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P. Di Vecchia, F. Drago and M. L. Paciello : KINEMATICAL CONSTRAINTS, FACTORIZATION AND KINEMATICAL ZEROS, I.

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P. Di Vecchia, F. Drago and M. L. Paciello[(x)]: KINEMATICAL CONSTRAINTS, FACTORIZATION AND KINEMATICAL ZEROS, I. -

ABSTRACT. -

The kinematical constraints at $t = 0$ are derived for an important class of reactions. The factorization requirements for the Regge pole residue functions are studied. The "minimal" behaviour at $t = 0$ of the Regge residue functions, consistent with the kinematical constraints and the factorization, is given. The results are compared with those obtained in the group theoretical approach.

I. INTRODUCTION. -

The success of the Regge pole theory in the description of the strong interaction physics has given rise during the last years to an intense analysis of the crossing symmetry and the analyticity properties of the helicity amplitudes. Starting from the results obtained in this field, the kinematical singularities and zeros can be investigated. Using these two properties and following different ways Y. Hara[(1)] and L. L. Wang[(2)] evaluated the kinematical singularities of the parity conserving helicity amplitudes. The

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part of the program which was not completed by these authors was to obtain a prescription for all kinematical zeroes. These further zeroes of some suitable combinations of helicity amplitudes were found by Cohen Tannoudji et al.[3] in a very extensive paper using the formalism of the transversity amplitudes.

In this paper we consider the series of reactions:

$$\begin{aligned} &S + N \rightarrow J + N \\ &N + N \rightarrow N + N \\ &S + S \rightarrow J + J \end{aligned} \qquad \text{(I. 1)}$$

where N is a nucleon and J(S) is a spin J(S) and mass $m_J(m_S)$ particle with $m_J \neq m_S \neq$ nucleon mass. Then, using Wang's formalism, we evaluate the "minimal" behaviour of the Regge pole residue functions satisfying analyticity, crossing symmetry and factorization.

In Section II we write down, in terms of the helicity amplitudes, the kinematical constraints at $t = 0$ for the reactions of type of (I. 1). The details of the calculations are given in the Appendices A and B.

In sec. III we write down the factorization requirements near $t = 0$ for the Regge pole residue functions free from any kinematical singularity. Finally in Sec. IV we study the consequences of the factorization and of the kinematical constraints. We find a finite number of "minimal" solutions characterized by a number M for the Regge pole residue functions. Then we compare our results with those of Cosenza, Sciarrino, Toller (CST)[4] and find full agreement, at least in the limit where we neglect the daughter trajectories.

In this work we will unevitably cover some arguments partially treated previously in the literature[5].

This program, extended to any type of reaction takes into account in a very general way of the constraints imposed by analiticity, crossing symmetry and factorization, and therefore submits the Regge pole theory to more and more stringent tests.

II. CONSPIRACY RELATIONS. -

We shall use the formalism of L.L. Wang[2] denoting by $f^t_{c'A';D'b'}$ a helicity amplitude for the t-channel reaction $D' + b' \rightarrow c' + A'$. Helicity amplitudes free from kinematical singularities in s and t must be used in the derivation of the constraints. The first step is to define amplitudes free from kinematical singularities in s:

$$\text{(II. 1)} \qquad \bar{f}^{\,t}_{c'A';D'b'} = \left(\text{sen}\,\frac{\theta_t}{2}\right)^{-|\lambda'-\mu'|}\left(\cos\frac{\theta_t}{2}\right)^{-|\lambda'+\mu'|} f^{\,t}_{c'A';D'b'}$$

where $\lambda' = D'-b'$, $\mu' = c'-A'$.

Then the works of Hara(1) and Wang(2) show how one can remove the t kinematical singularities from the amplitudes formed into "parity conserving" combinations. The Wang result can be written:

$$\text{(II. 2)} \qquad \bar{f}^{\,t}_{c'A';D'b'} \pm \bar{f}^{\,t}_{-c'-A';D'b'} = K^{\pm}_{c'A';D'b'}\,(\tilde{f}^{\,t\pm}_{c'A';D'b'})$$

where $K^{\pm}(t)$ is a known factor containing the kinematical singularities at $t=0$. The analyticity requirements and the crossing symmetry will provide additional kinematical zeroes at $t=0$ in certain linear combinations of the parity conserving helicity amplitudes. We consider first the set of the reactions $S+N \rightarrow J+N$ where N is a nucleon and J(S) is a spin J(S) and mass $m_J(m_S)$ particle with $m_J \neq m_S \neq$ nucleon mass. We suppose in the following that $J \geqslant S$.

