Laboratori Nazionali di Frascati

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M. Puglisi and F. Tazzioli : A NEW TYPE OF CAVITY RESONATOR.

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A NEW TYPE OF CAVITY RESONATOR

M. PUGLISI - F. TAZZIOLI (*)

The geometrical dimensions, and particulary the external sizes of a cavity resonator, depend both upon the selected resonant wave-length and on the maximum voltage required. In this paper we describe a method which allows a remarkable reduction in the size of a resonator as compared with that of the resonators usually employed, having the same resonant wave-length and voltage.

1. – Introduction.

Among the greatest difficulties one meets when dealing with resonant cavities, are those relative to the size of the resonator with respect to the annexed apparatus. When at least one of the geometrical dimensions of the resonator is of the same order as the free-space resonant wavelength, the resonator is referred to as free. When all its dimensions are much smaller than such a wave length, the resonator is said to be loaded.

The above distinction doesn't make much sense when considering ideal resonators, i.e. made up from perfect conductors and dielectrics.

If however we consider real resonators, that distinction becomes important because the reduction in the size of the resonator obtainable by loading it has to be paid with a considerable increase in losses and with a reduction in the maximum allowable gap voltage in the resonator.

Usually a resonator is loaded by placing into it suitable electrodes that in most cases behave as capacitances. Fig. 1 *a* shows a free cylindrical resonator, while figs. 1 *b*, *c* show two types of loaded resonators that are usually derived from the former.

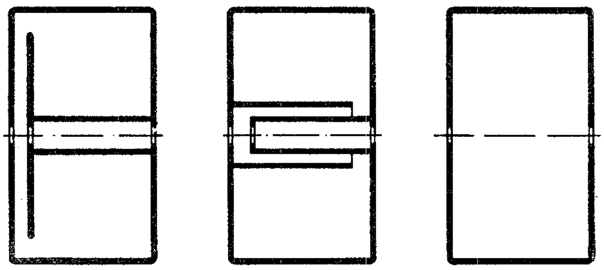

Fig. 1. — *a*) Free resonator; *b*), *c*) Loaded resonators derived from *a*).

In this paper we shall show how it is possible to transform a loaded resonator so as to lower remarkably its resonant frequency without reducing the maximum voltages allowable in the starting configuration. We shall also show how one may fix the design value of the resonant frequency in the lowest mode and in some of the upper modes.

The concept on which is based the foresaid transformation is that of lengthening as much as possible the path of electromagnetic waves in the resonator. This can be achieved by constructing a meander in the resonator, as fig. 2 shows. The cylinder or plate shaped electrodes are the boundaries of some circuit elements which,

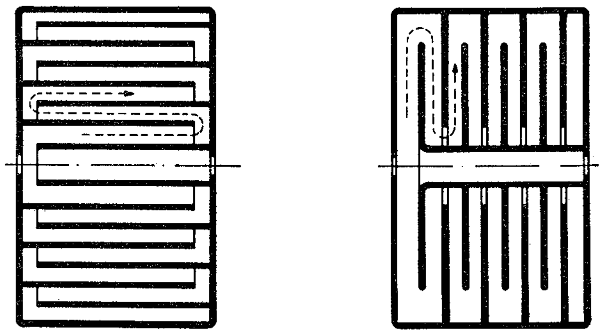

Fig. 2. — Cross sections of corrugated resonators: *a*) coaxial type, *b*) radial type. The dotted lines show the paths of waves.

under not very restrictive assumptions that will be specified, behave like coaxial or radial transmission line sections. The series of such line elements may resonate in a fundamental mode and in an infinite number of upper modes. The fundamental mode is of the TEM type.

The possibility of representing the resonator with a series of line elements in the fundamental mode depends on the geometry of such elements. For example, for a resonator to be representable as made up from coaxial line elements, the distance between two successive cylinders must be small with respect to their diameter and length and to the wave length of the fields that propagate between them.

The fringing electric field in the transition region between two successive elements may be accounted for by a shunt capacitance, that may be evaluated by graphical or analytical relaxation methods or by analog measurements, for example by the electrolytic tank. We shall see however that these capacitances modify only quantitatively the behaviour of the structure.

2. – Analytical statement of the problem.

In § 1 we have seen under which assumptions we may represent the elements that make up the cavity (coaxial cylinders or parallel discs) with line sections (respectively coaxial or radial) loaded with suitable capacitances at the ends. Under those assumptions we may therefore study the behaviour of these resonators by analyzing the circuit drawn in fig. 3.

The fundamental part of this circuit is a series of line sections, that may be different from each other,

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The last element is short-circuited at terminals (o-o) while the first one is loaded, at terminals (n-n) by the fringing capacitance of the terminals themselves (1).

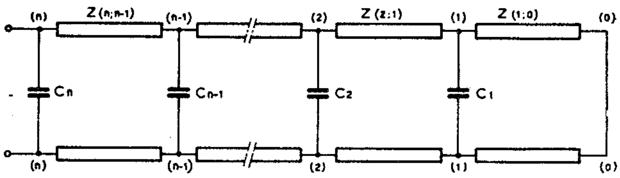

Fig. 3. — Equivalent circuit of a corrugated resonator.

The analytical study of the shown network is very complex, but it may be simplified by supposing the various elements lossless. This hypothesis is not far from reality, at least for the evaluation of the resonant frequencies of the structure, as the line elements that are employed to make a practical resonator must have a very high quality factor. A first step in the calculation is to represent the given circuit with a series of two-port cells and this may be done in various ways. For example, if we substitute each one of the capacitors C_n, C_{n-1} ... C_1, with the parallel of two capacitors having each one a value respectively of $C_n/2$, $C_{n-1}/2$ ... $C_1/2$, the m^{th} two-port of the ladder, except the terminal ones, may be represented as in fig. 4. The terminal cells are representable with the two-ports shown in figs. 5 *a* and *b*.

