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The Infra-Red Radiative Corrections for Colliding Beam (Electrons and Positrons) Experiments.

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Summary. — The infra-red corrections to be applied to the results expected from an electron positron colliding beam experiment are determined with the help of the Bloch-Nordsieck theorem. Experiments are characterized by a resolution function $\varrho(k)$ of a four-dimensional timelike energy-momentum vector, which represents the probability that a four-momentum loss k escapes detection. The results are applicable to a class of experiments in which the statistical error is matched to the error of the energy-momentum resolution. Various approximations which allow a rapid and accurate estimate of radiative corrections are discussed.

1. — Introduction.

Any search for a possible breakdown of quantum electrodynamics leads inevitably to the necessity of applying important radiative corrections to the measured results. This is due to the fact that renormalization theory makes the form of electrodynamics at low momentum (and energy) transfer a matter of definition, so that any discrepancy with the existing theory has to be searched for by either studying processes in which the transferred momentum is very large or by making high-precision measurements on processes with a moderately high momentum transfer. In either case radiative corrections are important: in the first because high-energy charged particles are created and destroyed—a process which leads to large currents and therefore to the liberation of a considerable portion of the energy in the form of relatively soft electromagnetic radiation—in the second case because the high precision of the experiment requires for its interpretation a high precision in the determination of any correction.

It has long been recognized (compare SCHWINGER ⁽¹⁾, JAUCH and ROHR-
LICH ⁽²⁾, YENNIE ⁽³⁾, LOMON ⁽⁴⁾, ERIKSON ⁽⁵⁾) that straightforward pertur-
bation theory does not lend itself easily to dealing with the flood of soft pho-
tons which emerges from a high-energy collision between charged particles.
The reason for this is that the picture of an experimenter as of one counting
soft photons is not entirely realistic: existing perturbation theory works in a
representation in which the number of photons is diagonal and the emission
of any additional photons requires a further step in the perturbation procedure.
The experimenter on the other hand does not see single photons, but rather
an unbalance of energy and momentum between the incident and emergent
particles.

This unbalance is attributed to electromagnetic radiation which escapes
direct observation partly because in many experiments the detectors are sepa-
rated from the region in which the interaction takes place by some form of
container in which low-energy radiation disappears tracelessly, partly because
the detector is not designed to register soft photons individually.

A reflection of the incompatibility between the pictures of an experiment
drawn by theory and reality is the fact that no two experiments on the same
subject, carried out with different apparatus, can be compared with one
another before the radiative corrections have been determined (and determined
in a form which is applicable to both experiments).

The main purpose of this paper is to supply the experimenter with tools
for applying the radiative corrections himself. We shall confine our considera-
tions to high-energy reactions of the type

$$(1) \quad e^+ + e^- \rightleftharpoons A + \bar{A}$$

where \bar{A} is the antiparticle of A and we assume $m_A \neq 0$, because of the interest
of these reactions in view of the experiments on electron-positron colliding
beams now in preparation, but we think that the method here discussed is
more general than what the restriction to interactions of the type (1) might
suggest.

Perturbation theory defines a cross-section $d^2\sigma_0(\theta, \varphi)$ for reaction (1) (θ and φ
are the polar angles of the A particle and we neglect here and in the following
some complications which may arise from the spin of the particles involved

⁽¹⁾ J. SCHWINGER: *Phys. Rev.*, **76**, 790 (1949).

⁽²⁾ J. M. JAUCH and F. ROHR-
LICH: *Helv. Phys. Acta*, **27**, 613 (1954); *The Theory
of Photons and Electrons* (Cambridge, 1955).

⁽³⁾ D. R. YENNIE and H. SUURA: *Phys. Rev.*, **105**, 1378 (1957); D. R. YENNIE,
S. C. FRAUTSCHI and H. SUURA: *Ann. of Phys.*, **13**, 379 (1961).

⁽⁴⁾ E. L. LOMON: *Nucl. Phys.*, **1**, 101 (1956); *Phys. Rev.*, **113**, 726 (1959).

⁽⁵⁾ K. E. ERIKSON: *Nuovo Cimento*, **19**, 1010 (1961).

in the reaction). This cross-section cannot be directly compared with experiment, since reaction (1) can never take place without the production of photons. Indeed if the cross-section for (1) were calculated accurately it should be zero.

What can actually be compared with an experiment is a cross-section $d^6\sigma(\theta, \varphi, k)$ for reaction (1), in which the four-vector k represents the momentum and the energy carried away in the form of electromagnetic radiation. Since this radiation always removes energy from the reaction and since, though the momenta of different photons may cancel one another, their energies may not, it follows that k will be confined to the positive light cone $k_0 = \omega \geq |\mathbf{k}|$.

Any individual experiment on reaction (1) can be characterized by a function $\varrho(k)$ of the vector k . This function describes the probability that a four-momentum loss k will remain unnoticed; one naturally has $\varrho(0) = 1$ and $0 \leq \varrho(k) \leq 1$. The cross-section $d^2\sigma_{\text{exp}}(\theta, \varphi)$ that is measured in an experiment described by the resolution function $\varrho(k)$ can be expressed in terms of $d^6\sigma$

$$(2) \quad d^2\sigma_{\text{exp}}(\theta, \varphi) = \int d^4k \varrho(k) \frac{d^6\sigma(\theta, \varphi, k)}{d^4k}.$$

As long as the four-momentum loss is sufficiently small, $d^6\sigma$ can be factorized

$$(3) \quad d^6\sigma(\theta, \varphi, k) = d^4P(k) d^2\sigma_E(\theta, \varphi),$$

where $d^4P(k)$ is the probability of a four-momentum loss in d^4k , and $d^2\sigma_E(\theta, \varphi)$ differs from $d^2\sigma_0(\theta, \varphi)$ determined from lowest-order perturbation theory by an « ultra-violet » correction. (Compare Sect. 4).

The probability $d^4P(k)$ is determined by applying the methods of statistical mechanics to the predictions of the Bloch-Nordsieck⁽⁶⁾ theorem. This theorem states that as long the recoil effects of the emitted radiation on the emitting particles can be neglected the distribution of the number of photons is Poissonian and that the average number of photons with given momentum can be determined classically. A general formula for $d^4P(k)$ is given in Sect. 2.

The properties of the function $d^4P(k)$ are discussed in Sect. 5 and 6. In Sect. 7 we discuss what we retain to be useful approximations to the angular distribution of the momentum loss.

The separability assumed in eq. (3) requires that the function $\varrho(k)$ which is characteristic of an experiment is different from zero only for energy losses which are very small compared to the total energy available in the reaction. This requirement leads to the definition of a « balanced » experiment in which

(6) F. BLOCH and A. NORDSIECK: *Phys. Rev.*, **52**, 54 (1937).

the statistical error of the measurement is matched to the kinematical error in the observation of a single event. In Sect. 8 we shall discuss the possibility of a check of quantum electrodynamics by means of « balanced » experiments. We do not believe that « balanced » experiments are the only ones which provide such a check.

2. — $d^4P(k)$ derived from the Bloch-Nordsieck theorem.

If the momentum transfer is large we can consider process (1) as taking place in a very short time, which because of the uncertainty principle will be of the order $\Delta t = 1/2E$. The incoming particles can be pictured to be annihilated and the outgoing particles to be created in the interval Δt . Classically we can picture this process in exactly the same way as it was pictured by Sommerfeld in his theory of the production of X-rays. The incoming particles are slowed down to rest (or to nonexistence—in either case the final product can no longer radiate) and the outgoing particles are accelerated to their final velocities in a time Δt . This deceleration and acceleration produces—classically—an electromagnetic field of which we can calculate the energy from the Poynting vector formed from the field strengths. It is also possible to obtain a Fourier analysis of the Poynting vector, which allows one to determine the energy dW radiated by process (1) into a frequency interval $d\omega$. This energy will be given by

$$(4) \quad dW = \beta(E, \theta, \omega) d\omega.$$

As long as the slowing down and accelerating happens in a time interval which is very short compared to $1/\omega$, *i.e.* as long as $\omega \ll 2E$, we can assume β to be independent of ω , and that for the simple reason that in this case we can represent the process of deceleration and acceleration by a δ -function in time, the Fourier transform of which is a constant. For sufficiently small frequencies β will therefore only depend on the energy and the angle θ —it will be independent of θ if the A particle is neutral. The function $\beta(E, \theta)$ is calculated in Appendix I, in which we also show that β is an invariant function of the momenta. The fact that for low frequencies β does not depend on the frequency characterizes the spectrum (4) as a « noise-spectrum ». The task of determining radiative corrections therefore reduces to that of eliminating the noise due to the disappearance and creation of charged particles.

