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1) - In this note we want to evaluate the rate of increase of transverse beam size and momentum spread in ISR, due to multiple Coulomb scattering within one beam(1).

The model we can assume for the transverse motion is that of an oscillator subject to random variations of momentum Δq_T:

$$\ddot{y} + \omega^2 y = \sum_i \frac{\Delta q_T(t_i)}{m} \delta(t - t_i) \tag{1}$$

Neglecting the solution of the homogenous equation, the solution of (1) can be written as

$$y(t) = \omega^{-1} \int^t \sin \omega(t-t') \sum_i \frac{\Delta q_T(t')}{m} \delta(t'-t_i)dt'.$$

We are interested in the average value (average with respect to the random variable Δq_T) of $y^2(t)$. Assuming

$$(2) \qquad \left\langle \sum_{i,j} \Delta q_T(t_i) \Delta q_T(t_j)\, \delta(t-t_i)\, \delta(t'-t_j) \right\rangle = \left\langle \Delta q_T^2(t) \right\rangle \delta(t-t')$$

and that $\langle \Delta q_T^2(t) \rangle$ is slowly varying with respect to ω^{-1}, one has, neglecting fast oscillating terms,

$$(3) \qquad \langle y^2(t) \rangle = \left[2\,\omega^2 m^2\right]^{-1} \int^t \langle \Delta q_T^2(t') \rangle \, dt',$$

and

$$\frac{\partial \langle y^2(t) \rangle}{\partial t} = \left[2\,\omega^2 m^2\right]^{-1} \langle \Delta q_T^2(t) \rangle$$

$$(4) \qquad = \frac{(\beta C)^2}{2\,\omega^2} \left\langle \left(\frac{\Delta q_T(t)}{q}\right)^2 \right\rangle$$

$$\simeq \frac{1}{2} \beta^2 \left(\frac{R}{\nu}\right)^2 \left\langle \left(\frac{\Delta q_T(t)}{q}\right)^2 \right\rangle ,$$

where βC and q are the velocity and momentum of the particle, R is the machine radius and ν is the betatron wave number.

In the same way, writing the energy of a particle as

$$E = E_s(1 + p),$$

where E_s is the energy corresponding to the central orbit in the ring, we have

$$(5) \qquad \langle p^2(t) \rangle = \frac{\beta^2 C^2}{E_s^2} \int^t \langle \Delta q_1^2(t') \rangle \, dt'$$

and

$$(6) \qquad \frac{p^2(t)}{t} = \frac{{}^2C^2}{E_s^2} \quad q_1^2(t) \quad .$$

In what follows we will evaluate $\langle \Delta q_1^2(t) \rangle$, $\langle \Delta q_T^2(t). \rangle$.

Since these quantities will be function of the beam dimensions and energy spread we will limit ourselves to the evaluation of (4) and (6). This will allow us to see that the rate of variations of beam dimensions and energy spread are small on a time scale given by the beam lifetime and hence that the effect is negligible.

2) - Let us consider two particles whose displacement and momenta with respect to a reference trajectory (RT) of arc length "S" are defined by

(7) $$\delta P_i = \underline{\alpha}(s)\,\sigma_i(\sigma) + \underline{\beta}(s)\,x_i(s) + \underline{\gamma}(s)\,z_i(s), \qquad i = 1,2$$

(8) $$\underline{q}_i = q_s\left\{(1+p_i)\,\underline{\alpha}(s) + \underline{\beta}(s)\,x_i'(s) + \underline{\gamma}(s)\,z_i'(s)\right\},$$

Where $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ define an orthonormal triad on RT ($\underline{\alpha}$ is tangent to RT and $\underline{\beta}$ is the principal normal); q_s is the momentum of a particle following RT; x' and z' are the derivatives with respect to s of x and z, and p, x, z, x', z' are all assumed to be small compared to one.

Then in a scattering event defined, in the center of mass, (CM), system of the two particles, by the polar and azimuthal angles χ, φ the three components of the momentum q_l, q_r, q_v along the directions $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ change by the quantities

(9) $$\frac{\Delta q_l}{q_s} = \pm\frac{1}{2}\gamma_o\,\theta \sin\chi \sin\varphi,$$

(10) $$\frac{\Delta q_r}{q_s} = \pm\frac{1}{2}\,\theta_z \sin\chi \cos\varphi,$$

(11) $$\frac{\Delta q_v}{q_s} = \pm\frac{1}{2}\,\theta_x \sin\chi \cos\varphi,$$

where $\gamma_o = (q_s^2C^2 + m_o^2C^4)^{1/2}/(m_oC^2)$, ($m_o$ is the rest mass),

$\theta_x = |x_1' - x_2'|$, $\theta_z = |z_1' - z_2'|$, $\theta = (\theta_x^2 + \theta_z^2)^{1/2}$.

The relations (9), (10), (11) have been evaluated assuming that the velocity of the particles in the CM system is non relativistic, and that one can neglect second order terms in the quantities δP_i, $\underline{q}_i - q_s\underline{\alpha}$, χ.