The kinematical constraints for these processes at $t=0$ are given by:

$$\text{(II. 3)} \qquad i\,\tilde{f}^{\,(+)t}_{c'A';\frac{1}{2}-\frac{1}{2}} - \tilde{f}^{\,(-)t}_{c'A';\frac{1}{2}\frac{1}{2}} = 0\,(t)$$

for any c' and A' satisfying the inequality $c' \neq A'$ and by:

$$\text{(II. 4)} \qquad \frac{i}{2}\,\tilde{f}^{\,(-)t}_{c'A;\frac{1}{2}-\frac{1}{2}} - \tilde{f}^{\,(-)t}_{c'A';\frac{1}{2}\frac{1}{2}} = 0\,(t)$$

for any c' and A' satisfying the equality $c' = A'$. The details of the derivation of these constraints are given in App. A. The sign (±) refers to the dominant parity(x) exchanged in the t channel of the reactions S+N J+N, following the notations used in Ref. (6). The number of the independent constraints (II. 3) is given by J (2S+1), while the number of those (II. 4) is given by (2S+u)/2 with

$$\text{(II. 5)} \qquad u \begin{cases} = 0 & \text{if} \quad \sigma_J\,\sigma_S = 1 \\ = 1 & \text{if} \quad \sigma_J\,\sigma_S = -1 \end{cases}$$

where $\sigma_J = \eta_J(-1)^J$ and η_J is the intrinsic parity of the particle J.

We next, consider the reactions of the type of $S+S \rightarrow J+J$ and write down the constraints that must be satisfied at $t=0$:

$$\text{(II. 6)} \qquad f^{\,t(+)}_{c'A';D'b'} + f^{\,t(-)}_{c'A';D'b'} = 0\,(t^{m(\lambda',\mu')})$$

(x) Note that this (±) sign in general is different from that defined in Eq. (II. 2).

for λ' and μ' satisfying the conditions $\lambda' \neq 0$ and $\mu' \neq 0$. The details of the calculations are presented in App. B.

Finally we write the constraint in $N+N \rightarrow N+N$

$$f^{t}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}} + f^{t}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} - f^{t}_{\frac{1}{2}\frac{1}{2};-\frac{1}{2}-\frac{1}{2}} - f^{t}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}} = 0 \qquad \text{(II. 7)}$$

The derivation can be found in Ref. (7).

If only Regge poles contribute to the scattering amplitudes the additional kinematical constraints can be satisfied in either of two ways:

a) Each of the two amplitudes involved in an equation has an additional factor of t in the residue: in this case the relation is trivially satisfied (evasion).

b) The amplitudes f involved in the kinematical constraint go to a constant when $t \rightarrow 0$; this implies a relation between the intercepts $\alpha(0)$ and the residues $\beta(0)$ of the various trajectories which contribute to the f (conspiracy). We can think of two mechanisms for the conspiracy:

1) There are two Regge poles whose trajectories and residues are related at $t = 0$.
2) Daughter trajectories[5, 8] can combine with the mother in order to satisfy the constraint relation.

In the group theoretical approach[9-12] a conspiracy between trajectories of different spin-parity is called Class III and this is needed in order to satisfy relations (II. 3) and (II. 6)[(x)]. A conspiracy between trajectories of the same spin-parity $P = (-1)^{J+1}$ is called class II and is needed for the relations (II-4).

Class I contains only non conspiring $P = (-1)^{J}$ trajectories.

III. THE REGGE RESIDUE FACTORIZATION NEAR t = 0. -

The partial wave expansion for the parity conserving helicity amplitudes, free from s kinematical singularities in given by[(6)]:

$$\bar{f}^{(\pm)t}_{c'A';D'b'} = \sum_{J} (2J+1) \left[e^{J+}_{\lambda\mu} F^{J\pm}_{c'A';D'b'} + e^{J-}_{\lambda\mu} F^{J\mp}_{c'A';D'b'} \right] \qquad \text{(III. 1)}$$

where the $e^{J\pm}$ functions are defined in Ref. (6).

(x) - We are considering here only the asymptotic behaviour in s.