Let us now indicate with the symbol $[M_q]^m{}_{m-1}$, the transfer matrix relative to the general two-port shown in fig. 4 and with the symbols $[M_t]_n{}^n$ and $[M_t]_1{}^1$ respectively the transfer matrices of the two-ports shown in figs. 5 *a* and *b*. The transfer matrix $[M_T]_1{}^n$ of all the system, that is the matrix that relates terminals (n-n) and (1-1), is the product of $n+2$ matrices, as given by the expression

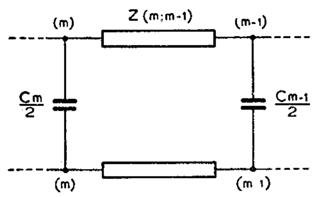

Fig. 4. — General two-port of the cascade that represents the resonator.

$$(1)\quad [M_T]_1^n = [M_t]_n^n\,[M_q]_{n-1}^n \cdots [M_q]_{m-1}^m \cdots [M_q]_1^2\,[M_t]_1^1$$

Each one of the matrices we have now considered has two rows and two columns. Therefore matrix $[M_T]_1{}^n$ also will be a square matrix of order two.

Calling T_{11}, T_{12}, T_{21}, T_{22} the elements of matrix $[M_T]_1{}^n$, the admittance Y_{nn} that appears between terminals (n-n) of the network is given by the relation

$$(2)\qquad Y_{nn} = \frac{T_{2,1}}{T_{1,1}}$$

Consequently all (and they are an infinite number) the resonant frequencies of the given network will be those satisfying the equation:

$$(3)\qquad \frac{T_{2,1}}{T_{1,1}} = 0$$

In the following paragraphs equation (3) will be discussed in detail.

3. – Resonant frequencies of coaxial cavities.

The transfer matrix of a lossless uniform line of length L and characteristic impedence Z is given by (4)

$$(4)\qquad [M_{\alpha z}] = \begin{vmatrix} \cos\alpha & j\,Z \operatorname{sen}\alpha \\ j\,\dfrac{\operatorname{sen}\alpha}{Z} & \cos\alpha \end{vmatrix}$$

where by λ and $\alpha = 2\,\pi\,(L/\lambda)$ we denote respectively the wave length and the phase shift of the fields that propagate along the line. Therefore matrix $[M_q]^m{}_{m-1}$, as defined in § 2, becomes

$$(5)\qquad [M_q]_{m-1}^n = \begin{vmatrix} 1 & 0 \\ \dfrac{j\,\omega\,C_m}{2} & 1 \end{vmatrix} \begin{vmatrix} \cos\alpha_{m,m-1} & j\,Z_{m,m-1} \operatorname{sen}\alpha_{m,m-1} \\ \dfrac{j \operatorname{sen}\alpha_{m,m-1}}{Z_{m,m-1}} & \cos\alpha_{m,m-1} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ \dfrac{J\,\omega\,C_{m-1}}{2} & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\alpha_{m,m-1} - \dfrac{\omega\,C_{m-1}\,Z_{m,m-1}}{2} \operatorname{sen}\alpha_{m,m-1} & j\,Z_{m,m-1} \operatorname{sen}\alpha_{m,m-1} \\ j\left[\dfrac{\omega\,(Cm+Cm-1}{2}\cos\alpha_{m,m-1} + \dfrac{1-(\omega^2\,C_m\,C_{m-1}\,Z^2{}_{m,m-1}/4)\operatorname{sen}\alpha_{m,m-1}}{Z_{m,m-1}}\right] & \cos\alpha_{m,m-1} - \dfrac{\omega\,C_m\,Z_{m,m-1}\operatorname{sen}\alpha_{m,m-1}}{2} \end{vmatrix}$$

We should now work out all the matrix products shown in the preceding paragraph and we would obtain a matrix which is very complicated if the line elements we consider are many.

Without impairing much the generality of the results we may suppose identical all the two-ports that make up the network. In this case matrix $]M_q]^m{}_{m-1}$ becomes independent from the indices and is given by (6)

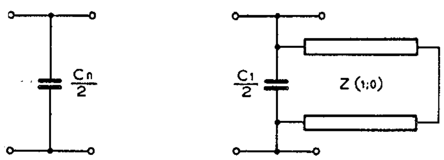

Fig. 5. — Terminating two-ports of the cascade.

$$(6)\qquad [M_q] = \begin{vmatrix} \cos\alpha - K \operatorname{sen}\alpha & j\,Z \operatorname{sen}\alpha \\ \dfrac{j}{z}\left[2\,K\cos\alpha + (1-K^2)\operatorname{sen}\alpha\right] & \cos\alpha - K\operatorname{sen}\alpha \end{vmatrix}$$

(1) Terminals (n-n) must be open in resonant cavities that are to interact with charged particles, as for example in klystrons and in particle accelerators.

where we have set $K = \dfrac{\omega C Z}{2}$

Matrices $[M_t]_n{}^n$ and $[M_t]_1{}^1$ are given by (7)

$$(7)\quad [M_t]_n^n = \begin{vmatrix} 1 & 0 \\ \dfrac{j\omega C}{2} & 1 \end{vmatrix} \qquad [M_t]_1^1 = \begin{vmatrix} 1 & 0 \\ j\,\dfrac{\dfrac{\omega C Z}{2}\operatorname{tg}\alpha - 1}{Z \operatorname{tg}\alpha} & 1 \end{vmatrix}$$

Therefore, if the equal cells connected in series are n, matrix $[M_T]_1{}^n$ results from the expansion of expression (8)

$$(8)\qquad [M_T]_1^n = [M_t]_n^n \left([M_q]\right)^n [M_t]_1^1$$

in which appears the n^{th} power of matrix $[M_q]$.