Notice that, to first order in δP_i, $\underline{q}_i - q_s\underline{\alpha}$, the momentum variations do not depend on p_i.

Let us call

(12) $$\mathcal{E} = f(\chi, \varphi, \overline{\theta})$$

any one of the $\Delta q/q$ defined by (9), (10), (11).

We are interested in the average square value of momentum variation per unit time. This can be evaluated by using the function $P(\varepsilon)d\varepsilon$ which gives the number of scattering events per unit time producing a momentum variation ε in the interval $\varepsilon - \varepsilon + d\varepsilon$.

$P(\varepsilon)$ is defined by

$$P(\varepsilon) = \int 2v_2(\theta)\rho(\bar{\theta}) \frac{d\sigma}{d\Omega} \delta \left[\varepsilon - f(\chi, \varphi, \bar{\theta})\right] \times \tag{13}$$
$$\times \sin\chi \, d\chi \, d\varphi$$

where $2v_r(\theta)$ is the relative velocity in the CM system of two particles defining an angle θ and

$$\rho(\bar{\theta}) = \int d(\delta P_1) d(\delta P_2) dq_1 dq_2 P(\delta P_1, q_1) P(\delta P_2, q_2) \times \tag{14}$$
$$\times \delta(\delta P_1 - \delta P_2) \delta\left[\bar{\theta} - \bar{\theta}(x_1', x_2', z_1', z_2')\right],$$

$P(\delta P_i, q_i)$ being the distribution function of the positions and momenta.

Assuming the beam to have a gaussian distribution in the transverse phase-space and a uniform distribution of longitudinal positions, one obtains, for $\bar{\theta} = \theta_x$, $\bar{\theta} = \theta_z$

$$\rho(\theta_x) = \frac{N^2}{V\left[4\pi\langle x'^2\rangle\right]^{1/2}} e^{-\theta_x^2/(4\langle x'^2\rangle)}, \tag{15}$$

$$\rho(\theta_z) = \frac{N^2}{V\left[4\pi\langle z'^2\rangle\right]^{1/2}} e^{-\theta_z^2/(4\langle z'^2\rangle)}, \tag{16}$$

where V, the beam volume, is given by

$$V = 4\pi L\left[\langle x^2\rangle \langle z^2\rangle\right]^{1/2} \tag{17}$$

N is the number of particles per beam, $\langle x^2\rangle$, $\langle z^2\rangle$, $\langle x'^2\rangle$, $\langle z'^2\rangle$ are average square values of transverse position and velocity, and L is the ISR circumference.

In the case of $\bar{\theta} = \theta = (\theta_x^2 + \theta_z^2)^{1/2}$ one obtains for $\rho(\theta)$ a more complicated expression which implies the use of a computer to obtain $\langle(\Delta q_1/q)^2\rangle$.[2]

Hence, for simplicity, we assume the beam width to be larger than the height, in which case, with good approximation, $\rho(\theta)$ reduces to $\rho(\theta_x)$.

Then one obtains,

$$(18) \qquad \left\langle \left(\frac{\Delta q_l}{q}\right)^2 \right\rangle = \frac{\pi^{1/2} N r_o^2 C}{V \langle x'^2 \rangle^{1/2} \gamma_o^3 \beta^3} F(\lambda_l)$$

$$(19) \qquad \left\langle \left(\frac{\Delta q_r}{q}\right)^2 \right\rangle = \frac{\pi^{1/2} N r_o^2 C}{V \langle z'^2 \rangle^{1/2} \gamma_o^5 \beta^3} F(\lambda_r)$$

$$(20) \qquad \left\langle \left(\frac{\Delta q_v}{q}\right)^2 \right\rangle = \frac{\pi^{1/2} N r_o^2 C}{V \langle x'^2 \rangle^{1/2} \gamma_o^5 \beta^3} F(\lambda_v)$$

where r_o is the classical proton radius.

In this formulas the function F is given, whithin approximations valid for the ISR case[3], by

$$(21) \qquad F(\lambda) = \int_1^\infty dt \, \frac{e^{-\lambda t^2}}{t^3} \left[\frac{1}{4} + t^2(\ln t - \frac{1}{4})\right]$$

This function is plotted in fig. 1.