The contribution of a single Regge pole to the previous amplitudes is given by:

(III.2) $$\frac{2\alpha^{\pm}(t)+1}{\mathrm{sen}\,\pi\alpha^{\pm}(t)}\,\bar{K}^{\pm}_{c'A';D'b'}(t)\left(\frac{P_{D'b'}\,P_{c'A'}}{S_o}\right)^{\alpha^{\pm}(t)-N}\gamma^{\pm}_{c'A';D'b'}\,(\cos\theta_t)^{\alpha^{\pm}(t)-N}$$

where $K^{\pm}$ is Wang's kinematical factor, N is the maximum between $|\lambda'|$ and $|\mu'|$, and $\gamma^{\pm}_{c'A';D'b'}$ is the residue free from kinematical singularities at $t=0$.

Here and in the following we omit the signature factor which is not essential for our considerations and we assume that the Regge representation survives at $t=0$ because the daughter trajectories come into play[8].

Now we impose the factorization conditions for the residue functions of the Regge poles.

The objects which factorize are the residues of the individual poles in $F^{J\pm}_{c'A';D'b'}$; therefore in our case the residue is:

(III. 3) $$\bar{K}^{\pm}_{c'A';D'b'}(t)\;\gamma^{\pm}_{c'A';D'b'}(t)\;(P_{c'A'}\,P_{D'b'})^{\alpha^{\pm}(t)-N}$$

Hence if reaction 1 is $D'+b'\rightarrow c'+A'$, reaction 2 is $D'+b'\rightarrow D'+b'$ and reaction 3 is $c'+A'\rightarrow c'+A'$ the factorization theorem can be stated as:

(III. 4) $$\left[\bar{K}^{\pm}_{1\,c'A';D'b'}(t)\;\gamma^{\pm}_{1\,c'A';D'b'}(t)\;(P_{c'A'}\,P_{D'b'})^{\alpha^{\pm}(t)-N_1}\right]^2 = \left[\bar{K}^{\pm}_{2\,D'b';D'b'}(t)\;\gamma^{\pm}_{2\,D'b';D'b'}\,(P_{D'b'})^{2(\alpha^{\pm}(t)-N_2)}\right]\times\left[\bar{K}^{\pm}_{3\,c'A';c'A'}(t)\,\gamma^{\pm}_{3\,c'A';c'A'}\,(P_{c'A'})^{2(\alpha^{\pm}(t)-N_3)}\right]$$

In Table I we give the behaviour of the factors $K^{\pm}_i(t)$, while the behaviour near $t=0$ of both sides of equation (III. 4) is given in Table II a for the reaction $S+N\rightarrow J+N$ and in Table II b for the reactions $S+S\rightarrow J+J$ and $N+N\rightarrow N+N$.

IV. RESULTS AND COMPARISON WITH THE CST APPROACH. -

In this section, under the assumption that only moving poles[x] are present in the complex J plane, we give the behaviour of the Regge pole residues near $t=0$ allowed by the kinematical constraints and the factorization theorem.

In the following discussion we will consider only the asymptotic term in s and we will neglect the daughter trajectories: this assumption implies, as we will show below, that no class II conspiracy exists[o].

In general, from a given solution to all the constraints, one can obtain other solutions by increasing the number of powers of some of the Regge pole residues in the original solution. If a given solution cannot be obtained from another solution in this way, we shall call it "minimal".

The "minimal" solutions for the Regge pole residue functions for the reactions with unequal masses of the type $S+S \to J+J$ are given by (Appendix C):

$$\text{(IV.1)} \qquad \gamma^{\pm}_{c'A';c'A'} \sim t^{\left|\,|\mu'| - M\right|}$$

while those for the process $N+N \to N+N$ can be written as:

$$\text{(IV.2)} \qquad \gamma^{\pm}_{D'b';D'b'} \begin{cases} \sim t^{M} & \text{if} \quad \sigma(-1)^{\lambda'} = 1 \\ \sim t^{|M-1|} & \text{if} \quad \sigma(-1)^{\lambda'} = -1 \end{cases}$$

M is a number which we introduce to label the "minimal" solutions. It can assume all the integer values between 0 and the Maximum (1;J+S). When $M=0$ the expression (IV.2) for the (−) functions must be slightly modified (see Appendix C). The behaviour of the other residues involved in the set of reactions (I.1) can be obtained using the factorization conditions (see Table IIIa for $M=0$ and Table IIIb for $M \neq 0$).

We can thus generalize to the set reactions (I.1) the results that we obtained in the case of the vector meson photoproduction[13] and the other related channels.

