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## On the Extent to which Coherent Bremsstrahlung from Crystals can be Monochromatized.

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High-energy bremsstrahlung spectra showing several linearly polarized spikes, like those of Fig. 1 (dot-dashed curves), were first obtained at the Frascati 1 GeV electrosynchrotron by using a single crystal as a radiator <sup>(1)</sup>.

The reason of such non-Bethe-Heitler behaviour of the bremsstrahlung process is seen to be due to the restricted range of crystal recoils. As far as Laue's law is obeyed, only those recoils are permitted which are normal to a set of crystal lattice planes.

Now a further restriction of the allowed recoils can be kinematically obtained, as proposed by MOZLEY and DE WIRE <sup>(2)</sup>, by a strong collimation of the photon beam. As a result, the spikes are sharpened as shown for instance by the dashed curve of Fig. 1. Essentially monoenergetic  $\gamma$ -ray beams with a linear polarization close to 100% can in principle be obtained in this way.

Unfortunately both the divergence and the multiple scattering of primary electrons play a major role in masking the effect of collimation, so that monoenergetic bremsstrahlung was not observed till now.

As such beams can be of importance for high-energy particle physics, a work was undertaken by the author in order to investigate what degree of monochromaticity can be preserved by considering the actual experimental conditions.

Typical results, to be explained in what follows, are shown by the continuous curves of Fig. 1 and 2, as well as by the dashed curve of Fig. 2.

The following steps were accomplished:

1) Evaluation of the coherent and polarized bremsstrahlung cross-section, differential both in angle and energy. In previous works <sup>(1,3,4)</sup> cross-sections differential in energy only were given.

<sup>(1)</sup> G. BARBIELLINI, G. BOLOGNA, G. DIAMBRINI and G. P. MURTAS: *Phys. Rev. Lett.*, **8**, 454 (1962).

<sup>(2)</sup> R. F. MOZLEY and J. DE WIRE: *Nuovo Cimento*, **27**, 1281 (1963).

<sup>(3)</sup> H. ÜBERALL: *Zeits. f. Natur.*, **17**, 332 (1962).

<sup>(4)</sup> G. BOLOGNA, G. LUTZ, H. D. SCHULZ, U. TIMM and W. ZIMMERMANN: *Nuovo Cimento*, **42 B**, 844 (1966).

- 2) Folding of the differential cross-section with the electron distribution for multiple scattering and beam divergence.
- 3) Integration, extended to the angular region accepted by the collimator, of the expressions thus obtained.
- 4) Best choice of the crystal.

