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Effect of the Coherent Radiation on the Phase Distribution of a Relativistic Bunch.

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Summary. — The particles of a bunch circulating in an accelerator are submitted to a force originating from the coherent radiation. It is of interest to determine the effect of this force on the particle synchrotron motion. We do this from the point of view of the phase distribution function. We show that the coherent radiation gives rise to an increase of the longitudinal dimension of the bunch.

1. – We shall denote by $\eta = \varphi - \varphi_s$ the phase difference (referred to the radiofrequency, RF) between a generic particle and a synchronous one.

If we neglect the damping due to incoherent radiation and introduce the force due to coherent radiation, then the equation of synchrotron motion is

$$\ddot{\eta} = -\Omega^2 \eta + \frac{cq\alpha\omega_s}{E_s}(F - F_s)\,, \tag{1}$$

where

$$\Omega^2 = \frac{(eV\sin\varphi_s)q\alpha\omega_s^2}{2\pi E_s}\,; \tag{2}$$

F, F_s are the coherent radiation forces acting upon a generic particle and a synchronous one respectively; they are negative quantities, because the forces act in the direction opposite to the motion of the bunch; $(eV\sin\varphi_s)$ is the energy supplied by the RF to the synchronous particle; q is the harmonic number of the RF; α is the momentum compaction; ω_s is the revolution frequency; E_s is the synchronous energy.

2. – For the force we assume the expression given by Iogansen and Rabinovich (1) which is based upon the following hypotheses:

a) the electromagnetic fields are the Liénard-Wiechert ones, *i.e.* there is no shielding;

b) the bunch is extremely thin;

c) the effective angular dimension η_0 of the bunch and the energy of the particles are such as to satisfy the relation $\gamma^{-3} \ll \eta_0 \ll 1$.

Hence the force depends on the particle angular position referred to a certain origin (see formula (14) in (1)).

If we take as origin the phase φ_s of the synchronous particle (obviously, doing that we neglect possible effects of the coherent radiation on the motion of the origin itself), the force can be written in the form

$$F = -\frac{2}{3^{\frac{1}{3}}}\frac{Ne^2}{R^2}q^{\frac{4}{3}}\frac{\mathrm{d}}{\mathrm{d}\eta}\int\limits_0^\infty \varrho(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi\,, \tag{3}$$

where N is the number of particles in the bunch; e is the electron charge; R is the machine radius; $\varrho(\eta)$ is the phase distribution function.

The force acting upon the synchronous particle will be

$$F_s = -\frac{2}{3^{\frac{1}{3}}}\frac{Ne^2}{R^2}q^{\frac{4}{3}}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\int\limits_0^\infty \varrho\,(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi\right]_{\eta=0}. \tag{4}$$

It follows that eq. (1) can be written in the form

$$\ddot{\eta} = -\Omega^2\eta - \lambda^2\frac{\mathrm{d}}{\mathrm{d}\eta}\int\limits_0^\infty \varrho(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi + \lambda^2\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\int\limits_0^\infty \varrho(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi\right]_{\eta=0} \tag{5}$$

with

$$\lambda^2 = \frac{cq\alpha\omega_s}{E_s}\frac{2}{3^{\frac{1}{3}}}\frac{Ne^2}{R^2}q^{\frac{4}{3}}\,. \tag{6}$$

3. – Let us introduce the distribution function $f(\eta, \dot{\eta}, t)$ obeying the equation

$$\frac{\partial f}{\partial t} + \dot{\eta}\frac{\partial f}{\partial \eta} + \frac{\partial}{\partial \dot{\eta}}(\ddot{\eta} f) = 0\,. \tag{7}$$

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(1) L. V. Iogansen and M. S. Rabinovich: *Sov. Phys. JETP*, **37** (10), 83 (1960).

As the radiation damping has been neglected, there are not velocity-dependent forces. Then $\partial\ddot{\eta}/\partial\dot{\eta} = 0$, and the last equation transforms simply into the following one:

$$\frac{\partial f}{\partial t} + \dot{\eta}\frac{\partial f}{\partial \eta} + \ddot{\eta}\frac{\partial f}{\partial \dot{\eta}} = 0\,, \tag{8}$$

where $\ddot{\eta}$ is to be substituted by the expression (5).

