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A Note on the Infra-Red Divergence.

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Summary. — In this paper we show how the infra-red divergence can be eliminated in the matrix element, provided that physically true final states for an experiment involving creation and destruction of charged particles are used.

1. – Introduction.

It is well known that no physical process involving the creation and destruction of charged particles can take place without the emission of soft photons. This fact together with the observation that the number of soft photons emitted in a reaction can never really be measured, leads one to introduce a new definition of the final states. This definition is closer to physical reality than the use of states which are diagonal in the number of photons.

We shall show that with this redefinition of the final state it is possible to determine a matrix element M for the process, which is

1) finite and does not therefore present an infra-red divergence,

2) directly comparable to the experimental cross-section which results to be proportional to $|M|^2$,

3) and separable: $M = AM'$, where A depends on the cut-off for soft photons and M' is finite.

The present method therefore leads to a compensation of the infra-red divergence for the matrix element itself and does not require—as in the usual method—the compensation of the terms corresponding to real and virtual photons in the expression for the cross-section. This simplification is due to the fact that the Bloch-Nordsieck theorem allows one to predict the phases of the emitted photons.

The new final states are not eigenstates of the energy: this corresponds exactly to the experimental situation, which does not allow a check of energy conservation to any desired accuracy.

2. – Relativistic deformation of final states.

We consider a process

$$\mathrm{A}+\mathrm{B}\rightarrow \mathrm{C}+\mathrm{D}+\ldots\,, \tag{1}$$

in which at least part of the particles A, B, C, D, ... involved are charged. What one actually observes is the process

$$\mathrm{A}+\mathrm{B}\rightarrow \mathrm{C}+\mathrm{D}+\ldots+\Gamma\,, \tag{2}$$

where Γ stands for any number of photons, limited only by the condition that the total 4-momentum of the emitted photons should be contained in that part of the forward light cone, which is defined by the energy and momentum resolution of the experimental apparatus. In the following we shall assume that there is energy resolution only—and no momentum resolution—and we assume that $\Delta\omega \ll E$, where for example E is the centre-of-mass energy of an incident particle.

The final state of reaction (1) will be called $|f)$. For describing reaction (2) we introduce a state vector $|f')$ defined as

$$|f') = e^{iA_c}|f) \tag{3}$$

with

$$A_c = \int \mathrm{d}^4x\, j_\mu(x)\, A_\mu(x)\,, \tag{4}$$

where $A_\mu(x)$ is the 4-potential of the electromagnetic field in the interaction representation and $j_\mu(x)$ is a c-number current defined as

$$j_\mu(x) = (2\pi)^{-4}\int \mathrm{d}^4k\, j_\mu(k)\, e^{ikx} \tag{5}$$

and

$$\gamma_\mu(k) = ie(2\pi)^{-\frac{3}{2}} \sum_i \varepsilon_i P_{i\,\mu}(P_i k)^{-1} \qquad \text{for } K_0 \leqslant \Delta\omega \text{ and zero otherwise}\,. \tag{6}$$

The signature ε_i is $+1$ for positive outgoing and negative incoming particles and -1 otherwise and p_i with $p_{i,0} > 0$ is the energy-momentum vector of

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the i-th particle. We use the metrics $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$. The current defined by eq. (6) is the Fourier component of the classical current accompanying the destruction of the particles A, B and the creation of the particles C, D... The cut-off introduced in this equation is not completely realistic, since it concerns the energy of the single photons, rather than the total energy carried away by the electromagnetic field. We shall discuss this point in Sect. **5**.

The operator e^{iA_c} defined in eq. (3) is unitary. The new states $|f')$ are therefore normalized provided that the old states $|f)$ were normalized. In the new frame of reference the matrix element for the transition $i\to f$ is given by

$$\bar{M}=(f'|S|i)=(f|e^{-iA_c}T(e^{iL})|i)\;; \tag{7}$$

L is the action integral describing the interaction that leads to process (1).

The matrix element $\bar{M}$ can be formally written as

$$\bar{M}=\langle f|(e^{-iA_c}-\mathbf{1})S|i\rangle+\langle f|S|i\rangle=M_c+M_{el}\,. \tag{8}$$

Of course such a definition has no physical meaning; on the other hand mathematically M_c and M_{el} are clearly defined by eq. (8) and serve as means to compare the present formalism with perturbation theory.