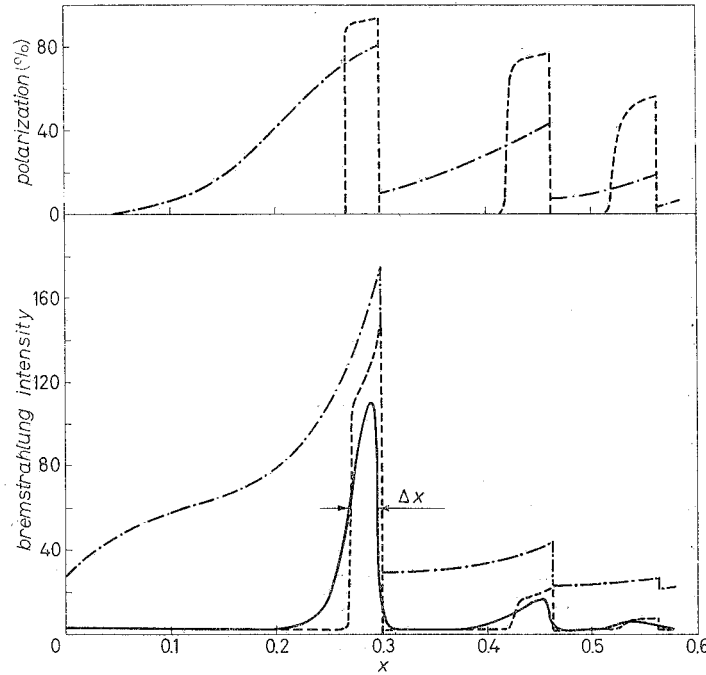


Fig. 1. - Bremsstrahlung intensity  $x(d\sigma/dx)/(Z^2\sigma_0N)$  and polarization  $[d\sigma_{\perp}^{(e)} - d\sigma_{\parallel}^{(e)}]/d\sigma$  from a single crystal of beryllium *vs.* photon fractional energy  $x = k/E_1$ .  $mc^2E_1 = 6$  GeV;  $b_1 \equiv [1\bar{2}10]$ ;  $b_2 \equiv [10\bar{1}0]$ ;  $b_3 \equiv [0001]$ ;  $\alpha = \beta = 1.5^\circ$ ;  $\theta/E_1 = 51.3$  mrad. Dot-dashed curves: no collimation, no scattering, no beam divergence. Dashed curves:  $U_0/E_1 = 3.4 \cdot 10^{-6}$  rad; no scattering; no beam divergence;  $\Delta x/x = 11\%$ . Continuous curve: same  $U_0$ ;  $T = 9.25 \cdot 10^{-3}$  g/cm<sup>2</sup>; no beam divergence; same  $\Delta x/x$ .

Several points are not considered. For instance, the influence of the source extension is not examined yet, because that makes the problem much more involved; on the other hand this effect can in a sense be simulated by suitably enlarging the collimator. Other points not considered are the technical difficulties in obtaining thin single crystals, like those needed here, and the related problem of crystal defects.

The results we present here were obtained by using Born approximation, as well as high-energy and small-angle approximation. The following units are used:  $mc^2$ , rest energy of the electron, for energies;  $mc$  for linear momenta;  $\lambda_C = h/mc$ , the electron Compton wave length, for lengths.

Let  $\mathbf{p}_1$  be the momentum of the incoming electron;  $E_1 \simeq p_1 \gg 1$  be its energy;  $E_2 \gg 1$  be the energy of the outgoing electron;  $\mathbf{k}$  be the momentum of the emitted photon,  $x = k/E_1$  its fractional energy;  $\mathbf{q}$  be the momentum transferred to the

atom,  $q_z$  and  $q_\perp$  the components of  $\mathbf{q}$  along and transverse to  $\mathbf{p}_1$ ,  $\delta$  the minimum momentum transfer;  $U$  be the emission angle of the photon (angle between  $\mathbf{k}$  and  $\mathbf{p}_1$ ) in  $1/E_1$  units ( $U/E_1 \ll 1$  radian),  $\gamma$  the angle of the planes  $(\mathbf{p}_1, \mathbf{q})$  and  $(\mathbf{p}_1, \mathbf{k})$ . We have

$$(1) \quad \begin{cases} \delta = \frac{k}{2E_1 E_2} = \frac{x}{2E_1(1-x)}, \\ q_z = \delta(1 + U^2) - 2\delta U q_\perp \cos \gamma + \frac{q_\perp^2}{2E_2}. \end{cases}$$

The last equation represents a parabolic surface in  $\mathbf{q}$  space. The relevant term on the right side of eq. (1) is the first one (the maximally contributing values of  $q$  and  $U$  for bremsstrahlung are of the order of  $10^{-2}$  and 1 respectively; furthermore we assumed  $E_2 \gg 1$ ). We have thus

$$(2) \quad q_z \simeq \delta(1 + U^2);$$

the paraboloid approximately degenerates in a plane perpendicular to  $\mathbf{p}_1$ , having a distance  $\delta(1 + U^2)$  from the origin.

If a round collimator is placed at the center of the bremsstrahlung beam,  $U$  assumes all the values between zero and a maximum value  $U_0$ ; thus the allowed kinematical region assumes the well-known shape of Überall's «pancake»<sup>(5)</sup>; this pancake, which is perpendicular to  $\mathbf{p}_1$ , has a distance  $\delta$  from the origin and a thickness  $\delta U_0^2$ .