We seek a stationary solution ($\partial f/\partial t = 0$) of eq. (8). By separating the variables, *i.e.* by putting

$$f(\eta, \dot{\eta}) = \varrho(\eta)w(\dot{\eta}),$$

one obtains two equations, one for the « momenta »

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\dot{\eta}} = -a^2\dot{\eta} \tag{9}$$

and the other one for the phases

$$\frac{\mathrm{d}\ln \varrho}{\mathrm{d}\eta} = a^2\ddot{\eta}\,, \tag{10}$$

with a^2 the separation constant.

From eq. (9) we obtain the « momenta » distribution $w = \overline{w}\exp\left[-\frac{1}{2}a^2\dot{\eta}^2\right]$, which is not influenced by the introduction of the new term of force.

By integrating eq. (10), taking account of (5), we obtain

$$\ln\varrho = -a^2\Omega^2\left\{\frac{\eta^2}{2} + \frac{\lambda^2}{\Omega^2}\int\limits_0^\infty \varrho(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi - \frac{\lambda^2}{\Omega^2}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\int\limits_0^\infty \varrho(\eta-\xi)\xi^{-\frac{1}{3}}\mathrm{d}\xi\right]_{\eta=0}\eta\right\} + C\,. \tag{11}$$

The solution of this equation gives the phase distribution function, if the coherent radiation is taken into account.

The applicability of eq. (11) to the case of real accelerators is limited essentially by the first of the hypotheses assumed in Sect. 2, *i.e.* by the fact that the actual shielding existing in the accelerators reduces (as demonstrated by NODVICK and SAXON (²)) the power irradiated by a bunch of particles. Hence there is also a decrease of the force acting upon the bunch.

Therefore the results, given by eq. (11), represent an upper limit for the coherent radiation effect on the phase distribution.

We give here a possible way to evaluate the order of magnitude of such a

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(²) J. S. NODOVICK and D. S. SAXON: *Phys. Rev.*, **96**, 180 (1954).

reduction of the effect. Let us assume that in the case of finite shielding the coherent radiation force decreases with respect to the no-shielding case in the same way as does the radiated power in the corresponding situations. Therefore we admit that there are no appreciable variations in the character of the force, in other words no spectral distortions.

Then if we employ in our calculations the formulae for the radiated power referring to a uniform distribution (see expressions (23), (24), (26) in ([2])) and express the radiated power P_{coh} (finite shielding) in terms of $P_{\text{coh}}^{(0)}$ (no shielding), we find the reduction factor

$$f = (6\eta_0/q)^{-\frac{2}{3}} \left\{ \frac{\sqrt{3}\,a}{2R} + \frac{32}{\pi} \left[\exp[-2\pi\delta_1] + \exp[-2\pi\delta_2] \right] S\left(\frac{2\pi R \eta_0}{qa} \right) \right\}, \tag{12}$$

where the new symbols are: a the distance separating the shielding plates,

$$\delta_1 = \frac{R - R_1}{a}, \qquad \delta_2 = \frac{R_2 - R}{a},$$

with R_1 inner radius and R_2 outer radius of the plates ($R_1 < R < R_2$);

$$S(y) = \tfrac{1}{2}[C + \ln y - \mathrm{Ci}(y)]$$

with C Euler's constant and $\mathrm{Ci}(y)$ the cosine integral.

The factor f is defined by the equality

$$P_{\text{coh}} = f P_{\text{coh}}^{(0)}.$$

(For example, we obtain for the storage ring Adone, with $\eta_0 = 5 \cdot 10^{-2}$, a reduction factor $f \approx \frac{1}{3}$.)

4. – We can derive an approximated solution of eq. (11) in the hypothesis that the perturbation method holds.

Then the zeroth-order solution is

$$\varrho_0 = \bar{\varrho}_0 \exp[-\tfrac{1}{2} a^2 \Omega^2 \eta^2]. \tag{13}$$

After defining the mean-square deviation of η, η_0 and by taking account of the normalization condition for ϱ_0, we obtain

$$\varrho_0 = \frac{1}{\sqrt{2\pi}\,\eta_0} \exp\left[-\frac{1}{2} \frac{\eta^2}{\eta_0^2} \right]. \tag{14}$$

