing we need is an explicit expression of ψ, which was defined by (8). Let us consider separately the various terms appearing in ψ starting from $\mathbf{R}$.

In $\mathbf{R}$ we can single out radiation effects by writting $\mathbf{R}$ as a sum of a term $\mathbf{R}_r$ due to the emission of synchrotron light plus other terms $\boldsymbol{\rho}^* E_s$.

The reason for this separation is that while the radiation reaction is known to be dependent on energy, position, and velocity, all other known random forces (mainly those due to collisions with gas atoms or with other particles of the circulating beams) are only energy-dependent.

Thus we will write

$$\frac{1}{E_s}\mathbf{R} = \frac{\mathbf{R}_r}{E_s} + \boldsymbol{\rho}_s^* + \mathbf{a}^* p, \tag{15}$$

where the subscript s indicates a quantity to be evaluated for $E = E_s$ and

$$\mathbf{a}^* = \left.\frac{\partial \boldsymbol{\rho}^*}{\partial p}\right|_{E=E_s}$$

The radiation reaction is given by (*6*)

$$\mathbf{R}_r = -\frac{2}{3}\frac{e^2}{c^5}\dot{\mathbf{r}}\gamma^4\left\{\ddot{\mathbf{r}}^2 + \frac{\gamma^2}{c^2}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})^2\right\},$$

where $\gamma = E/mc^2$. Introducing the quantity

$$w_s = \left(\frac{2}{3}e^2K^2\frac{\gamma_s^4}{E_s}\right) = \frac{2}{3}r_e\gamma_s^3K^2,$$

where r_e is the classical electron radius, $\mathbf{R}_r$ can be written, to the first order in the displacement, as

$$\begin{aligned}\frac{1}{E_s}\mathbf{R}_r = -w_s\Big\{\Big[1 + 2p + 2\left(Kn - \frac{H^2}{K}\right)x + \frac{2M}{K}z \\ - 2\frac{H}{K}z'\Big]\boldsymbol{\alpha} + (x' - Hz)\boldsymbol{\beta} + (z' + Hx)\boldsymbol{\gamma}\Big\}.\end{aligned} \tag{16}$$

We assume further that the fluctuation term $(1 - p)\mathbf{G}$, being the difference between the actual random value and the average value of the electron momentum variation, is such a small quantity that it can be evaluated directly on the RT; namely we assume (definition of $\mathbf{g}_s$)

$$(1 - p)\,\mathbf{G} \simeq \mathbf{G}_s \equiv \mathbf{g}_s E_s\,.$$

At last we consider the rf field, which is assumed to be longitudinal and dependent on a parameter σ measuring the phase of the particle with respect to the rf field and on s.

We further assume that the dependence on σ and s can be factorized so that we have

$$e\boldsymbol{\epsilon}/E_s = f(\sigma)\,\emptyset(s)\boldsymbol{\alpha}. \tag{17}$$

The field must be such as to satisfy the condition that, for fixed σ, $\oint \boldsymbol{\epsilon} \cdot d\mathbf{l}$ be independent of the path, so that one must have

$$\oint \boldsymbol{\epsilon} \cdot d\mathbf{l} = \oint_{\mathrm{RT}} \boldsymbol{\epsilon}_s \cdot d\mathbf{s}. \tag{17'}$$

Since $dl = (1 - Kx)\,ds$ it follows that $\boldsymbol{\epsilon}$ must either be of the form $\boldsymbol{\epsilon}_s \cdot (1 + Kx)$, in which case (17′) is satisfied up to terms of order x^2, or must be zero where K is different from zero, i.e., in the bending magnets. In the following we shall assume this second condition to be satisfied.

Now in a storage ring the average particle energy is kept constant so that the rf field must essentially balance the energy lost by radiation or other random effects. Hence we assume that a particle moving on RT has such a phase, σ_0, relative to the rf field that its energy loss is exactly compensated; namely,

$$f(\sigma_0) \oint \emptyset(s)\, ds = \oint (w_s - \rho_1^*)\, ds. \tag{18}$$

Equation (18) can be used as the definition of the synchronous phase σ_0.

A particle not moving on RT will have a phase displacement relative to the synchronous one given by

$$\sigma(s) - \sigma_0 = \int^s dl - \int^s ds = - \int^s Kx\, ds \tag{19a}$$

or

$$\sigma'(s) = -Kx. \tag{19b}$$

Notice that $\sigma(s)$ as defined here is just the longitudinal distance between the particle we are considering and the synchronous one; hence Eq. (19b) can be assumed as the one describing the longitudinal motion.

In most practical cases the rf field is obtained by means of several rf cavities distributed at equal distances along the machine and having a very short length compared to the cell of the magnetic lattice. It is then a good approximation to write

$$\frac{e\epsilon}{E_s} = \frac{eV_0}{N_R E_s} \sin(k\omega_0\sigma) \sum_{r=1}^{N_R} \delta\left(s - r\frac{L}{N_R}\right) \alpha \tag{20}$$

where L is the length of the rf, N_R is the number of rf cavities, $c\omega_0 = 2\pi c/L$ is the revolution frequency, $k\omega_0 c$ is the frequency of the rf cavity (k must be an integer number), and V_0 is the sum of the peak voltages of all the cavities.

We are now in a position to evaluate the order of magnitude of ψ. In fact from (18) it follows that the rf field must be of the same order of magnitude of $\mathbf{R}_r$, whose order of magnitude is given by w_s.

It is then easy to verify that the condition "a" mentioned at the beginning of this paragraph is satisfied as long as one has

$$r_e\gamma^3 K \ll 1. \tag{21}$$

For a machine in which the bending magnets have a field of the order of 10^4 Gauss this means $\gamma \lesssim 10^6$.

Hence all of the electron circular accelerators considered up to now satisfy (21) and, as a consequence, the condition "a".

Although it would be very interesting to study the situation in which condition "a" does not hold, we will not consider it in this paper.