We emphasize that M is formally the matrix-element of the operator $S'=\exp[-iA_c]S$ between the states $\langle f|$ and $|i\rangle$ which do not include outgoing soft photons: in other words the infinite number of soft photons created by S in the final state is destroyed by $\exp[-iA_c]$.

3. – Finiteness of $\bar{M}$.

We are now going to show that $\bar{M}$ is finite, *i.e.* that, apart from ultra-violet divergences, it does not present any infra-red divergence. More precisely we shall prove that the expression

$$|\bar{M}|^2=|M_c|^2+|M_{el}|^2+2\,\mathrm{Re}\,(M_cM_{el}^*)$$

is finite.

For this purpose we shall use the well-known theorem [1,2] which states that the infra-red divergences in the cross-section concerning the bremsstrahlung graphs are exactly compensated to any order in the fine-structure constant, by analogous divergences in the corresponding graphs with virtual photons. Then, as

$$M_{el}=\langle f|S|i\rangle\,,$$

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(1) J. M. JAUCH and F. RORHLICH: *Helv. Phys. Acta*, **27**, 613 (1954).

(2) J. M. JAUCH and F. RORHLICH: *The Theory of Electrons and Photons* (New York, 1955).

it will be sufficient to show that, in the limit $\omega_j \to 0$ (ω_j being the energy of of the j-th bremsstrahlung photon), the cross-section for the m-th order bremsstrahlung calculated with the usual methods of electrodynamics, coincides with the equal-order term of

$$|M_c|^2 + 2\,\mathrm{Re}\,(M_c M_{\mathrm{el}}^*)\,.$$

From eq. (7), with our definitions, we have

$$\sum_{n+t-s=2m=\mathrm{const}} \frac{(-i)^t}{t!} \Lambda_c^t \frac{(i)^n}{n!} T(L^n) = \tag{9}$$
$$= \frac{1}{m!}\left\{\frac{1}{2}\int \mathrm{d}^4k\,\delta(k^2)\theta(k_0) j_\mu(k) j_\mu^*(k)\right\}^m \langle f|\frac{(i)^s}{s!} T(L^s)|i\rangle\,.$$

Now we can easily find the term of order α^{m+s} in $|M_c|^2 + 2\,\mathrm{Re}\,(M_c M_{\mathrm{el}}^*)$:

$$[|M_c|^2 + 2\,\mathrm{Re}\,(M_c M_{\mathrm{el}}^*)]_{(m+s)} = \frac{1}{m!}\left\{\int \frac{\mathrm{d}^3k}{2k_0} j_\mu(k) j_\mu^*(k)\right\}^m \left|\langle f|\frac{(i)^s}{s!} T(L^s)|i\rangle\right|^2. \tag{10}$$

The definition of $j_\mu(k)$ (eqs. (5) and (6)) sets the upper limits on the integrals which appear in eq. (10).

The analogous term calculated with the usual final states is

$$\frac{1}{m!}\left\{\int \frac{\mathrm{d}^3k}{2k_0} j'_\mu(k) j'^*_\mu(k)\right\}^m \left|\langle f|\frac{(i)^s}{s!} T(L^s)|i\rangle\right|^2, \tag{11}$$

where now, adopting the notation already introduced, $j'_\mu(k)$ is defined by

$$j'_\mu(k) = ie(2\pi)^{-\frac{3}{2}} \sum_i \varepsilon_i P_{i\mu}(P_i k)^{-1} \tag{12}$$

without any restriction imposed on the values which k can assume. The upper limits in the integrals are fixed by the global condition

$$\sum \omega_k \leqslant \Delta\omega\,.$$

Although eqs. (10) and (11) would not seem to be exactly equal, we must recall that the description of the final states so far given in our formalism, is not completely realistic (compare Sect. **2**) and therefore actually we ought to cut also the integrals in (10) according to the above global condition.

As we shall discuss later (Sect. **5**) the difference between eqs. (10) and (11) is given only by a normalization factor very near to 1.

In this way we have shown the finiteness of $|\overline{M}|^2$ and consequently of $\overline{M}$.

In this demonstration we have not taken into consideration the emission of photons by propagators: indeed, it has been shown ([3,4]), that these terms do not lead to infra-red divergences.

4. – Separability of $\overline{M}$.