Natural parity Regge poles, like P and P', which are necessary to explain the high energy behaviour of the total nucleon-nucleon cross sec

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(x) - In principle there can be other contributions which are not requested to factorize such as branch cuts or Regge poles which become degenerate at $t=0$.

(o) - We are indebted to Prof. M. Toller and Dr. G. Cosenza for a very useful discussion about this point.

tion, must satisfy the constraints be evasion: only in this case in fact can they give a non vanishing contribution to the imaginary part of the amplitude $f^{(+)t}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$. In other words they are characterized by the value M = 0.

These results agree with those derived from the Lorentz or 0 (4) symmetry of the scattering amplitude.

Following the notations of ref. (10) such poles, would belong to a family of poles with M = 0, $\tau = 1$, $\sigma = 1$. These trajectories never conspiracy in the group theoretical approach. This is the class I.

We note that the constraints (II. 4) have, to be satisfied by evasion for any value of M; while, for the constraints (II. 3) and (II. 6), conspiracy and evasion are both allowed. In particular we find that for M = 1 all the constraints (II. 3) and (II. 6)[(x)] are satisfied by conspiracy. This is a class III conspiracy.

For values of M > 1 some constraints are satisfied by evasion, others by conspiracy. We remark once more that in this approach, the only possibility open is a parity doublets conspiracy mechanism: the class II conspiracy requires the presence of daughter trajectories, with singular residue functions, which are not included in our formalism.

Using 0 (3, 1) symmetry and some further hypothesis CST were able to give the behaviour of Regge pole residues near t = 0 satisfying the kinematical constraints and the factorization theorem in any kind of reactions.

In the case of the unequal mass scattering their result agree with (IV. 1), where now M is the Lorentz quantum number (see for instance Ref. (10)). By comparison of our results with those of CST we can understand the physical meaning of the number M that previously was introduced in order to classify the various "minimal" solutions.

In the case of the equal mass scattering the CST's result does not agree with (IV. 2). This is due to the following two reasons:

1) The CST scheme takes account automatically of the daughter trajectories, while we neglect them.
2) CST use too restrictive hypotheses which causes the identical vanishing of some Regge residues[(o)].

It would be interesting to use these results in the phenomenological fits in order to submit the Regge pole theory to more and more stringent tests.

(x) - i. e. those connecting amplitudes dominated by Regge poles of different spin-parity.

(o) - We were informed by Prof. M. Toller that very recently the obtained a more general solution in which no identically zero residues are present.

It is a pleasure to thank Prof. L. Bertocchi and M. Toller for many useful discussions. After the conclusion of this work we were informed by Prof. J. D. Jackson that relations (II. 3) and (II. 4) were derived independently by J. D. Stack.

APPENDIX A -

The method used for deriving the kinematical constraints in the channel $S+N \to J+N$ is due to Cohen-Tannoudji, Morel and Navelet[3]. Let us consider the process

$$\text{(A.1)} \qquad S\ (a) + N\ (b) \to J\ (c) + N\ (d)$$

where the expressions between brackets refer to the helicity of the corresponding particle. The t-channel of the reaction (A.1) is:

$$\text{(A.2)} \qquad \bar{N}\ (D') + N\ (b') \to J\ (c') + \bar{S}\ (A')$$

The helicity amplitudes of the processes (A.1) and (A.2) are related, through the crossing matrix[14], by

$$\text{(A.3)} \qquad \bar{f}^{s}_{cd;ab} = \left(\operatorname{sen}\frac{\theta_s}{2}\right)^{-|\lambda-\mu|}\left(\cos\frac{\theta_s}{2}\right)^{-|\lambda+\mu|} \sum_{A'b'c'D'} d^{S}_{A'a}(x_a)\, d^{\frac{1}{2}}_{b'b}(x_b)\, d^{J}_{c'c}(x_c)\ \times$$
$$\times\ d^{\frac{1}{2}}_{D'd}(x_d)\left(\operatorname{sen}\frac{\theta_t}{2}\right)^{|\lambda'-\mu'|}\left(\cos\frac{\theta_t}{2}\right)^{|\lambda'+\mu'|} \bar{f}^{t}_{c'A';D'b'}$$

where $\cos x_c$, $\cos x_a$, $\cos\theta_s/2$, $\operatorname{sen}\theta_s/2$, $\cos\theta_t/2$, $\operatorname{sen}\theta_t/2$ are regular functions near $t=0$, while $\cos x_b$ and $\cos x_d$ are singular near $t=0$.