4. As said in the previous section, to solve the equations of motion (14), we use a perturbation technique based on the assumption that condition "a" is satisfied. The procedure will be as follows:

(1) we solve (14) assuming $\psi = 0$; in this case one has p = constant and the equations determine the transverse motion to order $\psi = 0$, i.e., with no synchrotron oscillations, dampings, and fluctuation terms;

(2) using the preceding solution and assuming $\psi_1 \neq 0$ (but $\psi_2 = \psi_3 = 0$) we determine the longitudinal motion and the variation of energy;

(3) to the solutions obtained in this way we add a small term, due to the perturbation produced by ψ_2 and ψ_3 and then determine the complete solution of (14) by a perturbation technique.

Let us consider now the first step. Introducing the vectors

$$\hat{y}(s) = \begin{vmatrix} x \\ x' \\ z \\ z' \end{vmatrix} \qquad \hat{\chi}(s) = \begin{vmatrix} 0 \\ K \\ 0 \\ 0 \end{vmatrix}$$

and the matrix

$$A_0(s) = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -K^2(1-n) & 0 & M & H \\ 0 & 0 & 0 & 1 \\ M & -H & -K^2 n & 0 \end{vmatrix},$$

Eqs. (14) can be rewritten (when $\psi = 0$, and $p =$ constant),

$$\hat{y}'(s) = A_0(s)\,\hat{y}(s) - p\hat{\chi}(s) + \hat{y}_0\delta(s - s_0). \tag{22}$$

The vector $\hat{y}_0$ embodies the initial condition for $s = s_0$.

The solution of (22) is given by

$$\hat{y}(s) = \int_{-\infty}^{s} N_0(s, s')\{-p\hat{\chi}(s') + \hat{y}_0\delta(s' - s_0)\}\, ds', \tag{23}$$

provided that the matrix $N_0(s, s')$ satisfies the following equations:

$$\begin{aligned} (\partial/\partial s)\, N_0(s, s') &= A_0(s)\, N_0(s, s'); \\ (\partial/\partial s')\, N_0(s, s') &= -N_0(s, s')\, A_0(s'); \\ N_0(s, s) &= 1. \end{aligned} \tag{24}$$

The solution (23) can also be written in a simpler form if we choose the particular solution associated with the non homogenous term p to be periodic in s with the machine periodicity L.

Then we have

$$\hat{y}(s) = \hat{y}_\beta(s) - p\hat{\xi}(s),$$

where

$$\hat{y}_\beta(s) = N_0(s, s_0)\,\hat{y}_0 \tag{25}$$

and

$$\hat{\xi}(s) = \{1 - N_0(s, s - L)\}^{-1} \int_{s-L}^{s} N_0(s, s')\,\hat{\chi}(s')\, ds'. \tag{26}$$

The term $\hat{y}_\beta(s)$ describes the betatron oscillations and $p\hat{\xi}(s)$ the closed orbit, proportional to the energy displacement.

The frequencies of the betatron oscillations (usually of the same order of magnitude as ω_0) are defined by the characteristic equation

$$\det |\, N_0(s_0 + L, s_0) - \lambda I \,| = 0,$$

from which stability conditions can be investigated. This and other properties of the matrix $N_0(s, s')$ are briefly summarized in Appendix A.

We consider now the second step assuming for the moment ψ_1 to be given by

$$\psi_{10} = f(\sigma)\, \emptyset(s) - w_s + \rho_1^*,$$

and hence neglecting the fluctuation term and terms $\psi_{10}p$ or $\psi_{10}x$, which, as we shall see, give rise to secular effects.

Then the first of Eqs. (14) becomes

$$p' = f(\sigma)\, \emptyset(s) - w_s + \rho_1^*,$$

to which one must associate the Eq. (19b) defining σ, namely,

$$\sigma' = -Kx.$$

According to the decomposition of the transverse motion as given by (25) and (26), the equations for the energy variation and for the longitudinal motion can be rewritten as

$$\begin{aligned} p' &= f(\sigma)\, \emptyset(s) - w_s + \rho_1^*, \\ \sigma' &= K\xi_1 p - K y_{\beta_1}\,. \end{aligned} \tag{27}$$

Since we are mainly interested in the small amplitude motion around the synchronous particle we also linearize the function $f(\sigma)$:

$$f(\sigma) = f(\sigma_0) + \frac{\partial f}{\partial \sigma}\bigg|_{\sigma=\sigma_0} (\sigma - \sigma_0).$$

It must be noted however that $f(\sigma)$ contains the only relevant non linear terms: these terms have their full importance at injection when the dampings have not yet influenced appreciably the initial amplitudes.

Using the notation $F = \partial f(\sigma)/\partial\sigma\,|_{\sigma=\sigma_0}$ and introducing the vectors

$$\hat{\eta} = \begin{vmatrix} p \\ \sigma \end{vmatrix}, \qquad \hat{\tau} = \begin{vmatrix} -F\sigma_0\emptyset(s) + f(\sigma_0)\, \emptyset(s) - w_s + \rho_1^* \\ -K y_{\beta_1} \end{vmatrix},$$

$$\rho_0(s) = \begin{vmatrix} 0 & F\emptyset(s) \\ K\xi_1 & 0 \end{vmatrix},$$

the linearized equations obtained from (27) can be written as

$$\hat{\eta}'(s) = \rho_0(s)\, \hat{\eta}(s) + \hat{\tau}(s) + \hat{\eta}_0 \delta(s - s_0). \tag{28}$$

Here again the vector $\hat{\eta}_0$ gives the initial conditions.

The solution of (28) is

$$\hat{\eta}(s) = \int_{-\infty}^{s} M_0(s, s')\{\hat{\eta}_0 \delta(s - s') + \hat{\tau}(s')\}\, ds', \tag{29}$$

where the matrix $M_0(s, s')$ is defined in the same way as $N_0(s, s')$.