The raising of a matrix to the n^{th} power may be done by reducing the matrix to diagonal form (2). In order to do this we have to find two matrices, which we denote by $[S]$ and $[S^{-1}]$, that must satisfy identically the system of equations (9)

$$(9)\qquad \begin{aligned} &[S]\,[S^{-1}] = [1] \\ &[\lambda] = [S^{-1}]\,[M_q]\,[S] \end{aligned}$$

where by $[1]$ and $[\lambda]$ we denote, respectively, the unit matrix and a diagonal matrix whose elements are the eigenvalues of matrix $[M_q]$. Solving system (9) we find for the unknown matrices the following expressions:

$$[\lambda] = \begin{vmatrix} \cos\alpha - K\operatorname{sen}\alpha + \sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1} & 0 \\ 0 & \cos\alpha - K\operatorname{sen}\alpha - \sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1} \end{vmatrix}$$

$$(10)\qquad [S^{-1}] = \begin{vmatrix} \dfrac{1}{2} & \dfrac{j Z \operatorname{sen}\alpha}{2\sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1}} \\ \dfrac{j\sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1}}{Z\operatorname{sen}\alpha} & 1 \end{vmatrix}$$

$$[S] = \begin{vmatrix} 1 & \dfrac{-j Z \operatorname{sen}\alpha}{2\sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1}} \\ \dfrac{-j\sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1}}{Z\operatorname{sen}\alpha} & \dfrac{1}{2} \end{vmatrix}$$

It is well known that matrix $[S]$ is defined as the matrix of the eigenvectors of the matrix to be brought to diagonal from. Obviously there are as many uncertainties as in the eigen-vectors, that is to say, in our case, two uncertainties in the choice of the elements of matrix $[S]$. In order to simplify calculations we have set equal to unity element S_{11} as well as the determinant of $[S]$.

We shall begin by assuming that

$$\cos\alpha - K\operatorname{sen}\alpha < 1$$

This means that we take into account only those frequency bands in which the eigenvalues are complex (and have a modulus one) and the eigen-vectors are real. The frequency bands in which these conditions are fulfilled coincide with the frequency intervals in which propagation without attenuation is possible in the structure formed by an infinite series of two-ports like the one we are considering [2]. We shall see that the restriction now made doesn't place any important limits to the results of the theory.

Solving system (9) one obtains:

$$(11)\qquad [M_q]^n = [S]\,[\lambda]^n\,[S^{-1}] = [S]\,[\lambda_{kk}^n]\,[S^{-1}]$$

where $[\lambda^n{}_{kk}]$ is a new matrix whose elements are the n^{th} powers of the eigenvalues of $[M_q]$.

Now, as a consequence of the assumptions made on the eigenvalues of matrix $[M_q]$ we may set:

$$(12)\qquad \begin{aligned} &\cos\alpha - K\operatorname{sen}\alpha \pm j\sqrt{1 - (\cos\alpha - K\operatorname{sen}\alpha)^2} = \\ &\qquad = \cos\varphi \pm j\operatorname{sen}\varphi = e^{\pm j\varphi} \\ &\frac{\sqrt{(\cos\alpha - K\operatorname{sen}\alpha)^2 - 1}}{Z\operatorname{sen}\alpha} = jR \end{aligned}$$

The expression of $([M_q])^n$ becomes:

$$(13)\quad ([M_q])^n = \begin{vmatrix} 1 & -\dfrac{1}{2R} \\ R & \dfrac{1}{2} \end{vmatrix} \begin{vmatrix} e^{jn\varphi} & 0 \\ 0 & e^{-jn\varphi} \end{vmatrix} \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2R} \\ -R & 1 \end{vmatrix} = $$

$$= \begin{vmatrix} \cos n\varphi & \dfrac{j}{R}\operatorname{sen} n\varphi \\ jR\operatorname{sen} n\varphi & \cos n\varphi \end{vmatrix}$$

In order to obtain the transfer matrix $[M_T]_1{}^n$ we have to carry out the operations indicated by (8). Setting, for simplicity of notation

(2) For more details, see, for example [1].

$$X = \frac{\dfrac{\omega C Z}{2} \operatorname{tg} \alpha - 1}{Z \operatorname{tg} \alpha}$$

we obtain:

$$(14) \quad [M_T]_1^n = \begin{vmatrix} 1 & 0 \\ j \dfrac{\omega C}{2} & 1 \end{vmatrix} \begin{vmatrix} \cos n\varphi & \dfrac{j}{R} \operatorname{sen} n\varphi \\ jR \operatorname{sen} n\varphi & \cos n\varphi \end{vmatrix} \begin{vmatrix} 1 & 0 \\ jX & 1 \end{vmatrix} =$$

$$= \begin{vmatrix} \cos n\varphi - \dfrac{X}{R} \operatorname{sen} n\varphi & \dfrac{j}{R} \operatorname{sen} n\varphi \\ j\left[\dfrac{\omega C}{2}\left(\cos n\varphi - \dfrac{X}{R} \operatorname{sen} n\varphi\right) + X \cos n\varphi + R \operatorname{sen} n\varphi\right] & \cos n\varphi + \dfrac{\omega C}{2R} \operatorname{sen} n\varphi \end{vmatrix}$$