Let now  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  be three orthogonal reference axes (in general not coincident with the basis axes of the crystal), chosen in such a way that the angle between  $\mathbf{p}_1$  and  $\mathbf{b}_1$  is  $\theta/E_0 \ll 1$  radian. This angle

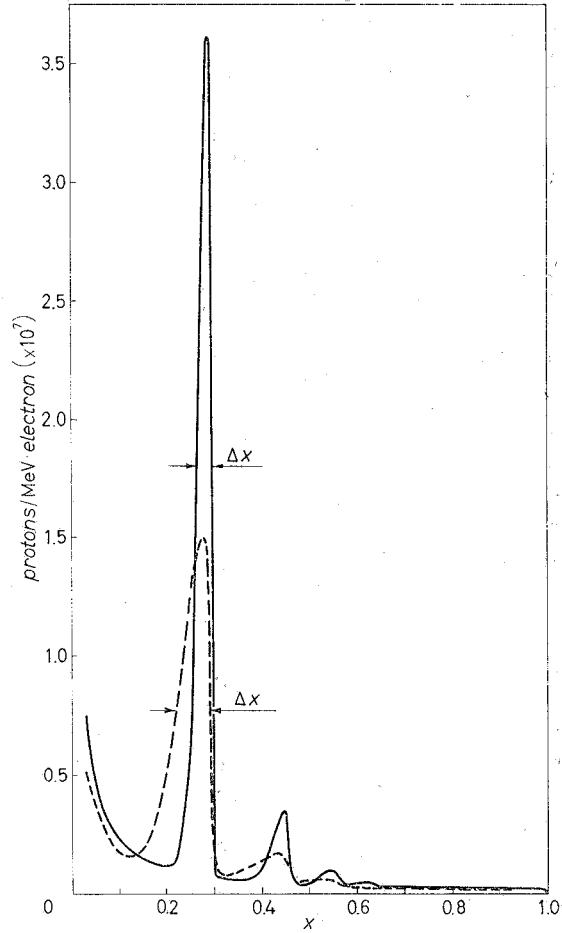


Fig. 2. — Number of photons per MeV emitted by an electron vs.  $x$ . The conditions for the continuous curve are the same as for the continuous curve of Fig. 1. Dashed curve: beam divergence  $\omega_0/E_1 = 4.3 \cdot 10^{-5}$  rad. Other conditions are unchanged ( $\omega_M(T)/E_1 = 2.1 \cdot 10^{-5}$  rad);  $\Delta x/x = 25\%$ .

<sup>(5)</sup> H. ÜBERALL: *Phys. Rev.*, **103**, 1055 (1956).

plays a fundamental role in what follows; with this choice many equations simplify.

If we take the origin of  $\mathbf{q}$  space coincident with that of  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  and consider the surface of eq. (1), we have a situation which is the exact analogous of that for coherent scattering of X-rays. If  $\mathbf{g}$  is a vector of the reciprocal lattice space, the coherent emission of a photon only takes place when  $\mathbf{q} = \mathbf{g}$  <sup>(5)</sup>, which is the same as the Laue law. With a geometrical picture we can say that the photon is emitted at the angle  $U$  when the paraboloid (1) passes through a reciprocal lattice point. This « coherence » paraboloid thus plays the same role as the Ewald sphere plays for X-ray scattering.

By calling  $g_i$  the components of  $\mathbf{g}$  along  $\mathbf{b}_i$ ,  $\alpha$  the angle of the planes  $(\mathbf{b}_1, \mathbf{b}_2)$  and  $(\mathbf{b}_1, \mathbf{p}_1)$ ,  $\beta$  the angle of  $(\mathbf{b}_1, \mathbf{b}_2)$  and  $(\mathbf{b}_1, \boldsymbol{\epsilon})$ , where  $\boldsymbol{\epsilon}$  is the direction of the photon polarization,  $\psi$  the angle of  $(\mathbf{p}_1, \mathbf{k})$  and  $(\mathbf{p}_1, \boldsymbol{\epsilon})$  (all oriented from the former to the latter plane), the condition  $\mathbf{q} = \mathbf{g}$  becomes

$$(3) \quad \begin{cases} q_z = g_1 + \frac{\theta}{E_1} (g_2 \cos \alpha + g_3 \sin \alpha), & q_{\perp}^2 = g_2^2 + g_3^2, \\ q_{\perp} \cos \gamma = -g_2 \cos (\psi - \beta) + g_3 \sin (\psi - \beta). \end{cases}$$

These are to be inserted in eq. (1) in order to obtain the emission angles  $U, \psi$  of the photon, when existing. These angles are completely determined by each reciprocal lattice point (*viz.*, by each direct lattice set of parallel planes) and by the electron and photon energies. The situation is thus completely different with respect to incoherent bremsstrahlung.