Assuming $\emptyset(s)$ to be given by the expression introduced in Section 3, namely,

$$\emptyset(s) = \sum_{r=1}^{N_R} \delta\left(s - r\frac{L}{N_R}\right),$$

it is easy to show by explicit integration of (28) with $\epsilon < s < \epsilon + L/N_R$, that the matrix $M(L/N_R + \epsilon, \epsilon)$ is given, in the limit $\epsilon \to 0$, by

$$\lim_{\epsilon \to 0+} M\left(\frac{L}{N_R} + \epsilon, \epsilon\right) = \begin{vmatrix} 1 + \dfrac{L\alpha F}{N_R} & F \\ \dfrac{L\alpha}{N_R} & 1 \end{vmatrix}, \tag{30}$$

where α, the momentum compaction, is given by

$$\alpha = \frac{N_R}{L} \int_{\epsilon}^{\epsilon + L/N_R} K(s)\, \xi_1(s)\, ds.$$

The frequencies of the energy and phase oscillations are determined by the characteristic equation

$$\lim_{\epsilon \to 0+} \det | M(L/N_R + \epsilon, \epsilon) - \lambda I | = 0.$$

Due to the simple structure of M [Eq. (30)],

$$\lambda = e^{\pm i\mu}$$

with

$$\cos \mu = 1 + \frac{1}{2} \frac{L\alpha F}{N_R} \tag{31}$$

Clearly the motion is stable only if $F < 0$. According to (20),

$$f(\sigma) = (eV_0/N_R E_S) \sin k\omega_0 \sigma,$$

and we have

$$F = \frac{eV_0 k\omega_0}{N_R E_S} \cos k\omega_0 \sigma_0$$

and

$$\cos \mu = 1 + \frac{1}{2} \frac{2\pi\alpha k e V_0}{N_R{}^2 E_S} \cos k\omega_0\sigma_0 . \tag{32}$$

As long as condition "a" (Section 3) is satisfied the second term on the right-hand side of (32) is much smaller than one. Hence we have

$$\mu^2 \simeq \left(\frac{2\pi}{N_R}\right)^2 \nu_s{}^2 = \left(\nu_s\omega_0 \frac{L}{N_R}\right)^2,$$

where

$$\nu_s{}^2 = -\frac{\alpha k e V_0}{2E_s\pi} \cos k\omega_0\sigma_0 .$$

Clearly ν_s measures the frequency of the longitudinal and energy oscillations in units of the revolution frequency. Notice that

$$\nu_s \ll 1.$$

To summarize the results of this section we can say that, in the limits of the approximations used, the particle performs fast betatron oscillations around the closed orbit $p\xi$, and phase and energy oscillations with frequency much smaller than ω_0.

At last we want to add a remark concerning the particular solution of (28) associated with the term $\hat{r}(s)$, namely,

$$\int_{-\infty}^{s} M_0(s, s')\, \hat{r}(s')\, ds'.$$

From the structure of $\hat{r}$ it follows that, when $\nu_s \ll 1$, this particular solution is small and can be neglected.

5. Let us write down (14) using the complete expression for ψ as given in Section 3, in particular [see Eqs. (15)–(17), (17')],

$$\begin{aligned} p' &= \emptyset_0(s)\, F(\sigma - \sigma_0) + f(\sigma_0)\, \emptyset_0(s) + \rho_1^* - w_s \\ &\quad - p\{f(\sigma_0)\, \emptyset_0(s) + \rho_1^* - a_1^* + w_s\} \\ &\quad - w_s \left\{2\left(Kn - \frac{H^2}{K}\right) x + \frac{M}{K} z - \frac{H}{K} z' - Kx\right\} - \rho_1^* Kx + g_{s1}, \end{aligned} \tag{33}$$

$$\begin{aligned} x'' + K^2(1-n)\, x - Mz - Hz' &= -Kp - w_s(x' - Hz) \\ &\quad + (1-p)\, \rho_z^* + a_2^* p + g_{s2}, \\ z'' + K^2 nz - Mx + Hx' &= -w_s(z' + Hx) + (1-p)\, \rho_3^* + a_3^* p + g_{s3}, \end{aligned}$$

where the notation $\emptyset(s)\,(1 - Kx) = \emptyset_0(s)$ has been used to simplify the notations [note that $K = 0$ whenever $\emptyset(s) \neq 0$ according to the statement in Section 3].

To (33) we add the other equation, defining σ [see (19b)]

$$\sigma' = -Kx.$$

On the basis of the results of the previous section we look for a solution of (33) of the form

$$\begin{aligned} p &= p_{s1} + p_\beta \\ \hat{y} &= \hat{y}_\beta - \hat{\xi}(p_{s1} + p_\beta), \end{aligned} \tag{34}$$

p_β being a quantity of order w_s and oscillating with the betatron frequency, and $\hat{\xi}$ being still defined by (26).

Actually an examination of the first of Eqs. (33) shows that these fast oscillating terms derive from the fluctuation term g_{s1}, and from the coupling with $\hat{y}_\beta$, and that this coupling occurs through terms proportional to w_s or ρ_1^*. Hence when we substitute (34) in the first of Eqs. (33), we can single out the fast and the slow oscillating parts of p and, neglecting all terms like $p_\beta f(\sigma_0)$ or $p_\beta w_s$, we obtain the two equations

$$p_\beta' = -w_s \left\{ 2\left(Kn - \frac{H^2}{K}\right) y_{\beta_1} + \frac{2M}{K} y_{\beta_3} - 2\frac{H}{K} y_{\beta_4} - Ky_{\beta_1} \right\} - \rho_1^* K y_{\beta_1}, \tag{35}$$

$$\begin{aligned} p_{s1}' = {} & \emptyset_0(s)\, F(\sigma - \sigma_0) + f(\sigma_0)\, \emptyset_0(s) + \rho_1^* - w_s - p_{s1} \Big\{ f(\sigma_0)\, \emptyset_0(s) \\ & + \rho_1^* - a_1^* + w_s - K\xi_1(\rho_1^* - w_s) - 2w_s \left[\left(Kn - \frac{H^2}{K}\right) \xi_1 \right. \\ & \left. + \frac{M}{K} \xi_3 - \frac{H}{K} \xi_4 \right] \Big\} + g_{s1} . \end{aligned} \tag{36}$$

Note that $w_s \sim K^2$ so that division by K causes no trouble.