Therefore the admittance between terminals (n-n) is:

$$(15) \quad Y_{n,n} = j \frac{\dfrac{\omega C}{2}\left(\cos n\varphi - \dfrac{X}{R} \operatorname{sen} n\varphi\right) + X \cos n\varphi + R \operatorname{sen} n\varphi}{\cos n\varphi - \dfrac{X}{R} \operatorname{sen} n\varphi} = j\, B_{n,n}$$

The values of ω that make Y_{nn} equal to zero are the resonant frequencies we are looking for. In fig. 6 is shown a plot of the susceptance B_{nn} versus the radian frequency, for a corrugated resonator of the coaxial type, having the following parameters:

L = Length of the line elements = $9 \times 10^{-2}\, m$,

Z = characteristic impedance of the line elements = = $30\ \Omega$,

C = shunt capacitance at the junctions = $5\, pF$. Number of line elements = 7.

It is to be noted that in a resonator made according to the hypothesis on which is based (15), the distances between electrodes must grow with the radius in order to keep the characteristic impedance constant. Furthermore the fundamental resonant frequency, 95 MHz, is considerably lower than that of a plain loaded cavity having only the two innermost electrodes. (See fig. 1 *b*) of the same dimensions. As a matter of fact the latter frequency may be calculated with well known approximate methods [3], and results 245 MHz.

In § 1 we stated that the shunt capacitances at the junction between the line elements have only a quantitative effect on the behaviour of the system. In order to evaluate the effect of these capacitances we have calculated the resonant frequencies of the various modes versus the parameter $\omega_0 C Z$ that is the ratio of the characteristic impedance of the lines to the reactance of each capacitance, at the resonant frequency ω_0 pertaining to the system without capacitances. The plot is shown in fig. 7. Let us note that the parameter $\omega_0 C Z$ is directly proportional to the chracteristic impedance and varies inversely as the length of the line sections.

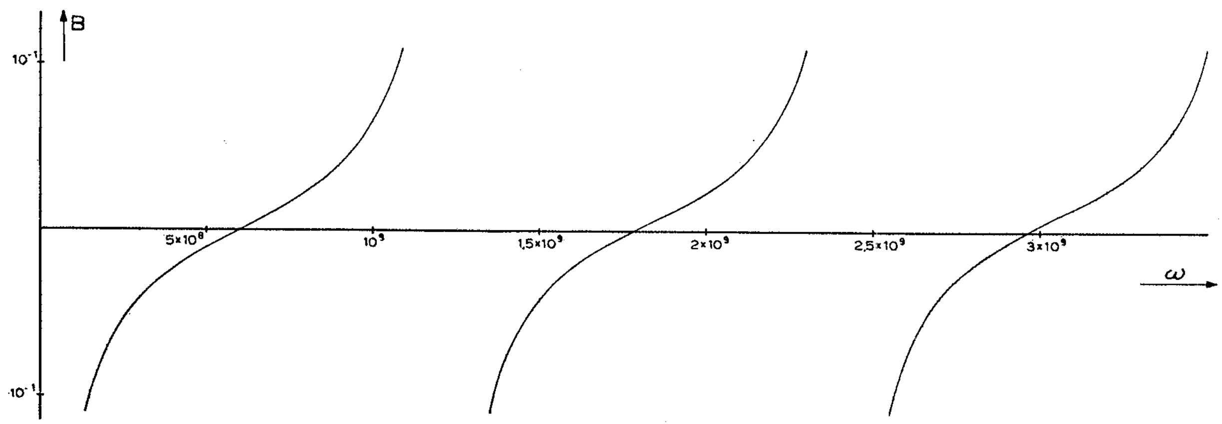

Fig. 6. — Input susceptance of the equivalent circuit of a corrugated resonator with 7 line elements.

Therefore the lesser is the distance between electrodes with respect to their length, the lesser will be the effect of the capacitances.

It is to be remarked that the effect of corrugations remains important even in the limit case of zero fringing capacitance. Because of its importance, theoretical as well as practical, we shall discuss analitically also this case, giving different values to the characteristic impedances of the line elements. This happens, for example, in practical cavities of the coaxial type, where the

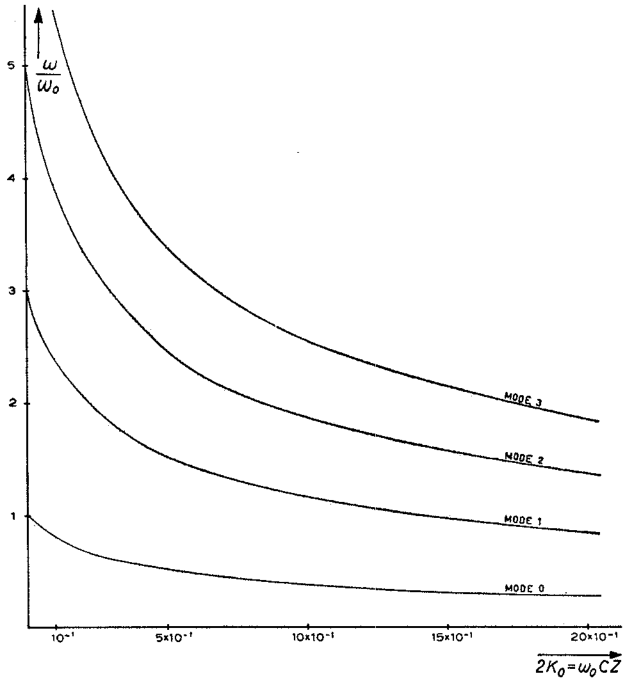

Fig. 7. — Effect of fringing capacitances on the resonant frequencies of a corrugated resonator.

electrodes are equidistant or their distance decreases towards the outer short-circuited element, since the distance between elctrodes is determined by the voltage on the electrodes themselves, and the latter decreases towards the short-circuit.