By substituting (14) into the right-hand side of eq. (11), we derive the first-order solution

$$\varrho_1 = \bar{\varrho}_1 \exp\left\{-\frac{1}{\eta_0^2}\left[\frac{\eta^2}{2} + \frac{\lambda^2}{\Omega^2}\big(I(\eta) - I'_0\eta\big)\right]\right\}, \tag{15}$$

where

$$I(\eta) \equiv \frac{1}{\sqrt{2\pi}\,\eta_0}\int_0^\infty \exp\left[-\frac{(\eta-\xi)^2}{2\eta_0^2}\right]\xi^{-1/3}\,\mathrm{d}\xi = \tag{16}$$

$$= \frac{1}{\sqrt{2\pi}}\,\eta_0^{-1/3}\exp\left[-\frac{\eta^2}{4\eta_0^2}\right]\Gamma\!\left(\frac{2}{3}\right)D_{-\frac{2}{3}}\!\left(-\frac{\eta}{\eta_0}\right)$$

($D_{-\frac{2}{3}}$ is a parabolic cylinder function; see, for example (³));

$$I'_0 \equiv \left[\frac{\mathrm{d}}{\mathrm{d}\eta}\,I(\eta)\right]_{\eta=0} = \frac{\Gamma^{(\frac{5}{9})}}{2^{\frac{4}{3}}\Gamma^{(\frac{4}{3})}}\,\eta_0^{-\frac{4}{3}}. \tag{17}$$

The constant $\bar{\varrho}_1$ is obtained from the normalization of ϱ_1.

The approximated behaviour of ϱ_1 can be simply derived. Expressing the parabolic cylinder function in (16) in terms of confluent hypergeometric functions we have for $I(\eta)$:

$$I(\eta) = 2^{-\frac{5}{6}}\Gamma\!\left(\frac{2}{3}\right)\eta_0^{-\frac{1}{3}}\exp\left[-\frac{\eta^2}{2\eta_0^2}\right]\left\{\frac{1}{\Gamma\frac{5}{6}}\,\Phi\!\left(\frac{1}{3},\frac{1}{2};\frac{1}{2}\,\frac{\eta^2}{\eta_0^2}\right) + \frac{\sqrt{2}\,\eta}{\Gamma(\frac{1}{3})\eta_0}\,\Phi\!\left(\frac{5}{6},\frac{3}{2};\frac{1}{2}\,\frac{\eta^2}{\eta_0^2}\right)\right\}.$$

From this expression it is clear that for large and small values of η the function ϱ_1 is well-approximated by a Gaussian function. As a matter of fact, for values $\eta^2/2\eta_0^2 \gg 1$ (say, $\geqslant 10$, *i.e.* for $|\eta| \leqslant 4.5\eta_0$) the asymptotic behaviour of the functions leads to a dependence $I(\eta) \sim (\eta/\sqrt{2}\eta_0)^{-\frac{1}{3}}$.

Then in expression (15) the term $\sim\eta^2$ will be the predominant one.

On the other hand, for values $\eta^2/2\eta_0^2 \ll 1$ (say, $\leqslant 1/10$, *i.e.* for $|\eta| \leqslant 0.45\eta_0$) we can expand in series both the exponential and the Φ-functions. If we neglect in this expansion terms of order higher than $\sim\eta^2$, we obtain for $(I(\eta) - I'_0\eta)$, which appears in (15), a constant (which may be incorporated in the normalization constant $\bar{\varrho}_1$) plus a quadratic term, $\sim\eta^2$.

In conclusion one obtains

$$\varrho_1 \sim \exp\left[-\frac{\eta^2}{2\eta_0^2}\left[1 - \frac{\lambda^2}{\Omega^2}\,\frac{2^{-\frac{5}{6}}}{3}\,\frac{\Gamma^{(\frac{2}{3})}}{\Gamma^{(\frac{5}{6})}}\,\eta_0^{-\frac{7}{3}}\right]\right]. \tag{18}$$

(³) *Higher Transcendental Functions*, vol. 2 (New York, 1955).

The exact behaviour of the distribution ϱ_1 shows that for the intermediate values of η the deviations of ϱ_1 from a Gaussian distribution are small. As examples, in Fig. 1 and 2 we have represented the distribution functions ϱ_0 (curves *a*) and ϱ_1 (curves *b*) with the Adone's parameters in two cases: $\eta_0 = 6\cdot10^{-2}$; $\eta_0 = 7\cdot10^{-2}$.

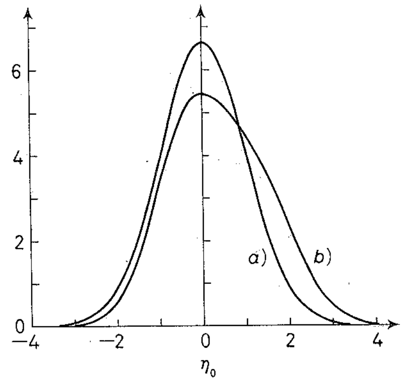

Fig. 1. – $\eta_0 = 6\cdot10^{-2}$.