Equation (36) differs from the corresponding Eq. (27) of section 4 only because of the addition of the fluctuation term and of the term proportional to p_{s1}. The coefficient of this term, which is the one responsible for the damping of the energy and phase oscillations, is small, being of order w_s, and periodic with the machine periodicity, so that it can be substituted by its average value D_s :

$$\begin{aligned} D_s = \frac{1}{L} \int_0^L \Big\{ & f(\sigma_0)\, \emptyset(s) + \rho_1^* + w_s - a_1^* - K\xi_1(\rho_1^* - w_s) \\ & - 2w_s \left[\left(Kn - \frac{H^2}{K}\right) \xi_1 + \frac{M}{K} \xi_3 \right] \Big\}\, ds. \end{aligned} \tag{37}$$

Notice that in the process of average the term $w_s(H/K)\,\xi_4$, which can be written as the derivative of a periodic function, drops out.

Hence Eq. (36), together with the other equation $\sigma' = -Kx$, can again be written in the form (28), namely

$$\hat{\eta}'(s) = \rho(s)\,\hat{\eta}(s) + \hat{\tau} + \hat{\eta}_0\delta(s - s_0) + \hat{\Gamma}_p \tag{38}$$

with

$$\rho(s) = \begin{vmatrix} -D_s & F\emptyset_0(s) \\ K\xi_1 & 0 \end{vmatrix} \tag{39}$$

and

$$\begin{aligned} \hat{\tau}(s) &= \begin{vmatrix} -F\emptyset(s)\,\sigma_0 + f(\sigma_0)\,\emptyset_0(s) + \rho_1^* - w_s \\ -Ky_1 + K\xi_1 p \end{vmatrix}, \\ \hat{\Gamma}_p(s) &= \begin{vmatrix} g_{s1} \\ 0 \end{vmatrix}. \end{aligned} \tag{40}$$

Next we come back to Eq. (33) to examine the behavior of betatron oscillations.

When substituting (34) in the last two of Eqs. (33) we make use of the following considerations:

(1) a part of the terms proportional to $p_{s1} + p_\beta$ drops out because of the definition of the vector $\hat{\xi}$;

(2) for the remaining term containing p_{s1} or its derivatives we assume, since we are now interested in fast oscillations,

$$p_{s1} = \text{const}, \qquad p'_{s1} = \bar{g}_{s1}\,;$$

(3) we neglect all terms proportional to w_s^2 or to $w_s\rho^*$;

(4) we neglect small closed orbit terms like ρ^* or $\rho^* p_{s1}$;

(5) we neglect all terms like $w_s x_\beta$, $w_s z_\beta$, $w_s x'_\beta$, $w_s z'_\beta$, when they are summed to one of the main focusing term like for instance in the sum $Mx_\beta + w_s x_\beta$.

As a result, with the help of (35), we obtain the following equations:

$$\begin{aligned} & x''_\beta + K^2(1 - n)\,x_\beta - Mz_\beta - Hz'_\beta \\ & \qquad + x'_\beta\{w_s[1 + 2Kn\xi_1 - K\xi_1] + \rho_1^* K\xi_1\} = \Gamma_1 \\ & z''_\beta + K^2 n z_\beta - Mx_\beta + Hx'_\beta \\ & \qquad + z'_\beta\left\{w_s\left[1 - 2\frac{H^2}{K}\xi_1 - 4\frac{H}{K}\xi_4 + 2\frac{M}{K}\xi_3\right]\right\} = \Gamma_2 \end{aligned} \tag{41}$$

where

$$\begin{aligned} \Gamma_1 &= g_{s2} + g_{s1}(2\xi_2 - H\xi_3) + g'_{s1}\xi_1\,, \\ \Gamma_2 &= g_{s2} + g_{s1}(2\xi_4 + H\xi_1) + g'_{s1}\xi_3\,. \end{aligned} \tag{42}$$

Since those coefficients of x_β' and z_β' which are proportional to w_s are small with respect to the main focusing forces, and are periodic with the machine periodicity, they can be substituted to a very good approximation with their average values which will be called D_{11}, D_{22}:

$$\begin{aligned} D_{11} &= \frac{1}{L}\int_0^L \{w_s(1 + 2nK\xi_1 - K\xi_1) + \rho_1^* K\xi_1\}\, ds, \\ D_{22} &= \frac{1}{L}\int_0^L \left\{w_s\left(1 - 2\frac{H^2}{K}\xi_1\right) + \frac{2M}{K}\xi_3\right\} ds. \end{aligned} \tag{43}$$

We again used the fact that the average value of $w_s(H/K)\,\xi_4$ is zero.

Eqs. (41) can now be written in vector form, namely,

$$\hat{y}_\beta' - A(s)\,\hat{y}_\beta = \hat{\Gamma} + \hat{y}_0\delta(s - s_0), \tag{44}$$

where

$$\hat{\Gamma} = \begin{vmatrix} 0 \\ \Gamma_1 \\ 0 \\ \Gamma_2 \end{vmatrix} \tag{45}$$

and

$$A(s) = A_0(s) + \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -D_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -D_{22} \end{vmatrix}. \tag{46}$$

We look now for solutions of (44) and (38) by the same procedure used in Section 4.

Defining a matrix $N(s, s')$ by means of Eqs. (24), but with A_0 substituted by A, the solution of (44) is

$$\hat{y}_\beta(s) = N(s, s_0)\,\hat{y}_0(s_0) + \int_{-\infty}^{s} N(s, s')\,\hat{\Gamma}(s')\, ds'. \tag{47}$$

The solution of (38) can also be written as

$$\hat{\eta}(s) = M(s, s_0)\,\hat{\eta}_0 + \int_{-\infty}^{s} M(s, s')\{\hat{\tau}(s') + \hat{\Gamma}_p(s')\}\, ds', \tag{48}$$

with $M(s, s')$ still defined by Eqs. (24) with $A_0(s)$ substituted by $\rho(s)$.