The transfer matrix of a network made up of a series of line elements short-circuited at one end and open at the other one, may be calculated in the same way as indicated by (1). In this case matrices $[M_q]$ are the transfer matrices of line elements and are formally identical with (4). Let us consider, for example, a corrugated coaxial cavity made up of seven line elements, as shown in fig. 8, where by Z_n and L_n we denote respectively

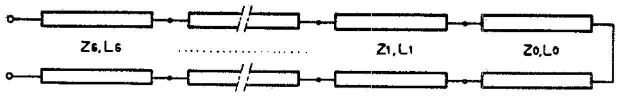

Fig. 8. — Equivalent circuit of a corrugated resonator, neglecting fringing capacitances.

the characteristic impedance and the length of the n^{th} line element.

The matrix $[M_T]_1{}^n$ pertaining to the considered circuit is obtained as shown by (16):

$$(16) \quad [M_T]_1^n = \begin{vmatrix} a_1 & j\,a_2 \\ j\,a_3 & a_4 \end{vmatrix} \begin{vmatrix} b_1 & j\,b_2 \\ j\,b_3 & b_4 \end{vmatrix} \begin{vmatrix} c_1 & j\,c_2 \\ j\,c_3 & c_4 \end{vmatrix} \cdot$$

$$\cdot \begin{vmatrix} d_1 & j\,d_2 \\ j\,d_3 & d_4 \end{vmatrix} \begin{vmatrix} e_1 & j\,e_2 \\ j\,e_3 & e_4 \end{vmatrix} \begin{vmatrix} f_1 & j\,f_2 \\ j\,f_3 & f_4 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ j\,y & 1 \end{vmatrix}$$

where j is the imaginary unit and, marking with the same subscript the parameters referring to the same line element, the meaning of the symbols is as follows:

$$(17) \quad \begin{array}{lll}
a_1 = a_4 = \cos\alpha_6 & a_2 = z_6 \operatorname{sen}\alpha_6 & a_3 = \dfrac{1}{z_6}\operatorname{sen}\alpha_6 \\
b_1 = b_4 = \cos\alpha_5 & b_2 = z_5 \operatorname{sen}\alpha_5 & b_3 = \dfrac{1}{z_5}\operatorname{sen}\alpha_5 \\
c_1 = c_4 = \cos\alpha_4 & c_2 = z_4 \operatorname{sen}\alpha_4 & c_3 = \dfrac{1}{z_4}\operatorname{sen}\alpha_4 \\
d_1 = d_4 = \cos\alpha_3 & d_2 = z_3 \operatorname{sen}\alpha_3 & d_3 = \dfrac{1}{z_3}\operatorname{sen}\alpha_3 \\
e_1 = e_4 = \cos\alpha_2 & e_2 = z_2 \operatorname{sen}\alpha_2 & e_3 = \dfrac{1}{z_2}\operatorname{sen}\alpha_2 \\
f_1 = f_4 = \cos\alpha_1 & f_2 = z_1 \operatorname{sen}\alpha_1 & f_3 = \dfrac{1}{z_1}\operatorname{sen}\alpha_1
\end{array}$$

By symbol Y we denote the input susceptance of the short-circuited line element. With the symbols adopted in fig. 9 we have:

$$(18) \qquad Y = -\frac{1}{Z_0 \operatorname{tg}\alpha_0}$$

Assuming that the line elements are of equal length and working out the operations indicated by (16) we

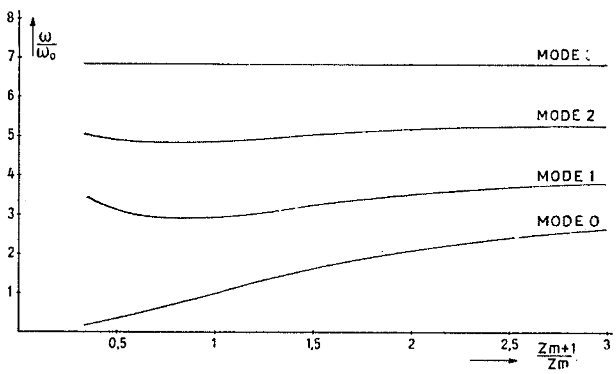

Fig. 9. — Resonant frequencies of a resonator vs. the ratio between the impedances of successive elements.

find that the admittance Y_{nn} is equal to zero when the following equation is satisfied:

$$(19) \quad Z_0 \operatorname{sen}\alpha_0 \left\{(P_3 Q_1 + P_4 Q_3) R_1 + (P_4 Q_4 - P_3 Q_2) R_3\right\} + \\ + \cos\alpha_0 \left\{(P_3 Q_1 + P_4 Q_3) R_2 - (P_4 Q_4 - P_3 Q_2) R_4\right\} = 0$$

where:

$$(20) \quad \begin{array}{ll}
P_3 = a_3 b_1 + a_4 b_3 & P_4 = a_4 b_4 - a_3 b_2 \\
Q_1 = c_1 d_1 - c_2 d_3 & Q_2 = c_1 d_2 + c_2 d_4 \\
Q_3 = c_3 d_1 + c_4 d_3 & Q_4 = c_4 d_4 - c_3 d_2 \\
R_1 = e_1 f_1 - e_2 f_3 & R_2 = e_1 f_2 + e_2 f_4 \\
R_3 = e_3 f_1 + e_4 f_3 & R_4 = e_4 f_4 - e_3 f_2
\end{array}$$

In fig. 9 is plotted the normalized value of the resonant frequencies of a corrugated cavity of the coaxial type (made up from 7 line elements) versus the ratio between the characteristic impedances of two successive elements.