From (4) one immediately concludes that the average number $d\bar{n}(\omega)$ of photons in the frequency interval $d\omega$ is given by

$$(5) \quad d\bar{n}(\omega) = dW/\omega = \beta d\omega/\omega.$$

(Here and in the following we put $\hbar = 1$.) The total average number of photons emitted in process (1) is seen to diverge at the lower limit $\omega = 0$. This is the infra-red divergence, which, it is seen, can have physical significance only if there existed an experiment which would enable us to count low-energy photons in the limit $\omega = 0$. This is of course not possible: we will only become aware of these photons if they draw some energy (or momentum, about which we shall talk later) from process (1). An inspection of (4) shows that this energy is always finite.

The correspondence principle only gives information about averages. In order to have the details of the distribution of the number of photons one has to apply the methods of second quantization to the electromagnetic field. As long as the photons considered are sufficiently soft and as long as the cross-sections show no violent dependence on energy and momentum transfer or on the total energy (this is the case near a resonance) one can neglect the quantal behaviour of the source particles. As we have already said in the Introduction, this is explained by the fact that the reaction (1) will in this case take place in a space-time region of linear dimensions of the order of $1/2E$, the details of which cannot be resolved by photons the energy of which is small compared to E .

The fundamental theorem concerning the quantum theory of the emission of soft photons from a classical source is due to BLOCH and NORDSIECK⁽⁶⁾. They have shown that the distribution of the number of photons is a Poisson distribution, *i.e.*

$$(6) \quad P(\{n_k\}) = \prod_k \frac{\bar{n}_k^{n_k}}{n_k!} \exp[-\bar{n}_k].$$

Here n_k is the number of photons which are emitted with momentum \mathbf{k} (we have assumed a discrete momentum spectrum for the photons, corresponding to the quantization of the electromagnetic field in a finite conducting box), \bar{n}_k is the average value of the number of photons with momentum \mathbf{k} —it can be determined by means of a procedure analogous to the step which leads from eq. (4) to (5) (compare Appendix I). $P(\{n_k\})$ is the probability of process (1) ending up with n_{k_1} photons with momentum \mathbf{k}_1 , n_{k_2} photons with momentum \mathbf{k}_2 , etc.

From eq. (6) one can determine the probability $d^4P(k)$ of observing a four-momentum loss k accompanying the reaction (1). This probability is given by

$$(7) \quad d^4P(k) = \sum P(\{n_{k'}\}) \delta_4\left(\sum_{k'} k' n_{k'} - k\right) d^4k,$$

where the sum \sum is carried out over all the values of all the $n_{k'}$. The four-dimensional δ -function selects the distributions $\{n_{k'}\}$ with the right energy-momentum loss k . k' in the argument of the δ -function is given by

$k' \equiv (k', k'_0 = |k'|)$. Equation (7) shows that $d^4P(k) \neq 0$ only if k is inside or on the future light cone. This follows from the fact that all the $n_{k'} \geq 0$ and that $k'_0 \geq 0$.

The sum over the distributions $\{n_k\}$ can be carried out by using the methods of statistical mechanics. To this end one introduces a four-vector selector variable x , so that δ can be replaced by

$$(8) \quad \delta_4\left(\sum_{k'} k' n_{k'} - k\right) = (2\pi)^{-4} \int d^4x \exp\left[i\left(x, \sum_{k'} k' n_{k'} - k\right)\right].$$

One can in this way invert the order between forming the product in $P(\{n_k\})$ (eq. (6)) and the summation over all the distributions. The sum over the n_k can be easily carried and one obtains

$$(9) \quad d^4P(k) = (2\pi)^{-4} \int d^4x \exp[-h(x) - i(k, x)] d^4k,$$

in which $h(x)$ is defined as

$$(10) \quad h(x) = \sum_k (1 - \exp[i(k, x)]) \bar{n}_k.$$

In spite of (5) the expression (10) has no infra-red divergence, since $1 - \exp[i(k, x)] = 0$ for $k = 0$. Without any further specification the integral (10) is indeterminate for $k \rightarrow \infty$ unless a suitable limit is defined. The simplest definition of such a limit would be obtained by agreeing to extend the product in eq. (6) only over soft photons $k_0 < K \ll E$, where K is a cut-off energy which separates the soft photons, which are observed with difficulty but for which the Bloch-Nordsieck theorem can be expected to hold, from the hard photons, which can be individually counted. We shall use a different approach here and put $K = E$. It will be seen in the following that the choice of the cut-off has no influence whatsoever on the prediction of the momentum and energy loss due to sufficiently soft photons. The choice $K = E$ need only be accompanied by the caution of not using the resultant form of d^4P for energy losses which are not small compared to E . The limit of trustworthiness of this procedure has been discussed by LOMON⁽³⁾ and ETIM and TOUSCHEK have shown⁽⁷⁾ that for energies of up to about 1.5 GeV and letting k_0 vary up to about 150 MeV this should not introduce an error greater than 1%.

A very important property of the integrand in eq. (9) follows from the fact that by its definition $d^4P(k) \neq 0$ only for $k_0 = \omega \geq 0$. Defining the scalar

(7) E. ETIM and B. TOUSCHEK: Laboratori Nazionali di Frascati, LNF-66/10 (1966).

product (k, x) as $(k\mathbf{x}) - \omega t$ ($t = x_0$) it follows that the function $\exp[-h(\mathbf{x}, t)]$ must be analytical in the lower half of the complex t -plane. (In this case the t -integration for $k_0 < 0$ can be completed by a circular path of integration (from $+\infty$ over $-i\infty$ to $-\infty$) and the analyticity of $h(\mathbf{x}, t)$ then ensures that the integral is zero.)

3. - The energy dependence of $d^4P(k)$.

Integrating $d^4P(k)$ defined in eq. (9) over all momenta one obtains a function $dP(\omega)$ which represents the probability of finding process (1) accompanied by an energy loss in $d\omega$. Using eq. (9) and observing that

$$\int d^3k \exp[-i(\mathbf{k}, \mathbf{x})] = (2\pi)^3 \delta_3(\mathbf{x}),$$

one gets in this way

$$(11) \quad dP(\omega) = (2\pi)^{-1} d\omega \int_{-\infty}^{+\infty} dt \exp[-h(0, t) + i\omega t].$$

Using (10) and (5), $h(0, t)$ can be easily evaluated and one obtains

$$(12) \quad h(0, t) = \beta \int_0^x \frac{dk}{k} (1 - \exp[-ikt]).$$

We want to study the behaviour of $dP(\omega)$ for $y = E/\omega > 1$ and to trust the result for $y \gg 1$. Introducing in (11) the dimensionless variable $x = \omega t$ instead of t we can write

$$(13) \quad dP(\omega) = \frac{d\omega}{\omega} M(y)$$

with

$$(14) \quad M(y) = (2\pi)^{-1} \int dx \exp[-h + ix],$$

where because of (12)

$$(15) \quad h = \beta \int_0^{xy} \frac{du}{u} (1 - \exp[-iu]).$$

Differentiating (14) with respect to y one gets

$$(16) \quad M'(y) = -\beta M(y)/y + (2\pi y)^{-1} \int dx \exp[-h + ix(1-y)].$$

The second term of the right-hand side is zero for $y > 1$, because of the analyticity of h in the lower half t -plane. For $y > 1$ one therefore remains with the first term in (16)—a differential equation for M with the solution $M = \text{const } y^{-\beta}$, so that from (13) we may conclude that $dP(\omega) = \text{const } d\omega \omega^{\beta-1}$ and we may conclude more specifically that

$$(17) \quad N dP(\omega) = \beta \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^\beta \quad \text{for } \omega < E.$$

N is a factor of normalization, which we will now discuss. It is easily seen that the right-hand side is normalized in such a way that its integral extended over ω from 0 to E is unity. On the other hand it follows from eq. (11) that the integral over dP extended from 0 to ∞ is unity. N is therefore defined as

$$(18) \quad N = \int_0^\infty dP(\omega) / \int_0^E dP(\omega)$$

and it is seen that $N > 1$. N can be evaluated exactly by following a procedure used by LOMON and described in detail by ERIKSON and which—for completeness—we summarize in Appendix II. The result is

$$(19) \quad N = \gamma^\beta \Gamma(1 + \beta),$$

where $\gamma = e^\gamma = 1.781$ is Euler's constant. For most practical applications of the near future N is very near to unity. For small β one has approximately $N = 1 + \pi^2 \beta^2 / 12$. Putting $N = 1$ will therefore involve an error of less than 1%. The difference $N - 1$ represents the probability that two or more photons of energy $< E$ combine to give an energy loss which is greater than E . This probability is quite small as the preceding consideration shows; that it should be proportional to β^2 can easily be understood from perturbation theory, which requires that the probability for the emission of two photons be proportional to the square of the fine-structure constant. (Compare Appendix I.) It must however be kept in mind that the preceding consideration must not be interpreted physically: the Bloch-Nordsieck approximation is certainly no longer valid for $k = E/2$, the minimum value for the energy of the bigger of the two quanta, if the energy loss should be greater than E . The smallness of $N - 1$ is only an indication—which will be borne out in Sect. 6—that the energy loss is mainly determined by a single photon.