Drawing out the singular behaviour at $t=0$ the rotation matrices related to the two nucleons can be written near $t=0$ in the form:

$$\text{(A.4)} \qquad d^{\frac{1}{2}}\ (\cos x_{b(d)}) \sim \frac{B(D)}{(-t)^{1/4}}\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

$$\text{(A.5)} \qquad B(D) = \left[\frac{M\,(m_J^2 - m_S^2)}{2\left[s-(m_{S(J)}-M)^2\right]\left[s-(m_{S(J)}+M)^2\right]}\right]^{\frac{1}{2}}$$

and M is the nucleon mass.

For sake of simplicity it is convenient to define, for any c' and A', the following system of equations:

$$G^{c'A'}_{bd} = \sum_{b'D'} d^{\frac{1}{2}}_{b'b}(x_b)\, d^{\frac{1}{2}}_{D'd}(x_d) \left(\text{sen}\frac{\theta_t}{2}\right)^{|\lambda'-\mu'|} \times$$

(A. 6)

$$\times \left(\cos\frac{\theta_t}{2}\right)^{|\lambda'+\mu'|} f^t_{c'A';D'b'} \sim \sum_{b'D'} d^{\frac{1}{2}}_{b'b}(x_b)\, d^{\frac{1}{2}}_{D'd}(x_d) \left(\tfrac{1}{2}\right)^{N(\lambda',\mu')} \bar{f}^t_{c'A';D'b'}$$

where $N(\lambda',\mu')$ = maximum $(|\lambda'|, |\mu'|)$ and the factor $(\frac{1}{2})^{N(\lambda',\mu')}$ comes out from the behaviour of $\cos\theta_t/2$ and $\text{sen}\,\theta_t/2$ at $t = 0$.

Since the rank of the system (A. 6) is one it is sufficient to consider only the equation which is obtained for $b = d = \frac{1}{2}$, so that we have

(A. 7)

$$G^{c'A'}_{\frac{1}{2}\frac{1}{2}} = -\frac{BD}{(-t)} \left[(\tilde{f}^t_{c'A';\frac{1}{2}\frac{1}{2}} - \tilde{f}^t_{c'A';-\frac{1}{2}-\frac{1}{2}}) \left(\tfrac{1}{2}\right)^{N(\mu',0)} - i \times \right.$$

$$\left. \times (\tilde{f}^t_{c'A';\frac{1}{2}-\frac{1}{2}} + f^t_{c'A';-\frac{1}{2}\frac{1}{2}}) \left(\tfrac{1}{2}\right)^{N(\mu',1)} \right]$$

Because all the parity conserving helicity amplitudes which enter in the expression (A. 7) have the same $t^{-\frac{1}{2}}$ singularity near $t = 0$ in the Wang formalism, we substitue $\tilde{f}$ for f. If now we put the relation (A. 7) into (A. 3) we see that we must have

(A. 8)

$$\sum_{c'A'} d^J_{c'c}(x_c)\, d^S_{A'a}(x_a) \left[\left(\tfrac{1}{2}\right)^{N(\mu',0)} (f^t_{c'A';\frac{1}{2}\frac{1}{2}} - f^t_{c'A';-\frac{1}{2}-\frac{1}{2}}) - \right.$$

$$\left. - i \left(\tfrac{1}{2}\right)^{N(\mu',1)} (f^t_{c'A';\frac{1}{2}-\frac{1}{2}} + f^t_{c'A';-\frac{1}{2}\frac{1}{2}}) \right] = 0\,(t)$$

in order to not have a kinematical pole at $t = 0$ which is forbidden because the left hand side of the equation (A. 3) contains quantities free from kinematical singularities in t.

Because of the presence of the $d_{\lambda\mu}$ functions the determinant of the system (A. 8) is equal to a finite value; therefore the only allowed solution is:

(A. 9) $$\left(\tfrac{1}{2}\right)^{N(\mu',0)} (f^t_{c'A';\frac{1}{2}\frac{1}{2}} - f^t_{c'A';-\frac{1}{2}-\frac{1}{2}}) - i \left(\tfrac{1}{2}\right)^{N(\mu',1)} (f^t_{c'A';\frac{1}{2}-\frac{1}{2}} + f^t_{c'A';-\frac{1}{2}\frac{1}{2}}) = 0(t)$$

for any c' and A'.

The conspiracy relations (II. 3) and (II. 4) follow from (A. 9).