The normalization frequency ω_0 is the one corresponding to characteristic impedances all equal to each other, that is to say $\frac{Z_{m+1}}{Z_m} = 1$.

We immediately remark that if we could make the value of the characteristic impedance to rise from the input terminals to the short-circui ed end, we would lower considerably the resonant frequency of the fundamental mode.

4. – Resonant frequency of corrugated cavities of the radial type.

The calculation of the resonant frequencies of corrugated cavities of the radial type may be made following the same outline as for coaxial ones. Referring to the

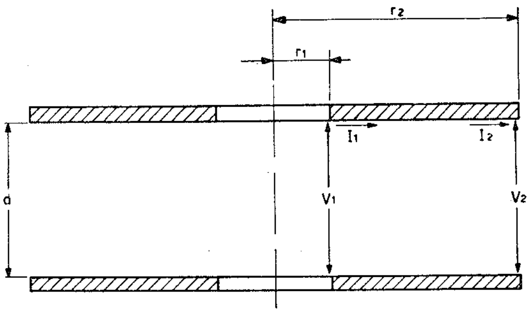

Fig. 10. — Radial line element.

notations shown in fig. 10, the elements of the transfer matrix of a radial-line element are [4]:

$$a_{11} = \frac{K \pi r_2}{2} \left[J_1 (K r_2) N_0 (K r_1) - N_1 (K r_2) J_0 (K r_1) \right]$$

$$a_{12} = \left(\frac{K \pi r_2}{2} \right) Z_0 (r_2) \left[N_0 (K r_2) J_0 (K r_1) - J_0 (K r_2) N_0 (K r_1) \right]$$

(21)

$$a_{21} = \left(\frac{K \pi r_2}{2} \right) \frac{1}{Z_0 (r_1)} \left[J_1 (K r_1) N_1 (K r_2) - N_1 (K r_1) J_1 (K r_2) \right]$$

$$a_{22} = \left(\frac{K \pi r_2}{2} \right) \frac{Z_0 (r_2)}{Z_0 (r_1)} \left[J_1 (K r_1) N_0 (K r_2) - N_1 (K r_1) J_0 (K r_2) \right]$$

where by J_0 and J_1 we denote respectively the Bessel functions of order zero and one, while by N_0 and N_1 we denote the Newman functions of order zero and one respectively.

The matrix characterized by the above elements correlates voltage V_2 and current I_2 (the latter directed as the radius r), that are assumed as known, to voltage V_1 and current I_1 (the latter also directed as the radius) taken as unknown.

The structure of the corrugated cavity of radial type is such that in two successive line elements the currents flow in opposite directions, that is the current flows alternatively from the inner radius to the outer one and vice-versa.

The elements $b_{11}, b_{12}, b_{21}, b_{22}$ that characterize the transfer matrix pertaining to the line elements in which the current flows from the outer radius to the inner one are obtained from those shown in (21) interchanging radiuses r_1 and r_2 and giving elements b_{12} and b_{21} the opposite sign of a_{12}, a_{21}.

For example, let us consider a resonant cavity of the radial type made up from 14 line elements all equal to each other. In fig. 11 is shown an axial section of the cavity in question.

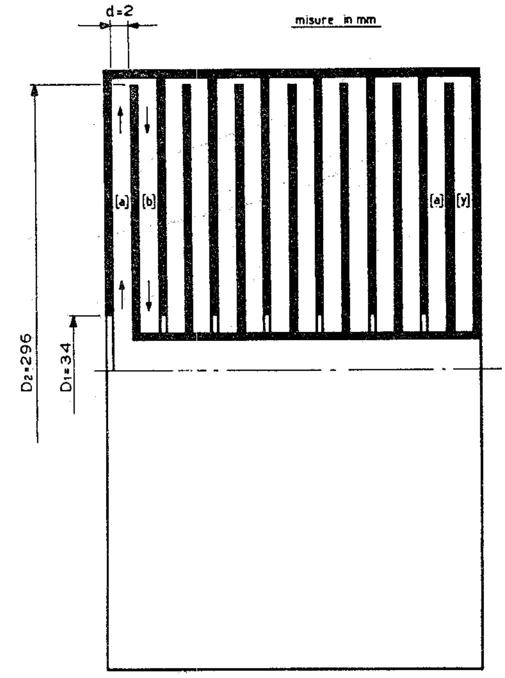

Fig. 11. — Cross section of a radial type corrugated resonator.