From eq. (9) we easily obtain for the matrix element of process (2), to the order e^{2m+s}

$$\overline{M}^{(2m+s)} = \frac{1}{m!}\left[\frac{\beta}{2}\int_{\lambda}^{\Delta\omega}\frac{\mathrm{d}k}{k}\right]^m \langle f|\,\frac{(i)^s}{s!}\,T(L^s)|i\rangle_\lambda\,, \tag{13}$$

where

$$\beta = \frac{1}{2}\int \mathrm{d}\Omega_k |k|^2 j_\mu(k)\, j_\mu^*(k)$$

and the auxiliary quantity λ, which has been introduced consistently in the two factors of eq. (13) as the lower limit, to the permitted energies of real and virtual photons, has to become equal to zero.

Summing over m in the eq. (13), we have

$$\overline{M}^{(s)} = \sum_m \overline{M}^{(2m+s)} = \sum_m \frac{1}{m!}\left(\frac{\beta}{2}\right)^m\left(\log\frac{\Delta\omega}{\lambda}\right)^m \langle f|\,\frac{(i)^s}{s!}\,T(L^s)|i\rangle_\lambda$$

and then

$$\overline{M}^{(s)} = \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2}\langle f|\,\frac{(i)^s}{s!}\,T(L^s)|i\rangle_\lambda\,. \tag{14}$$

As eq. (14) holds for any s, by summing over s we finally have

$$\overline{M} = \langle f|\,\exp[-iA]S|i\rangle = \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2}\langle f|S|i\rangle_\lambda\,. \tag{15}$$

But, as we have shown in the preceeding Section that $\overline{M}$ is finite, the apparent divergence in λ of the eq. (15) when $\lambda\to 0$, will be cancelled by $\langle f|S|i\rangle_\lambda$ which consequently must go to zero when $\lambda\to 0$.

We can therefore write

$$\overline{M} = \left(\frac{\Delta\omega}{E}\right)^{\beta/2} M_E\,. \tag{16}$$

([3]) D. R. YENNIE and H. SUURA: *Phys. Rev.*, **105**, 1378 (1957).
([4]) D. R. YENNIE, S. C. FRAUTSCHI and H. SUURA: *Ann. of Phys.*, **13**, 379 (1961).

The eq. (16) in which the auxiliary quantity λ does not appear any longer, defines formally M_E as a finite matrix element (whithout any infra-red divergence) for the perturbative series of process (1). Of course M_E may have ultraviolet divergences that, after renormalization, will give correction factors to the lowest-order matrix element for process (1).

The cross-section for process (2), will be proportional to $|\overline{M}|^2$

$$\text{d}\sigma \propto |\overline{M}|^2 = \left(\frac{\Delta\omega}{E}\right)^{\beta} |M_E|^2 = \left(\frac{\Delta\omega}{E}\right)^{\beta} \text{d}\sigma_E \,. \tag{17}$$

Actually the proportionality factor in eq. (17) can be completely determined if we take exactly account of the relation

$$\sum_{k=1}^{\infty} \omega_k \leqslant \Delta\omega \tag{18}$$

in defining final states.

5. – Correct expression of final states in an experiment with energy resolution $\Delta\omega$.

In order to determine the normalization resulting from (18) we shall adopt the following definition for the final states:

$$|f''\rangle = \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} \text{d}x \int_{-\infty}^{\infty} \exp[i\tau(H_s - x)]\text{d}\tau \,{:}\exp[iA_c]{:}\,|f\rangle \,, \tag{19}$$

where H_s represents the Hamiltonian of the electromagnetic field and where $j_\mu(k)$ is now defined by eq. (12).

The vectors $|f''\rangle$ in (19) represent the correct final states: in fact besides the detected particles, they contain also an indefinite number of photons with a total energy not greater than $\Delta\omega$.