Going back to the collimated bremsstrahlung beam, we can qualitatively reproduce the behaviour of the dashed spectrum of Fig. 1, for which  $U_0 = 0.4$ , by giving the pancake a translation along  $\mathbf{p}_1$  and looking at the points of the reciprocal lattice which escape or enter the pancake. The points which give the relevant contribution are those of the plane  $(\mathbf{b}_2, \mathbf{b}_3)$  through the origin <sup>(5)</sup>, for which  $g_1 = 0$ . From eq. (2) the range  $\Delta x$  for which a lattice point is contained within the pancake is easily evaluated. The « line width » at  $x = 0.3$  in Fig. 1 is thus approximately

$$(4) \quad \frac{\Delta x}{x} = (1 - x) U_0^2 = 11\%.$$

Let us call now  $d\sigma_{\parallel}^{(i)}$  and  $d\sigma_{\perp}^{(i)}$  the interference part of the differential cross-section for bremsstrahlung with polarization  $\boldsymbol{\epsilon}$  parallel and perpendicular to a fixed reference plane (this can be either  $(\mathbf{p}_1, \boldsymbol{\epsilon})$  or  $(\mathbf{b}_1, \boldsymbol{\epsilon})$ , as  $\mathbf{p}_1$  and  $\mathbf{b}_1$  are nearly parallel). The position of the reference plane with respect to the crystal is conveniently determined by the angle  $\beta$  already defined.

In terms of  $\beta$  and  $\mathbf{g}$  the angle  $\varphi$  between the planes  $(\mathbf{p}_1, \mathbf{q})$  and  $(\mathbf{p}_1, \boldsymbol{\epsilon})$  or the planes  $(\mathbf{b}_1, \mathbf{q})$  and  $(\mathbf{b}_1, \boldsymbol{\epsilon})$  is given by

$$(5) \quad \cos 2\varphi = \frac{(g_2^2 - g_3^2) \cos 2\beta + 2g_2 g_3 \sin 2\beta}{g_2^2 + g_3^2}.$$

By considering May's differential cross-section for polarized bremsstrahlung <sup>(6)</sup>

<sup>(5)</sup> M. MAY: *Phys. Rev.*, **84**, 265 (1951).

and by introducing the crystal interference factor<sup>(5)</sup>, we have found for the cross-sections relative to the plane through the origin ( $g_1 = 0$ )

$$(6) \quad \begin{cases} d\sigma_{\perp}^{(i)} + d\sigma_{\parallel}^{(i)} = d\sigma^{(i)} = \frac{\sigma_0 N}{x} I(U, \psi, x, E_1) \delta(U^2 - W^2) dU^2 d\psi dx, \\ d\sigma_{\perp}^{(i)} - d\sigma_{\parallel}^{(i)} = \frac{\sigma_0 N}{x} I_{-}(U, \psi, x, E_1) \delta(U^2 - W^2) dU^2 d\psi dx, \end{cases}$$

where  $\sigma_0 = (e^2/\hbar c)(e^2/mc^2)^2 = 5.78 \cdot 10^{-28} \text{ cm}^2$ ;  $N$  is the total number of atoms of the crystal;  $\delta(U^2 - W^2)$  is the Dirac  $\delta$ -function of the argument  $U^2 - W^2$ , and  $W^2$  is the value of  $U^2$  obtained from eq. (1) for  $\mathbf{q} = \mathbf{g}$ , viz. by using eqs. (3). After substitution we obtain

$$(7) \quad \begin{cases} J(U, \psi, x) = \frac{I(U, \psi, x, E_1)}{E_1} = \\ \quad = [1 + (1-x)^2] D\Phi_1(U, \psi, x) - \frac{2}{3}(1-x) D^2\Phi_2(U, \psi, x), \\ J_{-}(U, \psi, x) = \frac{I_{-}(U, \psi, x, E_1)}{E_1} = 2(1-x) D\Phi_3(U, \psi, x), \end{cases}$$

where

$$D = \delta E_1 = \frac{x}{2(1-x)};$$

$J, J_{-}, \Phi_1, \Phi_2, \Phi_3$  do not depend on  $E_1$ . We have<sup>(7)</sup>