4. — The physical interpretation of eq. (17) and its relation to perturbation theory.

The integration over the momenta, which leads to eq. (17), corresponds to an experiment in which the momentum resolution is zero and in which only

the energy can be measured with some precision. Such an experiment can be described in terms of a function $\varrho(k)$ (compare Sect. 1) which is unity for $k_0 = \omega < \Delta\omega$ and zero otherwise. In order that eqs. (17) and (5) be applicable we assume $\Delta\omega \ll E$.

Using eqs. (2) and (3) and inserting dP from eq. (17) we get for the experimental cross-section

$$(20) \quad d^2\sigma_{\text{exp}} = N^{-1} \left(\frac{\Delta\omega}{E} \right)^\beta d^2\sigma_E,$$

where we have suppressed the dependence on the polar angles of the Λ particles. We observe that the choice of the cut-off $K = E$ is indeed only a question of normalization. Had K been chosen $\neq E$ we would have had to replace $d^2\sigma_E$ by $d^2\sigma_K$ and $d^2\sigma_K$ would only have been defined by eq. (20). The choice $K = E$ is recommended by a comparison of (20) with perturbation theory. Expanding this equation in powers of β we obtain in first approximation

$$(20') \quad d^2\sigma_{\text{exp}} \approx (1 - \beta \lg(E/\Delta\omega)) d^2\sigma_E.$$

The lowest-order radiative correction obtained from perturbation theory can always be written in the form

$$(21) \quad d^2\sigma_{\text{exp}} = (1 - \beta \lg(E/\Delta\omega) + \lambda) d^2\sigma_0,$$

where $d^2\sigma_0$ is the cross-section obtained for process (1) in lowest nonvanishing order. The term λ may depend on the energy and on the angles θ and φ , but is independent of the accuracy $\Delta\omega$ of the experiment.

The term λ represents what may be called the genuine (as opposed to the infra-red) radiative correction. It is of purely theoretical concern and bears no relation to the details of the experimental arrangement. It has been calculated for some of the experiments proposed for ADONE⁽⁸⁾. We quote from LONGHI a typical result for the interaction $e^+ + e^- \rightarrow \mu^+ + \mu^-$. In this case one has

$$(22) \quad \lambda = \frac{2\alpha}{\pi} \left(\frac{13}{6} (\lg 2\gamma_e + \lg 2\gamma_\mu) - \frac{2\pi^2}{3} - \frac{28}{9} \right)$$

where

$$\gamma_e = \frac{E_e}{m_e} \quad \text{and} \quad \gamma_\mu = \frac{E_\mu}{m_\mu}.$$

⁽⁸⁾ Compare R. GATTO: *Ergebnisse der exakten Naturw.*, **39**, 106 (1965); G. LONGHI: *Proceedings of the Frascati Meeting on Storage Ring Experiments* (1966).

A recent calculation by ROSSI gives a slightly different result. According to this author $2\pi^2/3$ should be replaced by $\pi^2/6$ ⁽⁹⁾. This is about 6% for an energy of 1 GeV. Equation (22) bears out the general rule that the expansion parameter of high-energy quantum electrodynamics is $\alpha \lg 2\gamma$ rather than α !

In order to obtain an approximate definition of $d^2\sigma_E$ of eq. (3) in terms of $d^2\sigma_0$ of perturbation theory we only have to compare the two equally approximate equations (20') and (21). It is then immediately seen that the best one can do in order to make these two equations fit is choosing

$$(23) \quad d^2\sigma_E = (1 + \lambda) d^2\sigma_0 .$$

Inserting this into eq. (20) we get

$$(24) \quad d^2\sigma_{\text{exp}} = N^{-1} \left(\frac{\Delta\omega}{E} \right)^\beta (1 + \lambda) d^2\sigma_0$$

which coincides exactly with the results previously obtained by ERIKSON. The advantage of this formula—indeed its necessity—becomes clear if we consider the specific example of an « optimal » experiment with Adone. The best energy resolution obtainable is defined by the machine itself, which owing to the fluctuations of the radiation losses gives an energy spread of about 0.5 MeV. This makes $\beta \lg E/\Delta\omega = 0.6$, which would certainly make one doubt the possibility of applying unmitigated perturbation theory. The optimal measurement therefore reduces the cross-section by a factor 0.4 according to perturbation theory. The reduction factor deduced from eq. (20) is 0.54 and it is seen that the difference is quite considerable.

An important result of (20) is that it makes two experiments with different energy resolution directly comparable: all the theoretical work goes into $d^2\sigma_E$ and this is factored out in a comparison between two experiments.

5. — The separability of $d^4P(k)$.

High-energy experiments generally resolve the momentum better than the energy, since a momentum measurement can be carried out geometrically. Indeed the energy resolution may be only marginal as in the case of two spark chambers in coincidence. In this case all one knows about the energy is that it was sufficient to let the Λ particles penetrate the wall of the reaction chamber and the spark chamber. To apply radiative corrections to such a situation one has to determine d^4P as opposed to d^1P . In this Section we will show that the main features of d^4P are already contained in d^1P and precisely

⁽⁹⁾ G. Rossi: Laboratori Nazionali di Frascati, LNF-66/51 (1966).

that d^4P can be separated—for $\omega < E$ —into an energy-dependent part, which behaves like dP and a part which depends on the « velocity » $\mathbf{u} = \mathbf{k}/\omega$, which can be attributed to the four-momentum carried away by the electromagnetic radiation.

We shall show that one can write

$$(25) \quad d^4P = \beta N^{-1} \frac{d\omega}{\omega} \left(\frac{\omega}{E}\right)^\beta A(\mathbf{u}) d^3\mathbf{u},$$

where $A(\mathbf{u})$ is a normalized three-dimensional distribution function

$$(26) \quad \int A(\mathbf{u}) d^3\mathbf{u} = 1, \quad A(\mathbf{u}) \geq 0.$$

Since the four-momentum loss is confined to the positive light cone we have of course $0 \leq |\mathbf{u}| \leq 1$.

To prove (25) we go back to eq. (9), which after the substitution $\xi = \omega \mathbf{x}$, $\xi_0 = \tau$ becomes

$$(27) \quad d^4P(k) = (2\pi)^{-4} \frac{d\omega}{\omega} d^3\mathbf{u} \int d^4\xi \exp[-h + i\tau - i(\mathbf{u}, \xi)].$$

h —defined by eq. (10)—can be written as

$$(28) \quad h = \beta \int_0^y \frac{d\lambda}{\lambda} \int d^3\mathbf{n} f(\mathbf{n}) (1 - \exp[-i\lambda(\tau - (\mathbf{n}\xi))]).$$

The function $f(\mathbf{n})$ represents the angular distribution of single photons. Its detailed behaviour is given in Appendix I. $f(\mathbf{n})$ is normalized: $\int d^3\mathbf{n} f(\mathbf{n}) = 1$, $f(\mathbf{n}) \geq 0$. (28) is obtained from (10) by putting

$$\sum \bar{n}_k \dots = \beta \int \frac{dk}{k} \int d^3\mathbf{n} f(\mathbf{n}) \dots$$

By its definition $f(\mathbf{n})$ contains a factor $\delta(|\mathbf{n}| - 1)$, since \mathbf{n} can be pictured as the unit vector $\mathbf{k}/|\mathbf{k}|$ representing the direction of propagation of a single photon.