The states (19) are not normalized: in order to achieve that, we calculate their square modulus. We have

$$\begin{aligned} \mathcal{N} = {} & \frac{1}{4\pi^2} \int_{-\Delta\omega}^{\Delta\omega} \text{d}x \int_{-\Delta\omega}^{\Delta\omega} \text{d}x' \int_{-\infty}^{+\infty} \text{d}\tau \int_{-\infty}^{+\infty} \text{d}\tau' \langle f| \,{:}\exp[-iA_c]{:} \\ & {:}\exp[-i\tau(H_s - x)] \exp[i\tau'(H_s - x')\,{:}\exp[iA]{:}\,|f\rangle_c = \\ = {} & \frac{1}{4\pi^2} \int_{-\Delta\omega}^{\Delta\omega} \text{d}x \int_{-\Delta\omega}^{\Delta\omega} \text{d}x' \int_{-\infty}^{+\infty} \text{d}\tau \int_{-\infty}^{+\infty} \text{d}\tau' \exp[-i(\tau x + \tau' x')] \cdot \\ & \cdot \exp\left[\int \text{d}^4k\, \delta(k^2)\theta(k_0) j_\mu(k) j_\mu^*(k) \times \exp[i(\tau - \tau')k_0\right] = \frac{1}{\gamma^\beta} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta} \frac{1}{\Gamma(1+\beta)} \,, \end{aligned} \tag{20}$$

where $\gamma = e^{C} = 1.781$ is the Euler constant, β and λ were already defined above.

Now with the normalized states

$$(21) \qquad |f''\rangle = \frac{1}{\sqrt{N}} \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}\tau \cdot \exp[i\tau(H_s - x)] \colon \exp[iA_e] \colon |f\rangle$$

it is possible to define a new matrix element M:

$$(22) \qquad M = \frac{1}{\sqrt{N}} \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} \mathrm{d}x \int_{-\infty}^{+\infty} \mathrm{d}\tau \langle f| \colon \exp[-iA_e] \colon \exp[-i\tau(H_s - x)] S|i\rangle$$

and M is such that $|M|^2$ gives directly the cross-section for process (2).

By proceeding as in Sect. **3** and **4**, we find for M the expression

$$(23) \qquad M = \frac{1}{\sqrt{N}} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \frac{1}{\gamma_\beta} \frac{1}{\Gamma(1+\beta)} \langle f|S|i\rangle_\lambda$$

and taking into account eq. (20) we have

$$(24) \qquad M = \frac{1}{\gamma^{\beta/2}} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \frac{1}{\sqrt{\Gamma(1+\beta)}} \langle f|S|i\rangle_\lambda \,.$$

By the same reasoning which led to eq. (16), we can write

$$(25) \qquad M = \frac{1}{\gamma^{\beta/2}} \frac{1}{\sqrt{\Gamma(1+\beta)}} \left(\frac{\Delta\omega}{E}\right)^{\beta/2} M_E$$

so that we finally obtain for the cross-section

$$(26) \qquad \mathrm{d}\sigma = \frac{1}{\gamma^\beta} \frac{1}{\Gamma(1+\beta)} \left(\frac{\Delta\omega}{E}\right)^{\beta} \mathrm{d}\sigma_E$$

where M_E and $\mathrm{d}\sigma_E$ are defined by eq. (17).

Equation (26) is in agreement with the results obtained by ETIM, PANCHERI and TOUSCHEK [5].

As we can see comparing eqs. (25) and (16), the only difference between them is the factor

$$C = \frac{1}{\gamma^{\beta/2}} \frac{1}{\sqrt{\Gamma(1+\beta)}}$$

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[5] E. ETIM, G. PANCHERI and B. TOUSCHEK: Frascati, Internal Report LNF-66/38 (1966).

which for small values of β is $1-\pi^2\beta^2/24$, so it is very close to 1, as said before.

It can easily be shown that $(1/C^2)-1$ represents the probability that many photons, each with energy $<\Delta\omega$ combine to give a total energy loss $>\Delta\omega$.

Equation (25) obtained for the case of an experiment in which the momentum resolution is zero and only the energy resolution, $\Delta\omega\neq 0$ can be generalized if we replace the condition (18) by a more general condition which limits in some way also photon momenta: consequently eq. (15) too must be generalized. A method for doing his has been discussed in ref. (5).

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It is a pleasure to thank Prof. B. TOUSCHEK for having suggested this work as well as for this constant assistance.

RIASSUNTO

In questo lavoro si mostra come la divergenza infrarossa può essere eliminata nell'elemento di matrice purché si usino, come stati finali, stati fisici veri per un esperimento in cui siano create e distrutte particelle cariche.

Замечание об инфракрасной расходимости.

Резюме (*). — В этой статье мы показываем, как возможно устранить инфракрасную расходимость из матричного элемента, при условии, что для эксперимента, включающего уничтожение и рождение заряженных частиц, используются физически правильные конечные состояния.

(*) *Переведено редакцией.*

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