The transfer matrix $[M_T]_1{}^n$ has the following expression:

$$(22) \quad [M_T]_1^n = \left\{ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} \right\}^6 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} 1 & 0 \\ Y & 1 \end{vmatrix}$$

where Y is the admittance of the end short-circuited element. Working out the shown operations one obtains the elements of the overall transfer matrix and from these the value fo the admittance Y_{AB} at the input terminals.

$$(23) \quad Y_{AB} = j \frac{\left(a_{11} + a_{12} \frac{b_{22}}{b_{12}} \right) \xi_3 + \left(a_{21} - a_{22} \frac{b_{22}}{b_{12}} \right) \xi_4}{\left(a_{11} + a_{12} \frac{b_{22}}{b_{12}} \right) \xi_1 + \left(a_{21} - a_{22} \frac{b_{22}}{b_{12}} \right) \xi_2} = j B_{AB}$$

where the symbols ξ_1, ξ_2, ξ_3, ξ_4, have the following expressions

$$(24) \begin{cases} \xi_1 = A_1{}^3 - A_2 A_3 (2 A_1 + A_4) \\ \xi_2 = A_2 (A_1{}^2 - A_2 A_3 + A_1 A_4 + A_4{}^2) \\ \xi_3 = A_3 (A_1{}^2 - A_2 A_3 + A_1 A_4 + A_4{}^2) \\ \xi_4 = A_4{}^3 - A_2 A_3 (2 A_4 + A_1) \end{cases}$$

$$(24)\begin{cases} A_1 = (a_{11}\,b_{11} - a_{12}\,b_{21}) - (a_{11}\,b_{12} + a_{12}\,b_{22})\,(a_{21}\,b_{11} + a_{22}\,b_{21}) \\ A_2 = (a_{11}\,b_{12} + a_{12}\,b_{22})\,(a_{11}\,b_{11} - a_{12}\,b_{21} + a_{22}\,b_{22} - a_{21}\,b_{12}) \\ A_3 = (a_{21}\,b_{11} + a_{22}\,b_{21})\,(a_{11}\,b_{11} - a_{12}\,b_{21} + a_{22}\,b_{22} - a_{21}\,b_{12}) \\ A_4 = (a_{22}\,b_{22} - a_{21}\,b_{12}) - (a_{11}\,b_{12} + a_{12}\,b_{22})\,(a_{21}\,b_{11} + a_{22}\,b_{21}) \end{cases}$$

In fig. 12 is shown a plot of the susceptance B_{AB} versus ω. As already stated, the resonant frequencies are those for which the input admittance is equal to zero. From the above plot it appears that the ratios among the

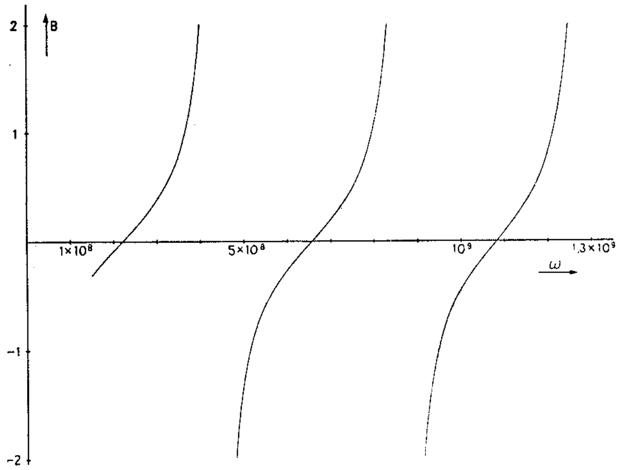

Fig. 12. — Input susceptance of a radial type corrugated resonator.

resonant frequencies are odd integers, that is all the cavity has a behaviour like that of a uniform line short-circuited at one end. This behaviour is explainable if we note that all the line elements are equal to each other.

Let us furthermore remark that the fundamental resonant frequency, 35 MHz, is much lower than that of a resonator having the same dimensions, but only the first capacitive disc. In fact the resonant frequency of such a resonator, evaluated with well known approximat methods [3], is 170 MHz.

In these calculations we have neglected the fringing capacitances at the junctions between adjacent elements as the effect of these capacitances is similar to that for coaxial cavities.

5. – Experimental checks.

In order to check the theoretical results, we have measured the resonant frequencies of some corrugated cavities, made on purpose so as to evidence the particular behaviours foreseen by the calculations. Here follows a list of some of these measurements:

a) Coaxial cavity made up from seven line elements of constant characteristic impedance equal to 25 Ω, each one 145 mm long; the outer diameter of the cavity is 307 mm. In fig. 13 is shown a photograph of the resonator.

The following resonant frequencies have been measured:

65,22 MHz; 175,15 MHz; 289,70 MHz; 401,20 MHz.

The frequencies calculated neglecting the fringing capacitances are:

73 MHz; 219 MHz; 365 MHz ...

The measured frequencies are lower than the calculated ones because of fringing capacitances.

b) Coaxial corrugated cavity made up from 3 line elements of constant impedance 25 Ω, each 400 mm long; the outer diameter of the cavity is 65 mm. In fig. 12 is shown a photo of the resonator.

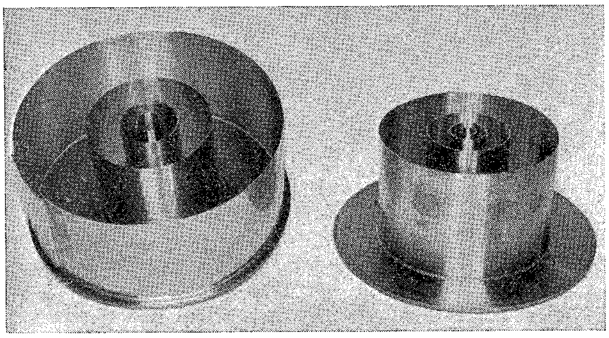

Fig. 13. — Coaxial type resonator with constant impedance line elements. The two parts are interpenetrating.