APPENDIX B -

The kinematical constraints between the t-channel helicity amplitudes of the reactions:

(B.1) $$S\,(a) + S\,(b) \rightarrow J\,(c) + J\,(d)$$

can be easily obtained following the method of Cohen-Tannoudji et al.[3]. In our case the crossing relation near $t = 0$ is given by:

(B.2) $$\bar{f}^{\,s}_{cd;ab} \sim \left(\mathrm{sen}\,\frac{\theta_s}{2}\right)^{-|\lambda-\mu|}\left(\cos\frac{\theta_s}{2}\right)^{-|\lambda+\mu|}\sum_{A'b'c'D'} d^S_{A'a}(x_a)\, d^S_{b'b}(x_b)\ \mathrm{x}$$
$$\mathrm{x}\ d^J_{c'c}(x_c)\, d^J_{D'd}(x_d)\,(+t)^{\frac{1}{2}|\lambda'-\mu'|}\ \bar{f}^{\,t}_{c'A';D'b'}$$

where $\cos\theta_s/2$, $\mathrm{sen}\,\theta_s/2$ and the $d_{\lambda\mu}$ functions are regular near $t = 0$. The factor $t^{\frac{1}{2}|\lambda'-\mu'|}$ arises from the behaviour of $(\mathrm{sen}\,\theta_t/2)^{|\lambda'-\mu'|}$ near $t = 0$. By standard resoning[1,2] the $\bar{f}^{\,t}_{c'A';D'b'}$ appearing in (B.2) have not kinematic singularities in t; therefore from the crossing relation (B.2) we cannot derive any constraint between the amplitudes $\bar{f}^{\,t}_{c'A';D'b'}$ at $t = 0$.

However it is possible to derive constraints between the parity conserving helicity amplitudes. In fact in terms of the parity conserving helicity amplitudes the expression (B.2) becomes:

(B.3) $$\bar{f}^{\,s}_{cd;ab} \sim \left(\mathrm{sen}\,\frac{\theta_s}{2}\right)^{-|\lambda'-\mu'|}\left(\cos\frac{\theta_s}{2}\right)^{-|\lambda'+\mu'|}\sum_{A'b'c'D'} d^S_{A'a}(x_a)\, d^S_{b'b}(x_b)\, d^J_{c'c}(x_c)\ \mathrm{x}$$
$$\mathrm{x}\ d^J_{D'd}(x_d)\,\frac{t^{\frac{1}{2}|\lambda'-\mu'|}}{2}\left[\tilde{f}^{\,t+}_{c'A';D'b'} + \tilde{f}^{\,t-}_{c'A';D'b'}\right] t^{-\frac{M(\lambda',\mu')}{2}}$$

where we have written explicitely Wang's singularities, introducing the amplitudes $\tilde{f}^{\,t}$ free from any kinematical singularity at $t = 0$. The factor $M(\lambda',\mu')$ is the Maximum between $|\lambda'-\mu'|$ and $|\lambda'+\mu'|$.

In order to not have additional singularities in the right hand side of equation (B.3), which are not present in the left hand side, we must have for t approaching zero:

(B.4) $$\tilde{f}^{\,t+}_{c'A';D'b'} + \tilde{f}^{\,t-}_{c'A';D'b'} = 0\,\left(t^{m(\lambda',\mu')}\right)$$

where $m(\lambda', \mu')$ is the minimum between $|\lambda'|$ and $|\mu'|$.

We remark here that in order to derive the constraints for reactions of the type (B. 1) it is really not necessary to use the crossing relation. In fact Frautchi and Jones[15] were able to obtain, the kinematical constraints (B. 4) using only the fact that the two sides of the identity:

$$\text{(B. 5)} \qquad \bar{f}^{\,t}_{c'A';D'b'} = \frac{1}{2}\left[\bar{f}^{\,+t}_{c'A';D'b'} + \bar{f}^{\,-t}_{c'A';D'b'}\right]$$

must have the same behaviour near $t = 0$.

APPENDIX C -

The residues $\gamma^{\pm}_{c'A';D'b'}$, are free from kinematical singularities near $t = 0$; so that they behave, for $t \to 0$, as t^m where m must be an integer positive number or zero.

Therefore without lack of generality we can assume for the Regge residues the behaviour presented in Tables III and IV when t goes to zero.