$$(8) \quad \begin{cases} \Phi_1 = \sum_g \frac{(\Phi g^2)}{Q_z^2}, \quad \Phi_2 = 12 \sum_g \Phi(g^2) \Lambda(\theta, \alpha) \Gamma^2(\psi), \\ \Phi_3 = \sum_g \Phi(g^2) \left\{ \frac{-\cos 2\psi}{Q_z^2} + 4D\Lambda(\theta, \alpha) \Gamma^2(\psi) \cos 2\psi + 2Q_z \Lambda(\theta, \alpha) \Lambda(\psi) \sin 2\psi \right\}, \end{cases}$$

where

$$(9) \quad \begin{cases} \Phi(g^2) = \frac{|F(l_1 l_2 l_3)|^2}{\pi^2 V n}, \\ F(l_1, l_2, l_3) = \sum_{j=1}^n \frac{[Z_j - f_j(g^2)]}{g} \exp \left[ -\frac{1}{2} A_j g^2 \right] \exp [2\pi i (l_1 u_{1j} + l_2 u_{2j} + l_3 u_{3j})], \\ Q_z = q_z E_1 = \theta (g_2 \cos \alpha + g_3 \sin \alpha), \quad \Lambda(\theta, \alpha) = \frac{Q_z - D}{Q_z^4 g^2}, \\ \Gamma(\psi) = q_{\perp} \cos \gamma = -g_2 \cos(\psi - \beta) + g_3 \sin(\psi - \beta), \\ \Lambda(\psi) = (g_2^2 - g_3^2) \sin(2\psi - 2\beta) + 2g_2 g_3 \cos(2\psi - 2\beta); \end{cases}$$

(7) The first of eqs. (6), (7) and the first and second of eqs. (8) were obtained independently for the particular case  $\alpha = \beta = 0$ ,  $n = 1$  also by G. BARBIELLINI (private communication).

$\cos 2\varphi$  is given by eq. (5). The summation in the  $\Phi_i$  must be performed only for those reciprocal lattice points which lie on the coherence paraboloid; i.e. for the values of  $g_2, g_3$  such that

$$(10) \quad Q_z = D(1 + U^2) - 2DUF(\psi) + \frac{g_2^2 + g_3^2}{2(1-x)}$$

The set of permitted values of  $\mathbf{g}$ , as well as the functions  $\Phi_i$  depend thus on  $U$  and  $\psi$ .

The meaning of the symbols which were not defined is as follows.  $V$  is the volume of the basis lattice cell, which contains  $n$  atoms of atomic numbers  $Z_j$  at positions  $u_{1j}, u_{2j}, u_{3j}$  ( $j = 1, \dots, n$ ) (expressed as fractions of the sides  $a_1, a_2, a_3$  of the cell). The triplet of integers  $l_1, l_2, l_3$  individuates a reciprocal lattice vector  $\mathbf{g}$ , whose components along  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  are  $g_1 = 0, g_2, g_3$ .  $f_j(g^2)$  is the atomic scattering factor of the  $j$ -th atom in the unit cell. The necessity of using an efficient model for its computation has been pointed out<sup>(8)</sup>. Extensive data of Hartree-Fock-Slater-Dirac calculations exist for free atoms in the form

$$(11) \quad f(g^2) = \sum_{i=1}^4 \alpha_i \exp[-\beta_i q^2] + c,$$

the coefficients  $\alpha_i, \beta_i$  and  $c$  being widely tabulated<sup>(9)</sup>. For atoms bound in a crystal corrections should be necessary<sup>(10)</sup>.

$A_j$  is in close relationship with the Debye-Waller thermal factor of the  $j$ -th atom. It was evaluated by ÜBERALL<sup>(5)</sup> and SCHIFF<sup>(11)</sup> for a monoatomic as well as a primitive cubic crystal. No method of calculating  $A_j$  exists for more complex cases. However it is sufficient to assume that a mean value  $A$  is appropriate; in general the thermal factor is treated as an empirical constant in X-ray crystallography; an extensive tabulation has been made<sup>(12)</sup>.

For the close analogy with X-ray scattering, we will call  $F(l_1, l_2, l_3)$  the « structure factor ». The function  $\Phi(g^2)$  depends only on the intrinsic characteristics of the crystal.

The functions  $D^2\Phi_1, D^3\Phi_2$  and  $D^2\Phi_3$  depend on the ratio  $\theta/D$ , not on  $\theta$  and  $D$  separately. The form which is given the  $\Phi_i$  is the most convenient for folding the electron angular distribution.