If one now multiplies eq. (27) by ω and differentiates with respect to ω , remembering that $y = E/\omega$ one obtains

$$\frac{\partial}{\partial \omega} \omega d^4P(k) = (2\pi)^{-4} d\omega d^3\mathbf{u} \frac{\beta}{\omega} \cdot \int d^4\xi \int d^3\mathbf{n} f(\mathbf{n}) (1 - \exp[-iy(\tau - (\mathbf{n}\xi))]) \exp[-h + i\tau - i(\mathbf{u}\xi)].$$

As in eq. (16) only the first term in the integral gives a contribution $\neq 0$ for $\omega < E$ and one therefore has—because of the normalization of f —

$$(29) \quad \frac{\partial}{\partial \omega} \omega d^4 P(k) = \beta d^4 P(k).$$

From this (25) immediately follows. For if one puts $d^4 P(k) = p(\omega, \mathbf{u}) d\omega d^3 \mathbf{u}$, (29) can be rewritten as $[\partial/(\partial \lg \omega)] \lg(\omega p(\omega, \mathbf{u})) = \beta$, so that $p(\omega, \mathbf{u})$ must be of the form $\omega^{-1+\beta} A(\mathbf{u})$. This apart from the normalization is exactly what is expressed in eq. (25). The normalization is chosen in such a way that $\int d^4 P$ extended over all energies and momenta is equal to 1.

A comparison of the expression (27) with (25) gives as a definition of A

$$(30) \quad \beta N^{-1} A(\mathbf{u}) = (2\pi)^{-4} y^\beta \int d^4 \xi \exp[-h + i\tau - i(\mathbf{u}\xi)].$$

The separability in the form (25) tells us that the right-hand side must be independent of y as long as $y > 1$. Equation (30) is therefore certainly valid in the limit $y \rightarrow \infty$. In this limit the λ -integration in (28) can be explicitly carried out. Putting $\tilde{h} = h - \beta \lg y$, we have

$$(31) \quad \tilde{h} = \beta \int d^3 \mathbf{n} f(\mathbf{n}) \log(\gamma(\tau - (\mathbf{n}\xi))) + i\beta \frac{\pi}{2}.$$

The logarithm under the integral is defined to be positive for large positive values of its argument. The analyticity of e^{-h} and with it of \tilde{h} in the negative imaginary half of the τ -plane, requires that the logarithm be defined in the complex plane of its argument cut from 0 to $+i\infty$. Equation (30) can now be replaced by

$$(32) \quad A(\mathbf{u}) = N\beta^{-1}(2\pi)^{-4} \int d^4 \xi \exp[-\tilde{h} + i\tau - i(\mathbf{u}\xi)].$$

This equation expresses a functional relationship between the velocity distribution $A(\mathbf{u})$ and the classical angular distribution $f(\mathbf{n})$. We have not succeeded to evaluate the integral (32) in a closed form, but we shall show in the next two Sections that it is possible to obtain satisfactory approximations for this function.

6. — The properties of the velocity distribution.

We shall first show that to order β we can approximate $A(\mathbf{u})$ by $f(\mathbf{u})$. This apart from the identity of the angular distributions of the classical radiation

and the 4-momentum loss also implies that in good approximation most of the energy is always carried away by a single photon. For $f(\mathbf{u}) \neq 0$ only if $|\mathbf{u}| = 1$, which means $|\mathbf{k}| = \omega$, and this is always true for a single photon. If more than one photon were involved we would in general have $|\mathbf{k}| < \omega$ unless all these photons were emitted in the same direction. To show

$$(33) \quad A(\mathbf{u}) = f(\mathbf{u}) + O(\beta),$$

we start from eq. (32) and substitute τ for $\tau - (\mathbf{u}\boldsymbol{\xi})$. (32) then becomes

$$A = N\beta^{-1}(2\pi)^{-4} \int d^4\xi \exp[-\bar{h} + i\tau].$$

\bar{h} is defined by

$$(34) \quad \bar{h} = \beta \int d^3\mathbf{n} f(\mathbf{n}) \lg(\gamma^i(\tau - (\mathbf{n} - \mathbf{u}, \boldsymbol{\xi}))).$$

Differentiating $A(\mathbf{u})$ with respect to u_i one gets

$$\frac{\partial A(\mathbf{u})}{\partial u_i} = -N(2\pi)^{-4} \int d^4\xi \xi_i \int d^3\mathbf{n}' f(\mathbf{n}') (\tau - (\mathbf{n}' - \mathbf{u}, \boldsymbol{\xi}))^{-1} \exp[-\bar{h} + i\tau].$$

If one now reverses the substitution for τ and replaces $\xi_i \exp[-i(\mathbf{u}\boldsymbol{\xi})]$ by $i(\partial/\partial u_i) \exp[-i(\mathbf{u}\boldsymbol{\xi})]$ one obtains an alternative for eq. (32), namely

$$(35) \quad A(\mathbf{u}) = -iN(2\pi)^{-4} \int d^4\xi \int d^3\mathbf{n}' f(\mathbf{n}') (\tau - (\mathbf{n}'\boldsymbol{\xi}))^{-1} \exp[-\tilde{h} + i\tau - i(\mathbf{u}\boldsymbol{\xi})].$$

This expression can be easily evaluated in the limit $\beta \rightarrow 0$. For in that limit we can neglect \tilde{h} in the exponent. The integration over τ has to be carried out in such a way that the integrand can be considered analytic in the lower half of the τ -plane. The path of integration must therefore avoid the pole at $\tau = (\mathbf{n}'\boldsymbol{\xi})$ by passing underneath. The τ -integration can now be carried out by closing the path of integration in the upper half of the plane, the pole giving the contribution $2\pi i \delta(\tau - (\mathbf{n}'\boldsymbol{\xi}))$. Integrating over $\boldsymbol{\xi}$ one then gets another δ -function: $(2\pi)^3 \delta(\mathbf{n}' - \mathbf{u})$. Since \tilde{h} is of the order β and N (compare eq. (19)) of the form $1 + O(\beta^2)$ this immediately gives eq. (33).

It is important to note that though (33) only holds $O(\beta)$ the normalization of A holds to order β^2 , *i.e.* $\int d^3\mathbf{u} A(\mathbf{u}) = 1 + O(\beta^2)$. In most practical applications (defined by the particularities of the function $\varrho(k)$) it turns out that the error introduced in the radiative corrections only depends on the error in the

normalization of A . This fortunate circumstance will be discussed in the following Section.

The most obvious violation of our expectations presented by (33) is that it makes $|\mathbf{u}|=1$. However, this error is of course also of $O(\beta)$. This can be shown more explicitly. Indeed

$$(36) \quad \langle \mathbf{u}^2 \rangle = \frac{1}{1 + \beta},$$

where the average is taken over all the directions and (36) is valid for $\omega < E$. To demonstrate (36) we go back to eq. (9), which we use to determine \mathbf{k}^2 in the average over $d^3\mathbf{k}$ and for a given energy loss ω . One has

$$\begin{aligned} \langle \mathbf{k}^2 \rangle \int d^4x \int d^3\mathbf{k} \exp[-h - i(k, x)] &= \\ = \int d^4x \int d^3\mathbf{k} \mathbf{k}^2 \exp[-h - i(k, x)] &= - \int d^4x \int d^3\mathbf{k} \exp[-h] \Delta \exp[-i(k, x)], \end{aligned}$$

where Δ is the Laplace operator in \mathbf{x} -space. Now, since for processes of the type (1) one obviously has $n_{\mathbf{k}} = n_{-\mathbf{k}}$ it follows that h is an even function of \mathbf{x} , so that $\text{grad } h(0, t) = 0$. One therefore gets after an integration by parts and carrying out the integration over $d^3\mathbf{k}$

$$\langle \mathbf{k}^2 \rangle \int dt \exp[-h(0, t) + i\omega t] = \int dt \Delta h(0, t) \exp[-h(0, t) + i\omega t].$$

Now since—because of (10) and (12)—

$$\Delta h(0, t) = \sum \bar{n}_{\mathbf{k}} \mathbf{k}^2 \exp[-i\mathbf{k}_0 t] = \beta \int d\lambda \lambda \exp[-i\lambda t],$$

one finds

$$\langle \mathbf{k}^2 \rangle \int dt \exp[-h(0, t) + i\omega t] = \beta \int d\lambda \lambda \int dt \exp[-h(0, t) + i(\omega - \lambda)t].$$

Because of the analyticity properties of h , the λ -integration needs to be extended only from 0 to ω . From eq. (17) we know that the integral over t on the left-hand side is proportional to $\omega^{\beta-1}$, the integral on the right-hand side is therefore proportional to $(\omega - \lambda)^{\beta-1}$ so that

$$(37) \quad \langle \mathbf{k}^2 \rangle \omega^{\beta-1} = \beta \int d\lambda \lambda (\omega - \lambda)^{\beta-1}.$$

Integrating the right-hand side one immediately obtains $\langle \mathbf{k}^2 \rangle = \omega^2(1 + \beta)^{-1}$ and

this is exactly the result (36). As long as β is small this result signifies that the bulk of the energy loss is always due to a single photon. Multiphoton processes are responsible only for a fraction β of the total energy loss.