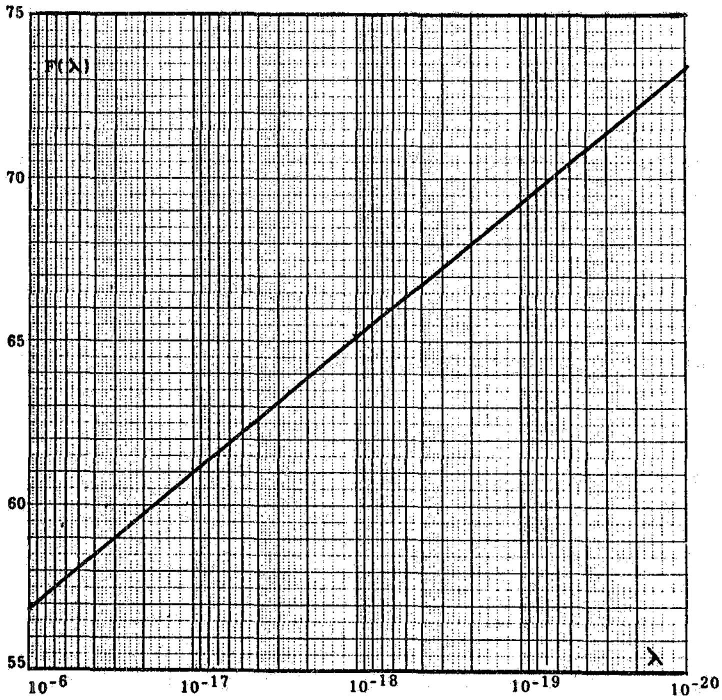

FIG. 1

The quantities λ_1, λ_r, λ_v are related to the minimum momentum variations which can occur in a scattering event.

From the Heisenberg Principle one has

$$\varepsilon_m = \left(\frac{\Delta q}{q_s}\right)_{min} = \frac{\hbar}{q_s\, d_m} = \frac{\lambda\!\!\!^{-}_c}{\gamma_o \beta\, d_m} \tag{22}$$

where d_m is the maximum impact parameter, which we assume to be equal to one half of the average distance between particles:

$$d_m = \frac{1}{2}\left(\frac{V}{N}\right)^{1/3}, \tag{23}$$

and $\lambda\!\!\!^{-}_c \simeq 2 \times 10^{-14}$ cm is the proton Compton wave-length.

Then the λ's are defined as

$$\lambda_1 = \frac{\varepsilon_m^2}{\gamma_o^2 \langle x'^2 \rangle} \tag{24}$$

$$\lambda_r = \frac{\varepsilon_m^2}{\langle z'^2 \rangle} \tag{25}$$

$$\lambda_v = \frac{\varepsilon_m^2}{\langle x'^2 \rangle} \tag{26}$$

3) - We make now a numerical evaluation of (18), (19), (20). Since the effect decreases strongly with energy we assume $\gamma_o = 10$, $\beta \simeq 1$; for the other parameters $L \simeq 9.4 \times 10^2$ m,

$\langle x^2 \rangle^{1/2} \simeq 3$cm, $\langle z^2 \rangle^{1/2} \simeq 1$ cm, $\langle x'^2 \rangle^{1/2} \simeq 5 \times 10^{-4}$, $\langle z'^2 \rangle^{1/2} \simeq$ $\simeq 1.5 \times 10^{-4}$, $\nu \simeq 8.75$ (betatron wave number), $N = 2 \times 10^{14}$.

From this we obtain

$d_m \simeq 1.3 \times 10^{-3}$ cm; $\varepsilon_m \simeq 1.5 \times 10^{-12}$ cm;
$\lambda_1 \simeq 9 \times 10^{-20}$ cm; $\lambda_r \simeq 10^{-16}$ cm;
$\lambda_v \simeq 9 \times 10^{-18}$ cm.

From fig. 1 it follows

$$F(\lambda_1) \simeq 69.4; \quad F(\lambda_r) \simeq 56.8; \quad F(\lambda_v) \simeq 61.3.$$

Hence we obtain

$$\langle (\frac{\Delta q_l}{q_s})^2 \rangle \simeq 9.4 \times 10^{-12}$$

$$\langle (\frac{\Delta q_r}{q_s})^2 \rangle \simeq 2.56 \times 10^{-13}$$

$$\langle (\frac{\Delta q_v}{q_s})^2 \rangle \simeq 8.3 \times 10^{-14}$$

Finally from (4), (6) we obtain

$$\frac{\partial}{\partial t} \langle x^2 \rangle \simeq 3.7 \times 10^{-7} \text{ cm}^2/\text{sec}$$

$$\frac{\partial}{\partial t} \langle z^2 \rangle \simeq 1.2 \times 10^{-7} \text{ cm}^2/\text{sec}$$

$$\frac{\partial}{\partial t} \langle p^2 \rangle \simeq 9.4 \times 10^{-12} \text{ sec}^{-1}$$

a result which shows that the effect is completely negligible for a time scale of the order of ten hours.

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REFERENCES -

(1) - H. Bruck et J. Le Duff, Labor. Accel. Lin., Orsay, Rap. Techn. 24-64 (1964).

(2) - C. Pellegrini, Proceeding of the International Symposium on electron and positron storage rings, Saclay, 1966.

(3) - The function $F(\lambda)$ is obtained as a particular case of the function $F(\xi_m, \xi_M)$ introduced in reference 2, equation (9). In the ISR case it is a good approximation to assume $\xi_M \rightarrow \infty$, so obtaining $F(\lambda)$ from $F(\xi_m, \xi_M)$.