Now a few words about the incoherent part  $d\sigma^{(e)}$  of the differential cross section. Überall's formulation essentially holds<sup>(13)</sup>; the only improvement made was the introduction of the more accurate expression (11), which, however, is not so important for the incoherent as for the coherent part.

By adopting the approximate equation (2), integrations of eqs. (6) over  $U$  and  $\psi$  are made independent. Thus analytical integration over  $\psi$  from 0 to  $2\pi$  gives rise

<sup>(8)</sup> G. BOLOGNA, G. LUTZ, H. D. SCHULZ, U. TIMM and W. ZIMMERMANN: *Proc. of the V Int. Conf. on High-Energy Accelerators*, Frascati, September 1965, p. 567.

<sup>(9)</sup> D. T. CROMER and J. T. WABER: *Acta Cryst.*, **18**, 104 (1965).

<sup>(10)</sup> S. GÖTTLICHER and E. WÖLFEL: *Zeits. f. Elektrochimie*, **63**, 891 (1959).

<sup>(11)</sup> L. I. SCHIFF: *Phys. Rev.*, **117**, 1394 (1960).

<sup>(12)</sup> *International Tables for X-Ray Crystallography*, vol. 3, sect. 3.3.5.1 (Birmingham, 1962).

<sup>(13)</sup> H. ÜBERALL: *Phys. Rev.*, **107**, 223 (1957).

to the following functions, in place of the  $\Phi_i$  of eqs. (8):

$$(12) \quad \left\{ \begin{array}{l} \chi_1(x) = 2\pi \sum_{\mathbf{g}} \frac{\Phi(g^2)}{Q_z^2}, \quad \chi_2(x) = 12\pi \sum_{\mathbf{g}} \Phi(g^2) A(\theta, \alpha) g^2, \\ \chi_3(x) = -2\pi \sum_{\mathbf{g}} \frac{\Phi(g^2)}{Q_z^4} \cos 2\varphi. \end{array} \right.$$

By using the Dirac  $\delta$ -function, integration over  $U$  from 0 to  $U_0$  is obtained by performing the summation in eqs. (12) over values of  $\mathbf{g}$  for which

$$(13) \quad D \leq \theta(g_2 \cos \alpha + g_3 \sin \alpha) \leq D(1 + U_0^2).$$

The physical situation corresponds to a round collimator, having a half-aperture  $U_0 \ll 1$ , placed in the bremsstrahlung beam at the angle  $U=0$ . Other collimation geometries are of course possible; for instance, a small collimator placed at  $U=1$  with a suitable  $\varphi$  favours the linear polarization but reduces the number of photons collected. We systematically adopt the former collimation geometry, as the polarization can be made very high anyway by suitably choosing  $\mathbf{g}$  (4). In this case the linear polarization is

$$(14) \quad P = \frac{d\sigma_{\perp}^{(i)} - d\sigma_{\parallel}^{(i)}}{d\sigma^{(i)} + d\sigma^{(e)}},$$

as the difference of the incoherent cross-sections is vanishing. In eqs. (14) the symbols refer to cross-sections integrated over angles.

By using eq. (1) instead of (2), the integrations of coherent cross-sections over angles are no longer independent and should be performed numerically, while for incoherent cross-section integration is still analytical.

A typical result is shown in Fig. 1 by the dashed curves. One can see that the left edge of each «line» is not as sharp as the right edge. This is exactly the contribution given by the extra terms of eq. (1). If eqs. (12), (13) were used, the left corner had to fall abruptly like the right one (2). By placing  $U_0 = \infty$  in eq. (13), the dot-dashed curves of Fig. 1 are obtained.