One can summarily take account of (36) by improving eq. (33) by putting

$$(38) \quad A(\mathbf{u}) = (1 + \beta)^{\frac{3}{2}} f(\mathbf{u}(1 + \beta)^{\frac{1}{2}}).$$

This expression for $A(\mathbf{u})$ preserves the normalization and gives the right average $\langle \mathbf{u}^2 \rangle$.

7. — Determination of the infra-red correction factor.

We have shown in Sect. 3 that the energy dependence of the infra-red correction can be determined accurately. For the momentum distribution we have so far only derived approximations. In this Section we show—discussing some typical examples—that the approximations discussed in the previous Section are amply accurate for their application to the next generation of experiments with electron positron storage rings.

We shall introduce a resolution function $\varrho(k)$ of the special form

$$(39) \quad \varrho(k) = \exp[-\omega^2/2\Delta\omega^2 - a_{rs}k_r k_s/2\Delta p^2].$$

Here $\Delta\omega$ is the energy resolution of the experiment and we shall call Δp the maximum momentum resolution. a_{rs} is a numerical 3×3 matrix normalized in such a way that its biggest eigenvalue is unity.

We introduce the infra-red correction factor C to be

$$(40) \quad C[\varrho] = \int d^4P(k) \varrho(k)$$

and we note that it follows from the considerations of Sect. 3 that $C = 1$ if both $\Delta\omega$ and Δp are infinite.

The considerations of Sect. 2 and 4 show that a knowledge of C allows one to compare two experiments carried out on the same reaction but with different apparatus. Two experiments giving respectively $d^2\sigma_{\text{exp}}^{(1)}$ and $d^2\sigma_{\text{exp}}^{(2)}$ are in agreement with one another if within the experimental error in the determination of $d^2\sigma$ one has

$$(41) \quad d^2\sigma_{\text{exp}}^{(1)}/C(1) = d^2\sigma_{\text{exp}}^{(2)}/C(2).$$

Using the separation theorem (25) for $d^4P(k)$ and putting $x = \omega/\sqrt{2}\Delta p$

and $y = \Delta p / \Delta \omega$, eq. (40) can be transformed to give

$$(42) \quad C = \beta N^{-1} \left(\frac{\sqrt{2} \Delta p}{E} \right)^\beta \int \frac{dx}{x} x^\beta \exp[-x^2 y^2] \int d^3 \mathbf{u} A(\mathbf{u}) \exp[-a_r u_r u_s x^2].$$

The lower limit of the x -integration is 0; the upper limit is $E/\sqrt{2}\Delta p$. If the experimental arrangement is such that the x -integration converges rapidly the upper limit can be replaced by ∞ .

A first indication of the insensitivity of the radiative correction factor C to the details of the velocity distribution is obtained in the following way. Consider the correction factor $C(b, \Delta p)$ formed from a family of velocity distributions $b^3 A(b\mathbf{u})$. An inspection of eq. (42) then shows that

$$C(b, \Delta p) = C(1b, \Delta p).$$

If the momentum resolution is weak and the energy resolution is strong we approach the limit dealt with in Sect. 3: (41) will only depend on the normalization of A and its dependence on Δp will be negligible. If, on the other hand, the momentum resolution is good and the energy resolution is weak, C will be proportional to Δp^β . According to eqs. (36) and (38) we should choose $b = (1 + \beta)^{\frac{1}{2}}$, so that in the case of good momentum resolution we should expect $C(38)/C(33) = (1 + \beta)^{\beta/2}$. $C(38)$ is calculated with the distribution (38) and correspondingly $C(33)$ with the distribution (33). It is therefore seen that the more realistic formula (38) gives a correction which is approximately $1 + (\beta^2/2)$ times bigger than that given by the approximation (33). For practically all experiments the error is less than 0.5%.

As a further evidence of the insensitivity of C to the details of the distribution A we consider the following special case. We put the line of collision of electrons and positrons in the z -direction and assume that the A and \bar{A} particles are observed by means of two spark chambers placed at $Y = \pm a$ and looking at A particles which emerge at about 90° to the direction of the incident beam. We assume that there is no energy resolution (*i.e.* $y = 0$ in eq. (42)) and that the momentum resolution is given by $\Delta p_y = \infty$, $\Delta p_x = \Delta p_z = \Delta p \ll E$. We shall neglect the photons emitted in the creation of the A particles and only deal with those emitted in the annihilation of positrons and electrons. In this case the integral (42) can be evaluated $O(\beta^2)$ and one gets for C

$$(43) \quad C(33) = N^{-1} \left(\frac{\sqrt{2} \Delta p}{E} \right)^\beta \Gamma \left(1 + \frac{\beta}{2} \right) (1 + 1/137\pi) + O(\beta^2).$$

If, on the other hand, one chooses for $A(\mathbf{u})$ the very rough approximation

$$(44) \quad A(\mathbf{u}) = \frac{1}{2}(\delta(\mathbf{u} - \hat{p}) + \delta(\mathbf{u} + \hat{p})),$$

one gets

$$(45) \quad C(44) = N^{-1} \left(\frac{\sqrt{2} \Delta p}{E} \right)^\beta \Gamma \left(1 + \frac{\beta}{2} \right).$$

The difference between (45) and (43) is seen to be less than 0.25%!

Another approximation for the function $f(\mathbf{n})$, which we discuss in Appendix I and which takes account of the fact that some photons may be emitted far off the backward and forward directions, is

$$(46) \quad f(\mathbf{n}) = \frac{1}{2} b (\delta(\mathbf{n} - \hat{\mathbf{p}}) + \delta(\mathbf{n} + \hat{\mathbf{p}})) + \frac{1-b}{4\pi} \delta(|\mathbf{n}| - 1),$$

where a realistic choice of b is

$$(47) \quad b = 1 - \frac{4\alpha}{\pi\beta}.$$

For $\beta = \beta_0 = 0.072$ (corresponding to $E = 1000$ MeV) one gets $b = 0.87$. In this case the integral (42) can be evaluated exactly and one gets

$$(48) \quad C(46) = N^{-1} \left(\frac{\sqrt{2} \Delta p}{E} \right)^\beta \Gamma \left(1 + \frac{\beta}{2} \right) (1 + (1-b)X)$$

with

$$(48') \quad X = \frac{2^{-\beta} \Gamma(1 - \beta/2)^2}{1 - \beta \Gamma(1 - \beta)}.$$

It is seen from this formula that even if b were let to vary between 0 and 1—*i.e.* from a completely isotropic distribution to a distribution which is completely peaked in the forward and backward directions the variation introduced in C would only be $X = \beta(1 - \lg 2) = 0.306\beta$ *i.e.* 2.2%. If we insert the realistic value (47) for b we get $(1-b)X = (4\alpha/\pi)(1 - \lg 2)$, which is independent of β as the correction to the correction α/π in eq. (43). Taking as a base the simplest approximation (44) we therefore find

$$(49) \quad C(33) = 1.0023 C(44), \quad C(46) = 1.0028 C(44).$$

Approximation (46) is therefore seen to be good to 5 parts in 10000.

All the approximate correction factors $C(33)$, $C(44)$ and $C(46)$ should be multiplied by $(1+\beta)^{\beta/2} = 1.005$ if one wants to take account of eq. (36).

The preceding considerations show that one can take considerable liberties in approximating the function $A(\mathbf{u})$ as long as one does not violate its normalization. It will be a long time before storage ring experiments with their notoriously slow accumulation of data will permit experiments in which the statistical error in the determination of a cross-section is less than 1% and for these it will be still quite sufficient to use approximations for $A(\mathbf{u})$ of the type (44).

To illustrate our results we now discuss the radiative corrections for the reaction

$$(50) \quad e^+ + e^- \rightarrow \mu^+ + \mu^-,$$

in which the muons are observed at $\theta = 90^\circ$. At a sufficiently high energy of both electrons and muons there will be little interference between the radiation emitted by either type of particle. We can then separate the function $f(\mathbf{n})$ into $f(\mathbf{n}) = f_e(\mathbf{n}) + f_\mu(\mathbf{n})$. Correspondingly we shall have

$$(51) \quad \beta = \beta_e + \beta_\mu,$$

the values of which are listed in the Table I.