As remarked at the end of Section 4, the term $\int_{-\infty}^{s} M(s, s')\,\hat{\tau}(s')\, ds'$ can be neglected.

In this case, (48) becomes

$$\hat{\eta}(s) = M(s, s_0)\,\hat{\eta}_0 + \int_{-\infty}^{s} M(s, s')\,\hat{\Gamma}_p(s')\,ds', \tag{48a}$$

which has exactly the same structure as (46).

6. The solution of Eqs. (14) given by (47), (48) represents, in the case $\hat{\Gamma} = 0$, $\hat{\Gamma}_p = 0$, damped oscillations with the damping constants determined by the coefficients D_{11}, D_{22}, D_s, which satisfy the relationship

$$D_{11} + D_{22} + D_s = \frac{1}{L}\int_0^L (4w_s - a_1^*)\,ds.$$

This formula is well known apart from the term a_1^*.

While the damping constant for the synchrotron oscillations is simply given by (see Appendix 1):

$$1/\tau_s = -\tfrac{1}{2} D_s$$

the damping constants for the betatron oscillation are complicated functions of D_{11}, D_{22} (see Appendix 1). Only in the case when the radial and vertical motion are uncoupled we simply have

$$\frac{1}{\tau_x} = -\frac{1}{2} D_{11} = -\frac{1}{2L}\int_0^L w_s(1 + 2nK\xi_1 - K\xi_1) + \rho_1^* K\xi_1\,ds,$$

$$\frac{1}{\tau_r} = -\frac{1}{2} D_{22} = -\frac{1}{2L}\int_0^L w_s ds.$$

Let us now consider the case in which the fluctuation terms $\hat{\Gamma}$ and $\hat{\Gamma}_p$ are different from zero.

In particular, examination of solutions at regime, that is many damping constants later than injection, is relevant to the problem of the beam size.

Then we are no longer interested in the instantaneous values of the random vectors $\hat{y}_\beta$ or $\hat{\eta}$, but rather in the correlation functions $\langle y_{\beta r}(s)\, y_{\beta t}(s')\rangle$, $\langle \eta_r(s)\, \eta_t(s')\rangle$, where the averages now refer to the distribution of the random variables appearing in the fluctuation terms.

In particular we want to determine the quantities $\langle y_{\beta r}(s)\, y_{\beta t}(s)\rangle$ and $\langle \eta_r(s)\, \eta_t(s)\rangle$, which we shall write for brevity as

$$\langle y_{\beta r}\, y_{\beta t}\rangle, \qquad \langle \eta_r \eta_t\rangle.$$

In fact, once one has determined these quantities, the distributions of the energy and of $\hat{y}_\beta$ are given by

$$P_b(y_{\beta 1}y_{\beta 2}y_{\beta 3}y_{\beta 4}) = \frac{1}{\pi^2}\{\det m_{rt}\}^{+1/2} \exp\left\{-\sum_{r,t=1}^{4} m_{rt}y_{\beta r}y_{\beta t}\right\}, \tag{49}$$

$$P_c(\eta_1) = \frac{1}{(2\pi\langle\eta_1^2\rangle)^{1/2}} \exp - \frac{\eta_1^2}{2\langle\eta_1^2\rangle}, \tag{50}$$

where the quantities m_{rt} are related to $\langle y_{\beta r}\, y_{\beta t}\rangle$ by

$$\langle y_{\beta r}y_{\beta t}\rangle = \frac{1}{2}\frac{M_{rt}}{\det m_{rt}}, \tag{51}$$

and

$$\sum_r m_{rs}M_{rs} = \det M.$$

Both m_{rt} and $\langle y_{\beta r}\, y_{\beta t}\rangle$ are symmetric in r and t.

The folding of P_b and P_c according to (34) and neglecting p_β gives the distribution of the absolute positions and velocities.

Writing

$$\hat{y} = \hat{y}_\beta - \eta_1\hat{\xi},$$

one has

$$P(\hat{y}) = \int_{-\infty}^{+\infty} d\eta_1 p_c(\eta_1)\, P_b(\hat{y} + \eta_1\hat{\xi}).$$

The explicit result is

$$P(\hat{y}) = \frac{1}{\pi^2}(\det \mu_{rs})^{1/2} \exp\left\{-\sum \mu_{rs}y_ry_s\right\}, \tag{52}$$

where

$$\mu_{rs} = m_{rs} - \left\{\sum_{t,v} m_{rt}m_{sv}\xi_t\xi_v\left(\frac{1}{2\langle\eta_1^2\rangle} + \sum m_{tv}\xi_t\xi_v\right)^{-1}\right\} \tag{53}$$

Use has been made of the symmetry property $m_{rs} = m_{sr}$, which also holds true for the new coefficients $\mu_{rs} = \mu_{sr}$.

Local symmetry axes (somewhat like normal coordinates) can be introduced by diagonalization of the matrix $|\,\mu_{rs}\,|$; this can provide some information in connection with the observation of synchrotron light (see Appendix 2).

7. In this section we want to evaluate explicitely the quantities $\langle y_{\beta r} y_{\beta t}\rangle$, $\langle \eta_r \eta_t \rangle$. To do this we first notice that the quantity $\hat{\Gamma}_p(s)$, defined by (40) can be written as

$$\hat{\Gamma}_p(s) = \sum_j \sum_\alpha \begin{vmatrix} C_\alpha^{(j)} \\ 0 \end{vmatrix} \delta(s - s_\alpha) \tag{54}$$

and that, since the $\hat{C}^{(j)}$ are random variables (describing the random process "j"), one has the important property

$$\langle \Gamma_{p,r}(s)\, \Gamma_{p,t}(s')\rangle = \delta(s - s')\, K_{rt}^p(s), \qquad r, t = 1, 2, \tag{55}$$

where

$$K^p(s) = \begin{vmatrix} \sum_j \langle C^{(j)^2}(s)\rangle & 0 \\ 0 & 0 \end{vmatrix}. \tag{56}$$

This matrix will provide the relevant input data for the actual computation of $\langle \eta_r \eta_t \rangle$.