Let us now introduce the effects of multiple electron scattering as well as those of the natural divergence of the beam. In order to do this we fold eqs. (6), (7), (8) with the normalized angular-distribution function:

$$(15) \quad \left\{ \begin{array}{l} G(\omega) = \frac{1}{\pi\omega_M^2(T)} \left\{ -Ei\left(-\frac{\omega^2}{\omega_0^2 + \omega_M^2(T)}\right) + Ei\left(-\frac{\omega^2}{\omega_0^2 + \omega_M^2(t_0)}\right) \right\}, \\ Ei(-v) = -\int_v^\infty \frac{\exp[-x]}{x} dx, \end{array} \right.$$

where  $\omega$  represents the angle, measured in units of  $1/E_1$ , between the actual electron momentum  $\mathbf{p}_1$  and the momentum  $\mathbf{p}_1^*$  of the electron flying along the axis of the beam cone;  $T$  is the thickness of the crystal, measured in  $g/cm^2$ ;

$$t_0 = \frac{M}{390Z^{\frac{1}{2}}(Z+1)},$$



where  $M$  is the atomic mass, and  $Z$  is the atomic number (we consider one kind of atoms only; generalization for more than one atomic species is obvious);

$$\omega_M^2(t) = 0.6744 \frac{Z(Z+1)}{M} t \ln \left\{ 1.531 \cdot 10^4 \frac{Z^{\frac{1}{2}}(Z+1)}{M} \sqrt{T t_0} \right\},$$

where  $\omega_M$  is a characteristic angle obtained from Molière's theory of multiple scattering<sup>(14)</sup>; finally  $\omega_0$  is the root mean square deviation of the angular distribution (assumed to be Gaussian) due to the natural divergence of the electron beam only. The electron scattering in the crystal was supposed to be incoherent; furthermore several other approximation were made in the derivation of eq. (15).

The final cross-sections are obtained by introducing the new independent variables  $\omega$ ,  $\chi$ ,  $U^*$  and  $\psi^*$  in eqs. (9), and by considering the proper differential element. For instance, the following substitutions are performed:

$$(16) \quad \begin{cases} Q_z = \theta(g_2 \cos \alpha + g_3 \sin \alpha) - \omega [g_2 \cos (\beta - \chi) + g_3 \sin (\beta - \chi)], \\ U^2 = U^{*2} + \omega^2 - 2U^* \omega \cos (\chi - \psi^*); \end{cases}$$

here  $\chi$  is the angle of the planes  $(\mathbf{p}_1^*, \mathbf{p}_1)$  and  $(\mathbf{p}_1^*, \mathbf{e})$ ;  $U^*$  and  $\psi^*$  mean  $U$  and  $\psi$  referred to  $\mathbf{p}_1^*$ . The final results are obtained by performing a triple numerical integration over  $\omega$ ,  $\chi$  and  $\psi^*$ , and by summing over those values of  $\mathbf{g}$  for which  $U^*$ , as obtained from eqs. (10) and (16), is less than  $U_0$  (in this situation the emitted photon is collected from the collimator). A similar folding is also made for the incoherent cross-section. The contribution of inelastic processes in which the atom is excited (bremsstrahlung due to atomic electrons) is of course incoherent. This is taken into account in an approximate way by adding the contribution of  $1/Z$  times the incoherent cross-section for the corresponding elastic process.

As far as the choice of the crystal is concerned, we sought among all crystals of the elements which is the one which gives the largest value of  $\Phi(g^2)/Z^2$  (see eq. (9)). The answer is that the best one is diamond; next comes beryllium, for which the maximum is only 5% smaller. On the contrary, the incoherent part  $Z^{-2} d\sigma^{(e)}/dx$  is about 15% higher than for diamond. This unfavours the ratio line height to background a bit. This ratio very unlikely is higher for crystals of compounds than for diamond and beryllium.

The numerical results we present here are for a single crystal of beryllium, which, apart from defects, is perhaps obtained in form of thin plates more easily than diamond.

As regards the choice of the orientation, two alternatives are possible according to whether high line-to-background ratio or high polarization is required. The first one is obtained when the pancake intersects the plane  $g_1 = 0$  along a high-density reciprocal lattice row. The second alternative is obtained when the intersection contains a single reciprocal lattice point<sup>(4)</sup>. The latter has the advantage that the angle  $\theta/E_1$  can be chosen as large as  $(50 \div 100)$  mrad; if a mosaic spread of some mrad exists for the axes of the crystal, its smearing-out effect on the spectral lines is not so drastic as for small  $\theta/E_1$ . For this reason we present results for the second alternative.

<sup>(14)</sup> W. T. SCOTT: *Rev. Mod. Phys.*, **35**, 231 (1963).