TABLE I.

E (MeV)	β_e	β_μ	β
250	0.059	0.010	0.069
500	0.065	0.016	0.081
750	0.069	0.020	0.089
1000	0.072	0.023	0.095
1250	0.074	0.025	0.099
1500	0.076	0.026	0.102
1750	0.077	0.028	0.105
2000	0.078	0.029	0.107
2250	0.079	0.030	0.109
2500	0.080	0.031	0.111
2750	0.081	0.032	0.113
3000	0.082	0.033	0.115

To evaluate the radiative corrections for this process we put, in correspondence to eq. (44) and taking account of the normalization of A and of (51),

$$(52) \quad 2\beta A(\mathbf{u}) = \beta_e(\delta(\mathbf{u} - \hat{p}) + \delta(\mathbf{u} + \hat{p})) + \beta_\mu(\delta(\mathbf{u} - \hat{q}) + \delta(\mathbf{u} + \hat{q})),$$

where \hat{q} is the unit vector in the direction of flight of the negative muon. This approximation should be good for E.R. muons.

We assume in agreement with the properties of an experimental arrangement actually proposed for use with Adone that

$$(53) \quad \Delta p \ll \Delta \omega \ll E,$$

the energy resolution of this experiment being based on the existence of absorbers in the path of the muon and therefore being less accurate than the observation of transverse momentum afforded by a spark chamber.

A remarkable feature of this type of experiment is that the momentum resolution in the spark chambers does not give any information about the radiation lost in the creation of the muons. With regard to the second term in (52) we can therefore apply the rules of Sect. 3, valid for the case of pure energy resolution. For the first term we apply (45), so that

$$(54) \quad \beta NC = \Gamma \left(1 + \frac{\beta}{2}\right) \left[\beta_e \left(\frac{\sqrt{2}\Delta p'}{E}\right)^\beta + \beta_\nu \left(\frac{\sqrt{2}\Delta \omega}{E}\right)^\beta \right],$$

where

$$(55) \quad \Delta p' = \Delta p (1 + \Delta p^2/\Delta \omega^2)^{\frac{1}{2}}.$$

If the first part of the inequality (53) is satisfied one has of course $\Delta p' = \Delta p$. One can improve eq. (54) by taking account of (36). This requires that the first term in (54) be multiplied by $(1 + \beta)^{\beta/2}$. As an extreme example we consider a very accurate experiment in which $\Delta p/E = 0.001$ and $\Delta \omega/E = 0.01$. The momentum resolution corresponds to assuming that it is possible to determine the angle of the muons to within 1 mrad. In this extreme case one gets for the radiative correction factor the value $CN = 0.54$. We have assumed $\beta = 0.095$ which (compare Table I) corresponds to an energy of 1000 MeV.

Equation (54) can also be applied to the case in which the two spark chambers are used to look at events with, say, $45^\circ < \theta < 135^\circ$ and $-45^\circ < \varphi < 45^\circ$. (The centres of the spark chambers are assumed to have $\theta = 90^\circ$ and $\varphi = 0^\circ$ and 180° respectively.) Formula (54) then still holds if $\Delta p'$ is replaced by $\Delta p'/|\sin \theta \cos \varphi|$ and $\Delta \omega$ by $\Delta \omega/|\sin \theta \cos \varphi|$. The latter corresponds to assuming that the energy of the muons is measured by counting the gaps traversed by these particles. If the experiment is extended to angles of up to 45° it follows that over the whole range of the experiment C should vary from $C(90^\circ)$ to $C(45^\circ) = C(90^\circ) 2^{\beta/2} = C(90^\circ) \times 1.0336$.

It must be remembered that $d^2\sigma_{\text{exp}}/C$ defines (compare eq. (20)) $d^2\sigma_x$ —which we can term the infra-red corrected cross-section. In order to com-

pare this cross-section with $d^2\sigma_0$ obtained from lowest-order perturbation theory we still have to apply the ultra-violet correction. To this end we have to add a muon contribution to (22). (There are no mixed electron muon terms, owing to the Cabibbo-Putolu theorem ⁽¹⁰⁾.) For an energy of 1000 MeV one has $\lambda = 0.068$.

8. — Estimate of errors and definition of the balanced experiment.

Equation (3) is valid in the limit $k = 0$, but is of course supposed to be applied to finite values of k . In this Section we want to estimate the error introduced by the application of eq. (3) to an actual experiment with a finite energy resolution $\Delta\omega$. Such an estimate can be obtained by comparing eq. (4) with the formula for bremsstrahlung. Throughout the previous calculations we have assumed that $dW = \beta d\omega$ and that β is a constant. If $\Delta\omega/E \neq 0$ we can apply the Bethe-Heitler formula which will give $dW = \beta d\omega(1 - (\Delta\omega/E) + \frac{2}{3}(\Delta\omega/E)^2)$. If we therefore extend the validity of (3) to finite $\Delta\omega$ this can be considered as equivalent to over-estimating β . We can take account of this by introducing an average β , viz. $\bar{\beta} = \beta(1 - (\Delta\omega/2E)) = \beta - \Delta\beta$. We estimate that the error introduced in the extrapolation of eq. (3) is of the magnitude of the error caused by over-estimating β , so that the relative error in applying eq. (3) should be of the order

$$(56) \quad \frac{\delta d^2\sigma_{\text{exp}}}{d^2\sigma_{\text{exp}}} = \beta \frac{\Delta\omega}{2E} \lg(E/\Delta\omega).$$

This error is tolerably small: about 1% for $\Delta\omega/E = 0.1$ and $\beta = 0.1$.

Equation (56) shows that in order to be able to apply the formalism which has been developed in this paper to an experiment, the resolution of this experiment has to be good. The formalism thus requires that the statistical accuracy of an experiment be matched by the accuracy of its resolution. We shall call such an experiment « balanced ». In particular we require for a balanced experiment that

$$(57) \quad \varepsilon \approx \beta \frac{\Delta\omega}{2E} \lg(E/\Delta\omega),$$

where ε is the relative statistical error of the measurement.

It is very likely—compare eq. (22)—that the expansion parameter of $d^2\sigma_{\pm}$

⁽¹⁰⁾ A. PUTSOLU: *Rome Thesis* (1961).

in terms of e^2 is β , so that one will have

$$(58) \quad d^2\sigma_E = d^2\sigma_0(1 + \lambda_1\beta + \lambda_2\beta^2 + \dots),$$

where the λ_n are numbers of the order of magnitude of unity.

In order to check electrodynamics to the n -th order we therefore have to make the statistical error $\varepsilon \ll \beta^n$. The matched energy resolution is then given by eq. (57).

The advantage of a matched experiment is that its results can be very easily compared with the theory, since the experimental corrections can be separated from the theoretical ones.

It must be clear that balanced experiments form a class—and not the only one—of experiments which can be used as a check of quantum electrodynamics. As an example of an unbalanced experiment one could consider the process $e^+ + e^- \rightarrow \mu^+ + \mu^- + \Gamma$ integrated over all angles and without limitations imposed on the photon-spectrum Γ . Such an « open » experiment can be considered a perfect check of quantum electrodynamics, provided that the beam monitoring can be trusted to the statistical accuracy. In experiments whose significance is independent of the monitoring some angular dependence has to be observed. In this case the angular distribution has to be determined with some accuracy, so that even if it were not for radiative corrections the angular resolution would have in some way to be matched to the statistical error. The majority of experiments so far proposed for Adone are balanced experiments for this reason.

APPENDIX I

The destruction of the incident particles and the creation of the final particles define a four-current $j_\nu(x)$. This four-current in turn will give rise to the emission of radiation, the vector potential of which satisfies the equation

$$(A.1) \quad \square A_\nu(x) = -4\pi j_\nu(x).$$

This equation has to be solved assuming that $\lim_{t \rightarrow -\infty} A_\nu(x) = 0$ corresponding to the fact that we assume that at the beginning of the reaction there is no radiation present.