The fundamental frequency measured is 58,5 MHz, while the calculated one, neglecting fringing capacitances, is 62,5 MHz. In this case the greater length of the elements lessens the effect of the fringing capacitances with respect to the preceding case.

c) Coaxial corrugated cavity made up from 13 elements whose characteristic impedances decrease towards the outer short-circuit envelope. The ratio between

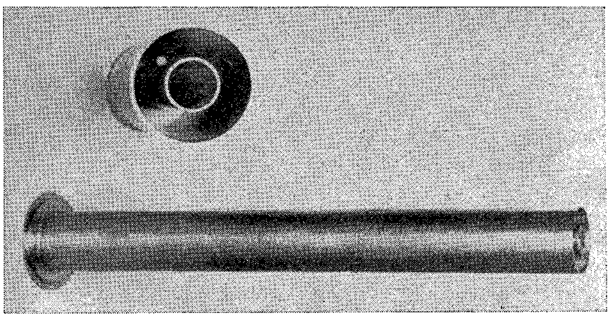

Fig. 14. — Long coaxial type resonator made to minimize the effect of fringing capacitances. The two parts are interpenetrating.

two successive impedances is $\dfrac{Z_{m+1}}{Z_m} = 1{,}4$. The length of each element is 80 mm. The outer envelope of the cavity has a diameter of 340 mm. In fig. 15 is shown a photo of the two interpenetrating parts of the resonator.

The measured frequencies are:

110 MHz; 217 MHz; 340 MHz.

The fundamental resonant frequency of a uniform coaxial line having a length equal to the sum of the lengths of the various elements is 70 MHz. Taking into account the correction factor that may be found in the diagram of fig. 9 for a ratio $\dfrac{Z_{m+1}}{Z_m} = 1{,}4$, the modified frequency results $70 \times 1.5 = 105$ MHz.

d) Radial corrugated cavity made up from 14 elements. Fig. 16 is a photo of the cavity. The distance between two successive discs is 2 mm. The outer radius of the discs is 148 mm, the inner one is 17 mm.

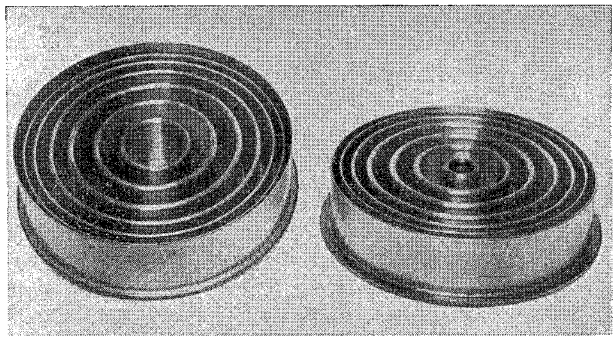

Fig. 15. — Coaxial type resonator with varying impedance of the line elements. The two parts are interpenetrating.

The measured frequencies are:

28,3 MHz; 86,06 MHz; 142,15 MHz; 196,15 MHz

while the calculated ones are:

35 MHz; 104 MHz; 172 MHz; 234 MHz.

The difference of 20% between each couple of values is due to the fact that in the measured cavity the thickness of the discs is of the same order as the distance between them and therefore the effect of the fringing capacitances is quite relevant.

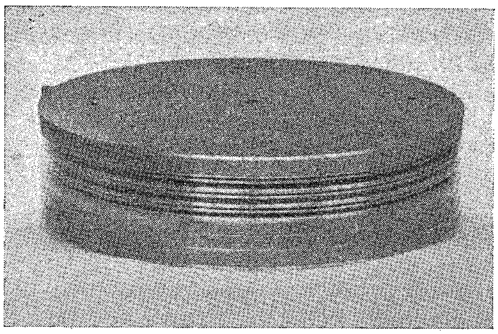

Fig. 16. — Radial type resonator. A part of the outer envelope has been removed to show the discs that form the corrugations.

6. – Conclusions.

The experimental checks we have reported in the preceding paragraph show that the worked out theory is applicable without too restrictive assumptions on the shape and dimensions of the electrodes that make up the corrugations. As we have seen, the treatment of the most general case, that is the one in which the line sections that make up the cavity have different characteristic impedances, is rather cumbersome. However from the numerical results of the treatment of various particular cases it is possible to argue that in most cases the fundamental resonant wavelength of a coaxial or radial corrugated resonator is very nearly equal to four times the overall length of the elements that form the resonator, if the characteristic impedances of the various elements are not too different from each-other. When such impedances grow or decrease uniformly from the input terminal to the short-circuit, the plot of fig. 9 shows the magnitude of the correction to be made on the resonant wavelength calculated as above.

We may therefore conclude by saying that the analytical treatment we have made gives a very simple criterion for the evaluation of the resonant frequencies of corrugated resonators.

In all the treatment till now we have always referred to ideal resonators, bounded by perfect conductors. However the results are valid also for real resonators provided that the quality factor is high enough.

Corrugated resonators may be employed in all the fields in which common resonators are, and have the advantage of a much smaller size at the same resonant frequency and under the same voltage. Against this technical and economical advantage is to be set the disadvantage of a greater power consumption, under the same performance, due to the lower input resistance of the corrugated resonator with respect to a common one. The low value of the input resistance of the corrugated resonator is due to the extent of its inner surface and may have more or less importance according to the use of the resonator.

At present super-conducting cavities are being studied, whose inner walls are coated with a thin layer of special alloys, that at very low temperature show the phenomenon of super-conductivity even under high magnetic fields. Such resonators may attain, at liquid Helium temperature, quality factors of the order of 10^8 and very high input impedances. An application of such cavities in particle accelerators is now under consideration. Corrugated resonators seem to be particularly suited for such an application as their small size facilitates their cooling while, in super-conducting conditions, their shunt impedance is always very high.

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