Then the factorization requirements in Table II a reduce to the linear system:

$$\text{(C. 1)} \qquad \begin{aligned} 2a_i + |\mu_i| - 1 &= A_i + \gamma \\ 2b_i + |\mu_i| - 1 &= B_i + \beta \\ 2c_i &= D_i + \delta \\ 2d_i &= 1 + D_i + \beta \\ 2e_i + |\mu_i| &= B_i + \delta \\ 2f_i + |\mu_i| &= A_i + \alpha \\ 2g_i + 1 &= C_i + \gamma \\ 2h_i &= C_i + \alpha \end{aligned}$$

The minimal solutions of the system (C. 1) are given by:

$$\alpha = \delta = M \qquad\qquad \beta = \gamma = |M - 1|$$

$$\text{(C. 2)} \qquad A_i = B_i = \begin{cases} |\mu_i| - M & \text{if } |\mu_i| \geq M \\ 0 & \text{if } |\mu_i| < M \text{ and } (-1)^{|\mu_i| - M} = 1 \\ 1 & \text{if } |\mu_i| \;\; M \text{ and } (-1)^{|\mu_i| - M} = -1 \end{cases}$$

$$\text{(C.2)} \qquad C_i = D_i = \begin{cases} 0 & \text{if} \quad (-1)^M = 1 \\ 1 & \text{if} \quad (-1)^M = -1 \end{cases}$$

where M is an integer positive number which can assume all the values between 0 and the Maximum $(J+S; 1)$.

The solutions (C.2) satisfy the factorization conditions in the reactions $S+N \rightarrow J+N$ and $N+N \rightarrow N+N$, but they are still inconsistent with the requirements of the factorization in the channel $S+S \rightarrow J+J$:

$$\text{(C.3)} \qquad \begin{aligned} A_i + A_j &\geq \big| |\lambda_i| - |\mu_j| \big| \\ A_j + C_i &\geq |\mu_j| \\ B_i + B_j &\geq \big| |\lambda_i| - |\mu_j| \big| \\ B_i + D_j &\geq |\mu_i| \end{aligned}$$

These further constraints give the behaviour (IV.1) and (IV.2), where till now M can assume the values $0, 1, \ldots$ Max $(J+S, 1)$.

These solutions satisfy all "class III" constraints. In order to satisfy the "class II" constraints for $M = 0$, we have to modify the value of σ, which must be equal to $\beta + 1$. This implies that some residues in the $J+N \rightarrow J+N$ process take additional powers of t.

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TABLE I_1

	Amplitude $\bar{N}N \rightarrow JS$	$\lvert\lambda\rvert$	$\lvert\mu\rvert$	Behaviour near t=0 $K_1(t)$
a	$\bar{f}^{t+}_{C'A';\frac{1}{2}-\frac{1}{2}}$	1	$\neq 0$	$t^{-\frac{1}{2}}$
b	$\bar{f}^{t-}_{C'A';\frac{1}{2}\frac{1}{2}}$	0	$\neq 0$	$t^{-\frac{1}{2}}$
c	$\bar{f}^{t-}_{C'A';\frac{1}{2}-\frac{1}{2}}$	1	0	$t^{-\frac{1}{2}}$
d	$\bar{f}^{t-}_{C'A';\frac{1}{2}\frac{1}{2}}$	0	0	$t^{-\frac{1}{2}}$
e	$\bar{f}^{t-}_{C'A';\frac{1}{2}-\frac{1}{2}}$	1	$\neq 0$	1
f	$\bar{f}^{t+}_{C'A';\frac{1}{2}\frac{1}{2}}$	0	$\neq 0$	1
g	$\bar{f}^{t+}_{C'A';\frac{1}{2}-\frac{1}{2}}$	1	0	1
h	$\bar{f}^{t+}_{C'A';\frac{1}{2}\frac{1}{2}}$	0	0	1

TABLE I_2

Amplitude $\bar{N}N \rightarrow \bar{N}N$	$\lvert\lambda - \mu\rvert$	Behaviour near t=0 $K_2(t)$
$\bar{f}^{t\pm}_{\mu;\lambda}$	even	1
	odd	t

TABLE I_3

Amplitude JS $\rightarrow$ JS	Behaviour near t=0 $K_3(t)$
$f^{t\pm}_{\mu;\lambda}$	$t^{-\frac{1}{2}N}$ $N = \mathrm{Max}(\lvert\lambda+\mu\rvert; \lvert\lambda-\mu\rvert)$