Then we follow the same procedure for the betatron oscillations. Notice first that the term $\hat{\Gamma}(s)$, defined by (42) (45), can be written as

$$\hat{\Gamma}(s) = \sum_j \sum_\alpha \{\hat{a}_\alpha^{(j)} f_1(s)\, \delta(s - s_\alpha) + \hat{b}_\alpha^{(j)} f_2(s)\, \delta'(s - s_\alpha)\}$$

and the insertion of this term in (47) is formally equivalent to the insertion of

$$\hat{\Gamma}^*(s) = \sum_j \sum_\alpha \{\hat{a}_\alpha^{(j)} f_1(s) - \hat{b}_\alpha^{(j)} f_2'(s) + A(s)\, \hat{b}_\alpha^{(j)} f_2(s)\}\, \delta(s - s_\alpha). \tag{57}$$

Then one also has, in the case of betatron oscillations like (55) for the phase motion, the following structure of the correlation functions of the random perturbing terms:

$$\langle \Gamma_r^*(s)\, \Gamma_t^*(s')\rangle = \delta(s - s')\, K_{rt}(s), \qquad r, t = 1, \ldots 4; \tag{58}$$

$K(s)$ will indicate the 4×4 matrix having elements $K_{rt}(s)$.

Let us now consider Eq. (47) which can also be rewritten in a form which makes explicit use of the periodicity of the magnetic structure, i.e.,

$$\hat{y}_\beta(s) = N(s, s - L)\, \hat{y}_\beta(s - L) + \int_{s-L}^{s} N(s, s')\, \hat{\Gamma}^*(s')\, ds'. \tag{59}$$

We shall abbreviate in the following $N(s, s - L)$ by $N(s)$.

It proves convenient to diagonalize $N(s)$; this can be accomplished by means of a matrix $U(s)$ such that (see Appendix 1)

$$\Delta(s) = U(s)\, N(s)\, U^{-1}(s)$$

is diagonal.

The transformed vector is

$$\hat{q}(s) = U(s)\, \hat{y}_\beta(s).$$

Also, due to the periodicity of $U(s)$,

$$\hat{q}(s - L) = U(s)\, \hat{y}_\beta(s - L)$$

so that

$$\hat{q}(s) = \Delta(s)\, \hat{q}(s - L) + \int_{s-L}^{s} U(s)\, N(s, s')\, \hat{\Gamma}^*(s')\, ds'. \tag{59a}$$

Next introduce the correlation functions

$$Q_{rs}(s_1, s_2) = \langle q_r(s_1)\, q_s(s_2)\rangle,$$
$$G_{rs}(s_1, s_2) = \langle q_r(s_1)\, \Gamma_s(s_2)\rangle;$$

Q and G will indicate in the following the corresponding 4×4 matrices.

Assuming that the machine is stable, a random-stationariety character [8] of Q can be recognized in the following property:

$$Q(s_1, s_2) = Q(s_1 + nL, s_2 + nL),$$

where n is any integer.

Then recalling the property (58), an equation for G is easily derived from (59a) in the form

$$G(s, s') = \Delta(s)\, G(s - L, s') + U(s)\, N(s, s')\, K(s')\, \theta(s, s').$$

Here $\theta(s, s')$ is a function (not a matrix) such that

$$\theta(s, s') = 1 \qquad \text{when} \quad s - L \leqslant s' \leqslant s,$$
$$= 0 \qquad \text{otherwise.}$$

It follows that, as required by causality,

$$G(s, s') = 0 \qquad \text{when} \quad s' > s,$$
$$= U(s)\, N(s, s')\, K(s) \qquad \text{when} \quad s - L \leqslant s' \leqslant s,$$

and this is all what we need together with the random-stationariety of Q to get the following equation for $Q(s) = Q(s, s)$:

$$Q(s) = \Delta(s)\, Q(s)\, \Delta(s) + R(s), \tag{60}$$

where

$$R(s) = U(s) \int_{s-L}^{s} N(s, s')\, K(s)\, N^{T}(s, s')\, ds'\, U^{T}(s).$$

Here U^{T} is the transpose of U, etc; use has been made of the obvious property $K(s) = K^{T}(s)$.

Introducing the diagonal matrix (see Appendix 1)

$$T(s, s') = U(s)\, N(s, s')\, U^{-1}(s')$$

and the matrix

$$Z(s, s') = U(s')\, K(s')\, U^{T}(s'),$$

$R(s)$ can also be written as

$$R(s) = \int_{s-L}^{s} T(s, s')\, Z(s')\, T^{\cdot}(s, s')\, ds'. \tag{61}$$

The solution of (60) is

$$Q(s) = \sum_{0}^{\infty} [\Delta(s)]^{k}\, R(s) [\Delta(s)]^{k}.$$

This series can be summed by virtue of the fact that Δ is diagonal so that

$$Q_{rs} = \frac{1}{1 - \Delta_r \Delta_s} R_{rs}\,.$$

It also easily follows that the quantities we need are given by

$$\langle y_{\beta r} y_{\beta s} \rangle = \sum_{r', s'} \frac{U^{-1}_{rr'} U^{-1}_{ss'}}{1 - \Delta_{r'} \Delta_{s'}} R_{r's'}\,. \tag{62}$$

This formula provides the formal solution of the problem. Its structure is better understood by writting

$$R(s) = U(s)\, \mathscr{R}(s)\, U^{T}(s)$$

and introducing the 4-indices simbol

$$M_{rtsv} = \sum_{r's'} \frac{U^{-1}_{rr'} U_{r't} U^{-1}_{ss'} U_{s'v}}{1 - \Delta_r \Delta_{s'}}.$$

Then $\langle y_{\beta r} y_{\beta s} \rangle$ can be expressed as the product of a factor depending on machine structure only (M) and a factor depending on radiation ($\mathscr{R}$):

$$\langle y_{\beta r} y_{\beta s} \rangle = \sum_{t,v} M_{rtsv} \mathscr{R}_{tv} .$$

In particular, resonant filtering of the radiation noise is exhibited in the denominators $1 - \Delta_{r'} \Delta_{s'}$ appearing in the definition of M_{rtsv} ; analysis of such denominators for a given machine structure is generally important in itself.