Beryllium crystal has a thermal factor  $A = 412$  at room temperature<sup>(12)</sup>, and a hexagonal close-packed structure. The basis cell has sides  $a_1 = a_2 = 94.02$ , at  $120^\circ$  to each other and  $a_3 = 147.44$ , normal to both  $a_1$  and  $a_2$ . A fourth redundant axis  $-(a_1 + a_2)$  is frequently used to display the hexagonal symmetry. The basis cell contains two atoms at positions  $0\ 0\ 0$  and  $\frac{2}{3}\ \frac{1}{3}\ \frac{1}{2}$ . If we assume our co-ordinate system coincident with three suitable zone axes, namely  $b_1 = [1\ \bar{2}\ 1\ 0]$ ,  $b_2 = [1\ 0\ \bar{1}\ 0]$ ,  $b_3 = [0\ 0\ 0\ 1]$ , the reciprocal lattice point  $0\ 0\ 2$ , for which  $\Phi(g^2)$  is a maximum, may be the unique point contained within the pancake. This is accomplished for instance by taking  $\alpha = 1.5^\circ$ ,  $\theta = 602$  and  $U_0 = 0.4$ , in such a way that the leading line of Fig. 1 (dashed curve) has the upper edge at  $x = 0.3$  and a full width given by eq. (4). The minor «lines» of Fig. 1 are due to reciprocal lattice points  $0\ 0\ 4$  and  $0\ 0\ 6$ .

Specifically, Fig. 1 was obtained for  $mc^2 E_1 = 6\text{ GeV}$  ( $\theta/E_1 = 51.3\text{ mrad}$ ). The figure shows both bremsstrahlung intensity  $x(d\sigma/dx)/(Z^2\sigma_0 N)$  and linear polarization given by eq. (14) for  $\beta = \alpha$ , as a function of  $x$  up to the value  $x = 0.6$ . Both the dashed and the dot-dashed curves are for no multiple scattering and no electron beam divergence, the former being obtained for  $U_0 = 0.4$ , the latter for  $U_0 = \infty$  (no collimation). Incoherent background for the former case is  $\sim 2\%$  of the line height, while polarization is  $91\%$ . By decreasing  $E_1$  as well as by maintaining  $U_0$  constant, both incoherent background and line width remain unchanged, while the height of the coherent line is proportionally decreased. For small  $E_1$ ,  $\alpha$  should be increased to make  $\theta/E_1$  sufficiently small for our approximation to hold.

The continuous curve of Fig. 1 shows the averaging effect of multiple scattering. We assumed beryllium crystal thickness  $T = 9.25 \cdot 10^{-3}\text{ g/cm}^2$  ( $\omega_M(T) = 0.24$ ) and no electron beam divergence ( $\omega_0 = 0$ ). Polarization is not evaluated yet, only to save computer time. By extrapolation the maximum value of  $P$  is argued to be  $\sim 80\%$ . In Fig. 1 the full width at half maximum is indicated as  $\Delta x$ . It is the same as for the dashed curve, but the energy distribution is of course more unfavourable and the maximum decreased.

In order to see whether such a beam may be used or not, the number of photons per MeV emitted by each electron is represented in Fig. 2 as a function of  $x$ . The continuous curve is computed in the previous conditions, while the dashed curve represents the opposite case in which angular divergence prevails on multiple scattering. We assumed  $\omega_0 = 0.5$ , other conditions being unchanged. We have  $\sqrt{\omega_0^2 + \omega_M^2(T)} = 0.56$  (see eq. (15)). Thus  $\Delta x/x = 25\%$ ; the maximum is reduced by a factor 2.5.

We could have chosen a smaller  $U_0$  in order to increase the monochromaticity. By looking at Fig. 2 we argue that this goal is made possible when the other conditions are the same as for the continuous curve, while it is impossible for the dashed curve. Investigation of more specific cases as well as exact evaluation of polarization will be postponed to another occasion.

We conclude by saying that the typical number of  $2 \cdot 10^{-7}$  photons/MeV·electron of Fig. 2 is largely sufficient for bubble chamber experiments, if a beam intensity of, say,  $10^{12}$  electron/second is available. (With this intensity, heating of the crystal by ionization loss is not a serious problem.) The noticeable reduction of the soft component of the spectrum with respect to a conventional bremsstrahlung beam as well as the high value of the polarization can be conveniently used; but we cannot hope to obtain an effective monochromaticity of a few percent with this method, when a high photon beam intensity is required.