TABLE IIa

	Factorization conditions in $\bar{N}N \to J\bar{S}$
a	$(\gamma^{+}_{C'A';\frac{1}{2}-\frac{1}{2}})^2\, t^{\vert\mu\vert-1} = (\gamma^{+}_{C'A';C'A'})(\gamma^{+}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}})$
b	$(\gamma^{-}_{C'A';\frac{1}{2}\frac{1}{2}})^2\, t^{\vert\mu\vert-1} = (\gamma^{-}_{C'A';C'A'})(\gamma^{-}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}})$
c	$(\gamma^{-}_{C'A';\frac{1}{2}-\frac{1}{2}})^2 = (\gamma^{-}_{C'A';C'A'})(\gamma^{-}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}})$
d	$(\gamma^{-}_{C'A';\frac{1}{2}\frac{1}{2}})^2 = t(\gamma^{-}_{C'A';C'A'})(\gamma^{-}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}})$
e	$(\gamma^{-}_{C'A';\frac{1}{2}-\frac{1}{2}})^2\, t^{\vert\mu\vert} = (\gamma^{-}_{C'A';C'A'})(\gamma^{-}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}})$
f	$(\gamma^{+}_{C'A';\frac{1}{2}\frac{1}{2}})^2\, t^{\vert\mu\vert} = (\gamma^{+}_{C'A';C'A'})(\gamma^{+}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}})$
g	$(\gamma^{+}_{C'A';\frac{1}{2}-\frac{1}{2}})^2\, t = (\gamma^{+}_{C'A';C'A'})(\gamma^{+}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}})$
h	$(\gamma^{+}_{C'A';\frac{1}{2}\frac{1}{2}})^2 = (\gamma^{+}_{C'A';C'A'})(\gamma^{+}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}})$

TABLE IIb

Factorization conditions in JS $\to$ JS
$(\gamma^{\pm}_{\mu;\lambda})^2\, t^{\vert\vert\mu\vert-\vert\lambda\vert\vert} = (\gamma^{\pm}_{\mu;\mu})(\gamma^{\pm}_{\lambda;\lambda})$

Factorization conditions in $\bar{N}N \to \bar{N}N$	$\vert\lambda-\mu\vert$
$(\gamma^{\pm}_{\mu;\lambda})^2 = (\gamma^{\pm}_{\mu;\mu})(\gamma^{\pm}_{\lambda;\lambda})$	even
$(\gamma^{\pm}_{\mu;\lambda})^2\, t = (\gamma^{\pm}_{\mu;\mu})(\gamma^{\pm}_{\lambda;\lambda})$	odd

TABLE IIIa

	Behaviour near t=0 of the Regge pole residue in the reaction N+J → N+S	
a	t^{a_i}	t
b	t^{b_i}	t
c	t^{c_i}	t
d	t^{d_i}	t
e	t^{e_i}	t
f	t^{f_i}	1
g	t^{g_i}	1
h	t^{h_i}	1

TABLE IIIb

	Behaviour near t=0 of the Regge pole residues in the reaction N+J → N+S	
a	t^{a_i}	$t^{(\|\|\mu_i\|-M\|+\|M-1\|-\|\mu_i\|+1)/2}$
b	t^{b_i}	$t^{(\|\|\mu_i\|-M\|+\|M-1\|-\|\mu_i\|+1)/2}$
c	t^{c_i}	t^{M}
d	t^{d_i}	$t^{(1+M+\|M-1\|)/2}$
e	t^{e_i}	$t^{(\|\|\mu_i\|-M\|+M-\|\mu_i\|)/2}$
f	t^{f_i}	$t^{(\|\|\mu_i\|-M\|+M-\|\mu_i\|)/2}$
g	t^{g_i}	$t^{(M+\|M-1\|-1)/2}$
h	t^{h_i}	t^{M}

TABLE IV

Residue	$\vert\lambda'\vert$	Behaviour near $t = 0$
$\gamma^{+}_{C'A';C'A'}$	$\neq 0$	t^{A_i}
$\gamma^{+}_{C'A';C'A'}$	0	t^{C_i}
$\gamma^{-}_{C'A';C'A'}$	$\neq 0$	t^{B_i}
$\gamma^{-}_{C'A';C'A'}$	0	t^{D_i}

Residue	Behaviour near $t = 0$
$\gamma^{+}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$	t^{α}
$\gamma^{+}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$	t^{β}
$\gamma^{-}_{\frac{1}{2}\frac{1}{2};\frac{1}{2}\frac{1}{2}}$	t^{γ}
$\gamma^{-}_{\frac{1}{2}-\frac{1}{2};\frac{1}{2}-\frac{1}{2}}$	t^{δ}