In fact, for a stable machine, one can write

$$\Delta_{1,2} = \exp[(-\epsilon_1 \pm i\nu_1) L]; \qquad \Delta_{3,4} = \exp[(-\epsilon_2 \pm i\nu_2) L],$$

with $\epsilon_{1,2} L \ll 1$ and $\nu_{1,2}$ real, so that the main terms of (62) are given by

$$\begin{aligned} \langle y_{\beta r} y_{\beta s} \rangle = {} & \frac{1}{2\epsilon_1 L} \{U^{-1}_{r1} U^{-1}_{s2} R_{12} + U^{-1}_{r2} U^{-1}_{s1} R_{21}\} \\ & + \frac{1}{2\epsilon_2 L} \{U^{-1}_{r3} U^{-1}_{s4} R_{34} + U^{-1}_{r4} U^{-1}_{s3} R_{43}\}. \end{aligned} \tag{63}$$

The evaluation of $\langle \eta_r \eta_t \rangle$ can, of course, be performed in the same way as for $\langle y_{\beta r} y_{\beta t} \rangle$. Calling $M(s, s - (L/N_R))$ the transfer matrix over one period defined by the rf cavities, and $\Omega(s)$ the diagonal matrix obtained by a similarity transformation generated by $V(s)$,

$$\Omega(s) = V(s)\, M(s, s - L/N_R)\, V^{-1}(s),$$

writing $\Omega_{1,2} = \exp[(-\epsilon_p \pm i\nu_s \nu_0) L/N_R]$, we have

$$\langle \eta_r \eta_t \rangle = \frac{1}{2\epsilon_p} \frac{N_R}{L} \{V^{-1}_{r1} V^{-1}_{t2} R^p_{12} + V^{-1}_{r2} V^{-1}_{t1} R^p_{21}\}, \tag{64}$$

where

$$R^p(s) = V(s) \int_{s-L/N_R}^{s} M(s, s')\, K^p(s')\, M^T(s, s')\, ds'\, V^T(s). \tag{65}$$

8. While (64) can be easily put in the usual, and more practical, form

$$\langle \eta_r \eta_t \rangle = \frac{\tau_s}{4} \frac{N_R}{L} \int_{s-L/N_R}^{s} \sum_j \langle c_j^2(s') \rangle \, ds' \cdot \begin{vmatrix} 1 - \frac{1}{4}\frac{L\alpha F}{N_R} & -\frac{1}{2}\frac{L\alpha}{N_R} \\ -\frac{1}{2}\frac{L\alpha}{N_R} & -\frac{L\alpha}{N_R F} \end{vmatrix}, \tag{66}$$

the computation of the transverse beam size by means of (63) is quite cumbersone and requires the use of a computer.

This task has been performed succesfully for the case of the (2×1.5)-BeV ring Adone, by M. Bassetti and M. Buonanni (unpublished), where useful numerical data on the beam size was indeed obtained.

However, in the case when the vertical and radial motion are uncoupled, (63) also can be reduced to a very simple form. We think it useful to show how can be done for instance for the case of the radial betatron motion. Of course the same procedure can be used to obtain (66) or the vertical dimension.

Using the notations of Ref. (7), the matrix $N(s, s')$ can be written in the one-dimensional case as

$$N(s, s') = \begin{vmatrix} \left[\frac{\beta(s)}{\beta(s')}\right]^{1/2} \{\cos\psi + \alpha(s') \operatorname{sen}\psi\} & [\beta(s)\,\beta(s')]^{1/2} \operatorname{sen}\psi \\ \frac{[\alpha(s') - \alpha(s)]\cos\psi - [1 + \alpha(s')\,\alpha(s)] \operatorname{sen}\psi}{[\beta(s)\,\beta(s')]^{1/2}} & \left[\frac{\beta(s')}{\beta(s)}\right]^{1/2} \{\cos\psi - \alpha(s) \operatorname{sen}\psi \end{vmatrix}.$$

The matrix $U(s)$ is given by

$$U(s) = \frac{1}{2i} \begin{vmatrix} i + \alpha & \beta \\ i - \alpha & -\beta \end{vmatrix}$$

and with the help of (67) one can write the matrix $T(s, s')$ as

$$T(s, s') = \left\{\frac{\beta(s)}{\beta(s')}\right\}^{1/2} \begin{vmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{vmatrix}$$

The next step is to write explicitely the matrix $K(s)$. To this end we assume that, in expression (42), the term g_{s2} can be neglected, as is usually possible, to a good approximation, and that g_{s1} is defined as in (54).

Then using (57), (58) one has

$$K(s) = \sum_j \langle c_j^2(s)\rangle \begin{vmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{vmatrix}. \tag{69}$$

Using (67), (69) we get for the off-diagonal elements of the matrix $Z(s, s')$

$$Z_{12}(s) = Z_{21}(s) = \tfrac{1}{4}\{(1 + \alpha^2(s))\, \xi_1^2(s) + 2\beta(s)\, \alpha(s)\, \xi_1(s)\, \xi_2(s) + \beta^2(s)\, \xi_2^2(s)\} \sum_j \langle c_j^2(s)\rangle. \tag{70}$$

We have now all the elements necessary to evaluate R_{12} and hence $\langle y_{\beta r}\, y_{\beta s}\rangle$ as defined by (63). The result is

$$\langle y_{\beta r} y_{\beta t}\rangle = \frac{\tau_{\beta r}}{4} \frac{1}{L} \int_{s-L}^{s} \sum_j \langle c_j^2(s')\rangle \left\{\frac{1 + \alpha^2(s')}{\beta(s')}\, \xi_1^2(s') + 2\alpha(s')\, \xi_1(s')\, \xi_2(s') + \beta(s')\, \xi_2^2(s')\right\} ds' D_{rt}(s). \tag{71}$$

Here $D_{rt}(s)$ is the r, t element of

$$D(s) = \begin{vmatrix} \beta(s) & -\alpha(s) \\ -\alpha(s) & \dfrac{1 + \alpha^2(s)}{\beta(s)} \end{vmatrix}.$$

This result agrees with the well known formulas in the literature (*2*).