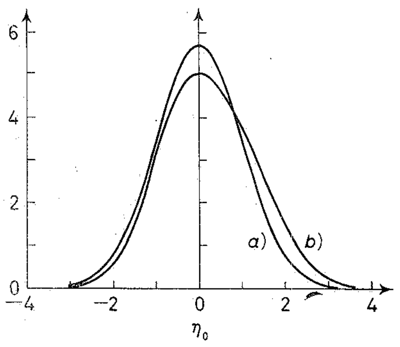

Fig. 2. – $\eta_0 = 7\cdot10^{-2}$.

Then we assume for evaluating purposes that ϱ_1 is again a Gaussian distribution, with a mean-square deviation, following from (18),

$$\eta_1 \simeq \eta_0\left(1 + \frac{\lambda^2}{\Omega^2}\,\frac{2^{-\frac{11}{6}}}{3}\,\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})}\,\eta_0^{-\frac{7}{3}}\right). \tag{19}$$

From the above equation one can conclude that the coherent radiation causes an increase of the longitudinal dimension of the bunch.

By making explicit the second term within parentheses in (19) and by observing that the numerical factor is $\simeq 1$, the mean-square deviation of the phase distribution becomes

$$\eta_1 \simeq \eta_0\left[1 + q^{\frac{4}{3}}\eta_0^{-\frac{7}{3}}\,\frac{Ne^2}{R(eV\sin\varphi_s)}\right]. \tag{19'}$$

Obviously this formula is applicable if the perturbation method is consistent. Then the second term in brackets is to be a small perturbation term (say, $\leqslant 1/10$). With assigned values of the machine parameters, formula (19′) is applicable for η_0 values such that

$$\eta_0^{\frac{7}{3}} \gg q^{\frac{4}{3}}\,\frac{Ne^2}{R(eV\sin\varphi_s)}\,.$$

The actual limitation on the unperturbed length of the bunch is obtained from the above expression by taking account of the reduction factor f. Eventually we have the condition

$$\eta_0^{\frac{7}{3}} \gg fq^{\frac{4}{3}} \frac{Ne^2}{R(eV \sin\varphi_s)}\,,$$

or

$$\eta_0^3 \gg 0.3q^2 \frac{Ne^2}{R(eV\sin\varphi_s)}\left\{\frac{\sqrt{3}\,a}{2R} + \frac{32}{\pi}\left[\exp\left[-2\pi\delta_1\right] + \exp\left[-2\pi\delta_2\right]\right] S\left(\frac{2\pi R\eta_0}{qa}\right)\right\}. \tag{20}$$

It must be stressed again that for the validity of the perturbation method the coherent radiation effect in any case must be smaller than ~10 %. In spite of this limitation the interest of the calculation is based essentially on two considerations:

a) the values of η_0 satisfying condition (20) lie in the operating region of actual accelerators. For example, in the case of Adone we must have $\sim\eta_0 \geqslant 5\cdot 10^{-2}$ (or the length of the bunch larger than ~50 cm);

b) with smaller values of η_0, when condition (20) is no longer satisfied (as in Adone in an intermediate-energy region), the perturbation method is not valid, however the equilibrium dimension of the bunch cannot exceed, as order of magnitude, the one allowed by our approximation (19′).

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I wish to thank F. AMMAN, M. BASSETTI and C. PELLEGRINI for helpful discussions.

RIASSUNTO

Le particelle di un fascio circolante in un acceleratore sono sottoposte ad una forza dovuta all'irraggiamento coerente. In questo lavoro si determina l'effetto di questa forza sul moto di sincrotrone di una particella, dal punto di vista della funzione di distribuzione delle fasi. Mostriamo che l'irraggiamento coerente porta ad un aumento della dimensione longitudinale del fascio.

Влияние когерентного излучения на фазовое распределение релятивистского сгустка.

Резюме (*). — Частицы сгустка, циркулирующего в ускорителе подвергаются действию силы, обусловленной когерентным излучением. Интересно определить влияние этой силы на движение частиц в синхротроне. Мы делаем это с точки зрения функции фазового распределения. Мы показываем, что когерентное излучение приводит к увеличению продольного размера.

(*) *Переведено редакцией.*