Introducing the Fourier amplitudes $A_\nu(k)$ and $j_\nu(k)$, given by

$$A_\nu(x) = \int d^4k A_\nu(k) \exp[i(kx)], \quad j_\nu(x) = \int d^4k j_\nu(k) \exp[i(kx)],$$

one has as a solution of (A.1) $A_\nu(k) = 4\pi j_\nu(k)/k^2$. In the integration over d^4k the poles at $k^2 = 0$ have to be circumnavigated in such a way that $A_\nu(x)$ vanishes for negative infinite times. For positive infinite times we will then have only a positive energy contribution, precisely

$$(A.2) \quad A_\nu^{\text{out}}(x) = (2\pi)^2 i \int d^3k j_\nu(\mathbf{k}, \omega) \exp [i(kx)]/\omega,$$

where $\omega = |\mathbf{k}|$. If one introduces a finite volume V of normalization (A.2) can be replaced by

$$(A.2') \quad A_\nu^{\text{out}}(x) = (2\pi)^5 i \sum_{\mathbf{k}} \frac{j_\nu(\mathbf{k}, \omega)}{\omega V} \exp [i(kx)].$$

In (A.2) as well as in (A.2') one has $(kx) = (\mathbf{k}\mathbf{x}) - \omega t$. To determine the number of photons emitted in process (1) we compare (A.2') with the expression

$$(A.3) \quad A_\nu^{\text{out}}(x) = \sum_{\mathbf{k}, \alpha} \left(\frac{\hbar}{\omega V} \right)^{\frac{1}{2}} e_\nu^\alpha(\mathbf{k}) a_\alpha^{\text{out}}(\mathbf{k}) \exp [i(kx)] + \text{h.c.},$$

in which $e_\nu^\alpha(\mathbf{k})$ is a three-dimensional polarization vector, $\alpha = 1, 2$ and $(e_\nu^\alpha(\mathbf{k}), \mathbf{k}) = 0$. $a_\alpha^{\text{out}}(\mathbf{k})$ is the amplitude of the outgoing field—its destruction operator in 2nd quantization. The average number of photons with polarization α and momentum \mathbf{k} is given by

$$(A.4) \quad \bar{n}_\alpha^{\text{out}}(\mathbf{k}) = |a_\alpha^{\text{out}}(\mathbf{k})|^2;$$

comparing (A.3) with (A.2') one therefore finds that

$$(A.5) \quad \sum_{\alpha} \bar{n}_\alpha(\mathbf{k}) = \bar{n}(\mathbf{k}) = (2\pi)^9 \frac{|\mathbf{j}_\perp(\mathbf{k}, \omega)|^2}{\hbar\omega V},$$

where \mathbf{j}_\perp is the component of \mathbf{j} perpendicular to \mathbf{k} . In continuous notation and in correspondence to eq. (5) of the text we can define $d^3\bar{n}(\mathbf{k})$ as the average number of photons in the momentum interval $d^3\mathbf{k}$. It follows immediately from (A.5) that

$$(A.6) \quad d^3\bar{n}(\mathbf{k}) = (2\pi)^6 \frac{d^3\mathbf{k}}{\omega} |\mathbf{j}_\perp(\mathbf{k}, \omega)|^2$$

(we have put $\hbar = 1$ in this expression).

In order to determine the coefficient β defined in eqs. (4) and (5) we consider the case of a particle which is born at time $t = 0$ and accelerated in a negligibly small time to the final velocity v . In this case we have $\mathbf{j}(x) = 0$ for $t < 0$ and

$$(A.7) \quad \mathbf{j}(x) = ev\delta(\mathbf{x} - \mathbf{v}t) \quad \text{for } t > 0.$$

The δ -function indicates that we deal with a point particle. To simulate the

experimental situation in which light emitted by the particle long after its creation cannot be observed, we imagine (A.7) to be multiplied by a convergence factor $\exp[-\varepsilon t]$ (with $\varepsilon > 0$) and we evaluate $\mathbf{j}(k)$ in the limit $\varepsilon \rightarrow 0$. This gives

$$(2\pi)^4 \mathbf{j}(k) = e\mathbf{v} \int_0^{\infty} dt \int d^3\mathbf{x} \delta(\mathbf{x} - \mathbf{v}t) \exp[-i(\mathbf{k}\mathbf{x}) + i\omega t - \varepsilon t]$$

and therefore

$$(A.8) \quad (2\pi)^4 \mathbf{j}(\mathbf{k}, \omega) = -ie \frac{\mathbf{v}}{((\mathbf{k}\mathbf{v}) - \omega)}.$$

The destruction of a particle gives a similar contribution to the current. The convergence factor has to be chosen as $\exp[+\varepsilon t]$ and the integration has to be extended from $-\infty$ to 0 instead of from 0 to $+\infty$. As a result the expression for (A.8) changes sign. It follows that the expression for \mathbf{j} corresponding to the creation and annihilation of n particles with velocities \mathbf{v}_i can be given as

$$(A.9) \quad (2\pi)^4 \mathbf{j}(\mathbf{k}, \omega) = -i|e| \sum_{i=1}^n \frac{\varepsilon_i \mathbf{v}_i}{((\mathbf{k}\mathbf{v}_i) - \omega)},$$

where the signature ε_i is positive for the creation of a positive or the destruction of a negative particle, it is negative for the creation of a negative and the destruction of a positive particle.

The denominators in (A.8) and (A.9) indicate a strong forward peaking for extreme relativistic particles. Indeed one can write for the denominator $\omega(1 - v \cos \theta)$ (where θ is the angle between the direction of flight of the emitting particle and the direction of flight of the photon). If v is close to 1 most of the radiation will be expected to go into an angle of order $1/\gamma$ ($\gamma = E/m$). This forward peaking is opposed by the transversality of the radiation, which via \mathbf{j}_{\perp} (compare (A.5)) introduces a factor $\sin^2 \theta$ into the angular distribution function. We shall now discuss this behaviour of the angular distribution for a special case.

The great majority of experiments proposed for $e^+ + e^-$ colliding beams deals with the destruction and creation of pairs of particles with opposite velocities. In many cases the energies of all the particles are extreme relativistic. We therefore consider first the current and the photon numbers created in the destruction (or creation) of a particle and an antiparticle in the centre-of-mass system. According to the rule given after (A.9) the signatures of the current contributions are opposite, but since also the velocities of the pair are opposed to one another the contributions of the two particles will add. Choosing $\mathbf{0Z}$ as the direction of flight of the negative particle and remembering that $\omega = |\mathbf{k}|$ we get for the current

$$(A.10) \quad (2\pi)^4 j_z = \frac{-2iev}{\omega} (1 - v^2 \cos^2 \theta)^{-1}.$$

Inserting this into eq. (A.6) one gets

$$(A.11) \quad d^3\bar{n}(\mathbf{k}) = \frac{4\alpha v^2}{(2\pi)^2} \frac{d\omega}{\omega} \frac{\sin^2 \theta}{(1 - v^2 \cos^2 \theta)^2} d \cos \theta d\varphi.$$

Comparing this equation with eq. (5) of this paper one immediately arrives at the following definition of β

$$(A.12) \quad \beta = \frac{2\alpha}{\pi} \int_{-1}^{+1} dx \frac{1 - x^2}{(1 - v^2 x^2)^2},$$

as a result of the integration over all the directions of the photon. The integral can be exactly evaluated and gives

$$(A.13) \quad \beta = \frac{\alpha}{\pi} \left(\frac{1}{v} (1 + v^2) \lg \frac{1 + v}{1 - v} - 2 \right),$$

which gives

$$(A.14) \quad \beta = \frac{4\alpha}{\pi} \left(\lg 2\gamma - \frac{1}{2} \right) \text{ e.r.}$$

in the extreme relativistic limit and

$$(A.15) \quad \beta = \frac{8\alpha}{3\pi} v^2 \text{ n.r.}$$

in the nonrelativistic limit. Table II covers the range of (A.13) in which neither approximation (A.14) nor (A.15) are valid.

TABLE II.

γ	1000β	γ	1000β
1.1	1.14	1.6	5.34
1.2	2.17	1.7	6.02
1.3	3.09	1.8	6.62
1.4	3.94	1.9	7.15
1.5	4.67	2.0	7.70

It is important to note that if the interference term is left out in the computation of (A.6), *i.e.* if $(j_+ + j_-)^2$ is approximated by $j_+^2 + j_-^2$, eq. (A.14) is only slightly changed. One obtains in this case $\lg 2\gamma - 1$ instead of $\lg 2\gamma - \frac{1}{2}$. The neglect of the interference term for e.r. particles therefore involves an error of $\Delta\beta = -2\alpha/\pi = 0.465\%$ —the interference is constructive since the currents of electron and positron add. In Sect. 7 we have neglected

the interference between the radiation emitted by the muons and the radiation emitted by the electrons. The inclusion of this interference term can now be estimated to give a contribution smaller than 0.5% to β , since for muons emitted at 90° there should be no interference at all.