APPENDIX 1

We collect here for completeness the relevant properties of the matrix $N(s, s')$ as defined by (23) and (24).

The first property we mention is

$$N(s'', s)\, N(s, s') = N(s'', s'). \tag{A.1}$$

This property is completely equivalent to the set of properties (24) since, when $s'' = s$, (A.1) gives $N(s, s) = 1$ and when $s'' = s + ds$ the equations of motion give

$$N(s + ds, s) = 1 + A(s)\, ds.$$

The second property concerns the diagonalizability of the matrix $N(s, s')$. First notice that $N(s, s')$ can be reduced to it's Jordan canonical form; then, from the continuity of $N(s, s')$ and the fact that $N(s, s) = 1$, it follows that $N(s, s')$ is diagonalizable.

The third property concerns the independence of the eigenvalues of $N(s)$ on s; this is easily shown by using property (A.1) as follows:

$$\det(N(s) - \lambda I) = \det\{N(s, s')\,[N(s') - \lambda I]\,N^{-1}(s, s')\} = \det(N(s') - \lambda I)$$

for every s, s'.

The fourth property is perhaps less evident; the matrix

$$T(s, s') = U(s)\,N(s, s')\,U^{-1}(s') \tag{A.2}$$

is diagonal. Here $U(s)$ is the transformation matrix such that, by definition,

$$\Delta(s) = U(s)\,N(s + L, s)\,U^{-1}(s) = U(s)\,N(s)\,U^{-1}(s)$$

is diagonal.

To prove the property (A.2) let us introduce the orthonormal eigenvectors $\hat{e}_i(s)$ and the eigenvalues λ_i (independent of s) of the matrix $N(s)$:

$$N(s)\,\hat{e}_i(s) = \lambda_i \hat{e}_i(s); \tag{A.3}$$

then put

$$N(s, s')\,\hat{e}_i(s') = \sum_l a_k^{(i)}(s, s')\,\hat{e}_k(s). \tag{A.4}$$

Now

$$N(s + L, s')\,\hat{e}_i(s') = N(s + L, s' + L)\,N(s')\,\hat{e}_i(s') = N(s, s')\,\lambda_i \hat{e}_i(s'),$$

so that multiplying (A.4) by $N(s + L, s)$ we get

$$N(s, s')\,\lambda_i \hat{e}_i(s') = \sum_l a_k^{(i)}(s, s')\,\lambda_k \hat{e}_k(s). \tag{A.5}$$

Next, multiplication of (A.4) by λ_i and substraction of (A.5) gives

$$\sum a_k^{(i)}(s, s')\,\hat{e}_k(s')\{\lambda_i - \lambda_k\} = 0. \tag{A.6}$$

We will assume nondegenerate eigenvalues for a properly operating machine; thus the linear independence of the eigenvalues requires

$$a_k^{(i)}(s, s') = a_k(s, s')\,\delta_{ki}\,.$$

It follows that

$$e_k^+(s)\,N(s, s')\,e_i(s') = a_i(s, s')\,\delta_{ki}\,, \tag{A.7}$$

where $e_k^+(s)$ indicates the Hermitian conjugate (one-row vector) of $e_k(s)$. But (A.7) is equivalent to (A.2) since the matrix $U(s)$ is build up by using the eigenvectors according to the well known technique for diagonalization.

Eventually, we note that the damping constants will be obtained from the secular equation for $N(s)$ and that only the eigensolution will have a simple exponential decay. Moreover, the homogeneous equation $y' = A(s)\,y$ will in general have a Wronskian proportional to

$$\exp\left\{-\int [\text{Trace } A(s)]\, ds\right\}.$$

APPENDIX 2

The distribution in positions as deduced from (52) by integrating over the angles can be expressed in an equivalent form in which local symmetry axes appear. To this end, let us introduce locally rotated axes $\bar{x}$, $\bar{z}$ by the transformation

$$\bar{x} = x \cos\theta + z \sin\theta, \qquad \bar{z} = -x \sin\theta + z \cos\theta.$$

By choosing

$$\operatorname{tg} 2\theta = 2\mu_{xz}/(\mu_{xx} - \mu_{zz})$$

and then putting

$$\bar{\mu}_{xx} = \frac{1}{2}(\mu_{xx} + \mu_{zz}) + \frac{\mu_{xz}}{\sin 2\theta},$$

$$\bar{\mu}_{zz} = \frac{1}{2}(\mu_{xx} + \mu_{zz}) - \frac{\mu_{xz}}{\sin 2\theta},$$

the distribution transforms into

$$P(\bar{x}, \bar{z}) = \pi^{-1}(\bar{\mu}_{xx}\bar{\mu}_{zz})^{1/2} \exp - (\bar{\mu}_{xx}\bar{x}^2 + \bar{\mu}_{zz}\bar{z}^2).$$

Thus, looking at the beam section (e.g., by the light), an elliptic spot appears having symmetry axes along the direction $\bar{x}$ and $\bar{z}$. Also, the inverse effective area S in the luminosity (for two crossing overlapping beams) is given by

$$\frac{1}{S} = \int P^2(\bar{x}, \bar{z})\, d\bar{x}\, d\bar{z} = \frac{1}{2\pi}(\bar{\mu}_{xx}\bar{\mu}_{zz})^{1/2} = \frac{1}{2} P(0, 0).$$

$1/S$ is thus half the maximum transverse density.

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