There are however glamorous exceptions to the rule that the interference term can be neglected: one of them is the backward electron-positron scattering. In this process we have full constructive interference between the currents of all the four particles involved in the process so that $\beta = 4\beta_e$, which with the values listed in Table I gives $\beta = 0.33$ at 3000 MeV!

Equation (A.11) allows one to determine the angular distribution function $f(\mathbf{n})$ introduced in eq. (28). Indeed it follows from (A.11) and (A.12) that we can put

$$(A.16) \quad f(\mathbf{n}) d^3\mathbf{n} = \frac{\alpha}{\beta\pi^2} \frac{\sin^2\theta}{(1 - v^2 \cos^2\theta)^2} d\cos\theta d\varphi.$$

That this function is properly normalized follows from (A.12).

The value of the function for $\cos\theta = 0$ is $\alpha/\beta\pi^2$. The choice of the constant b defined in eq. (46) was made on the basis $(1 - b)/4\pi = \alpha/\beta\pi^2$. This choice is of course only meaningful if it gives $1 - b < 1$, *i.e.* if β is sufficiently large. Indeed, a discussion of $f(\mathbf{n})$ shows a completely different behaviour for $v^2 < \frac{1}{2}$ and for $v^2 > \frac{1}{2}$. For $v^2 < \frac{1}{2}$ the function f has two minima at $\cos\theta = \pm 1$ and a maximum at $\cos\theta = 0$. For $v^2 > \frac{1}{2}$ the maximum at $\cos\theta = 0$ becomes a minimum and two new maxima are born at

$$(A.17) \quad \sin^2\theta_m = (\gamma^2 - 1)^{-1}, \quad \gamma^2 > 2.$$

With increasing energy these maxima form the backward and forward peaks placed at $\sin\theta_m \approx \pm 1/\gamma$. The value of f at the maximum is given by

$$(A.18) \quad f_m = \frac{\alpha}{4\beta\pi^2} \frac{\gamma^4}{\gamma^2 - 1}.$$

At high energies f_m therefore increases as γ^2 . The typical behaviour of $f(\mathbf{n})$ is shown in Fig. 1 for $\gamma = 3$.

We finally show that β is a relativistically invariant function of the momenta of the charged particles.

From (5), (A.6) and (A.9) it follows that one can write

$$(A.19) \quad \beta = \frac{\alpha}{(2\pi)^2} \int d^3\hat{n} \sum_{\hat{e}} \left| \sum_i \frac{(\mathbf{p}_i \hat{e}) \varepsilon_i}{(\mathbf{p}_i \hat{n}) - p_{0i}} \right|^2.$$

The summation over the polarization vector \hat{e} can be replaced by a summation over a four-vector e_μ . In place of the unitvector \hat{n} one can introduce a four-vector $n \equiv (\hat{n}, n_0)$ and one can replace $d^3\hat{n}$ by $2d^4n\theta(n)\delta(n^2)\delta(n_0 - 1)n_0$,

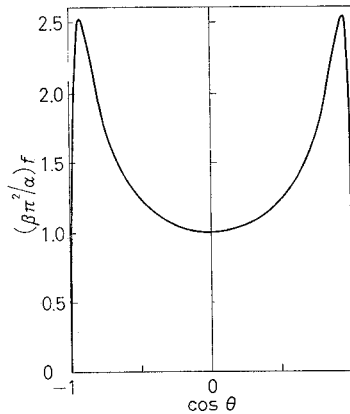


Fig. 1. - The angular distribution $f(\mathbf{n})$ for $\gamma = 3$.

where $\theta(n) = 1$ inside and on the positive light cone, and zero otherwise. This therefore gives

$$(A.20) \quad \beta = \frac{2\alpha}{(2\pi)^2} \int d^4n \theta(n) \delta(n^2) \delta(n_0 - 1) n_0 \sum_e \left| \sum_i \frac{(p_i e) \varepsilon_i}{(p_i n)} \right|^2.$$

We now define $\beta(\omega)$ by replacing $\delta(n_0 - 1)$ by $\delta(n_0 - \omega)$. A scale transformation shows that $\beta = \beta(\omega)$. The only not obviously covariant element in (A.20) is $n_0 \delta(n_0 - 1)$. Now, under the Lorentz transformation

$$n'_0 = \gamma(n_0 + v n_0 \cos \theta)$$

we have $n'_0 \delta(n'_0 - 1) = n_0 \delta(n_0 - \omega)$ with $\omega = (1 + v \cos \theta)^{-1} \gamma^{-1}$. It therefore follows that β is an invariant.

APPENDIX II

In this Appendix we discuss the evaluation of some typical integrals of infra-red electrodynamics. We first determine N defined in eq. (17) of the text. The simplest way of doing this is by the use of the separation theorem from which we had derived eq. (30) as a definition of $A(\mathbf{u})$. Equation (30) is valid for any $y > 1$ and must therefore also be valid for $y \rightarrow \infty$. Integrating (30) over $d^3\mathbf{u}$ and remembering that $A(\mathbf{u})$ is normalized one gets

$$(A.21) \quad 1 = \frac{N}{2\pi\beta} y^\beta \int d\tau \exp[-h(0, \tau) + i\tau].$$

Putting $k/\omega = x$ in eq. (12) one gets for $h(0, \tau)$

$$(A.22) \quad h(0, \tau) = \beta \int_0^y \frac{dx}{x} (1 - \exp[-ix\tau])$$

and, since we can evaluate (A.19) in the limit $y \rightarrow \infty$, we only need the asymptotic behaviour of h . In this way one obtains

$$(A.23) \quad h(0, \tau) = \beta \lg \gamma y \tau + i \frac{\beta \pi}{2}.$$

Since h must be analytic in the lower half on the complex τ -plane (compare Sect. 2) we think of the log as defined in a complex τ -plane which is cut along the positive imaginary axis. Inserting (A.21) into (A.19) one gets

$$(A.24) \quad 1 = \frac{N}{2\pi\beta} \gamma^{-\beta} \int_{-\infty}^{+\infty} d\tau \tau^{-\beta} \exp \left[i\tau - i \frac{\beta}{2} \pi \right].$$

The path of integration can now be deformed into a loop which starts from $+i\infty$, follows the left bank of the cut to 0 and rises along the right bank to $+i\infty$. It follows that we can write for the integral

$$\int_{-\infty}^{+\infty} = \int_0^{i\infty} d\tau \exp [i\tau] ((\tau^{-\beta})_r - (\tau^{-\beta})_l) \exp \left[-i \frac{\beta\pi}{2} \right],$$

where the subscripts r and l indicate respectively the right and left bank of the cut along the positive imaginary axis. One has

$$\tau_r = |\tau| \exp \left[i \frac{\pi}{2} \right] \quad \text{and} \quad \tau_l = |\tau| \exp \left[-i \frac{3\pi}{2} \right].$$

Using the definition

$$\Gamma(z) = \int_0^{\infty} dt \exp [-t] t^{z-1}$$

of the Γ -function as well as the identity $\Gamma(z) \Gamma(1-z) = \pi/\sin \pi z$ one obtains eq. (19) from (A.22).

RIASSUNTO

Le correzioni infrarosse da applicare ai risultati di un'esperienza con fasci incrociati di elettroni e positroni sono determinate tramite il teorema di Bloch-Nordsieck. Un esperimento viene caratterizzato da una funzione $\varrho(k)$ di un quadrivettore energia-impulso di tipo tempo, che rappresenta la probabilità che la perdita di un quadrimpulso k sfugga alla rivelazione. I risultati si possono applicare a un tipo di esperimenti per i quali l'errore statistico è opportunamente proporzionato a quello relativo alla risoluzione in energia e impulso. Sono discusse diverse approssimazioni che permettono una stima rapida ed accurata delle correzioni radiative.

Инфракрасные радиационные поправки для экспериментов на встречных (электронных и позитронных) пучках.

Резюме (*). — С помощью теоремы Блока-Нордсика определяются инфракрасные поправки для результатов, ожидаемых от экспериментов на встречных электронных и позитронных пучках. Эксперименты характеризуются функцией разрешения $\varrho(k)$, функцией от четырехмерного времениподобного вектора энергии-импульса, которая представляет вероятность того, что потери четырех-импульса k не регистрируются. Полученные результаты могут быть применимы к классу экспериментов, в которых статистическая ошибка соответствует ошибке для разрешения энергии-импульса. Обсуждаются различные приближения, которые позволяют дать быструю и аккуратную оценку радиационных поправок.

(*) Переведено редакцией.