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M. Bassetti and R. M. Buonanni : AN IMPROVED NEWTON-RAPHSON METHOD. -

(Nota interna: n. 346)

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We describe in the present work an improvement of Newton-Raphson method, for the numerical solution of n simultaneous (generally non linear) equations in n unknowns.

By means of such an improvement one can choose the initial point in a more extended neighborhood.

1. -

Let

$$f_i(x_1, x_2, \ldots\ldots, x_n) = 0 \qquad i = 1, 2 \ldots, n \tag{1}$$

be the system of n equations in n unknowns to be solved.

Let $P(x_1, x_2 \ldots, x_n)$ be a point in E_n - euclidean n dimensional space.

We define the functions of P:

$$t(P) = \left[\sum_{i=1}^{n} f_i^2(P) \right]^{1/2}, \tag{2}$$

and:

$$r_i(P) = f_i(P)/t(P) \qquad i = 1, 2, \ldots\ldots, n \tag{3}$$

Let P_o be a trial initial point. We consider the equations:

(4) $$f_i(P) = r_i(P_o) \quad t \qquad i = 1, 2, \ldots\ldots, n$$

whose solution, as t varies, describes a curve, passing through P_o for $t = t(P_o)$.

Different initial points will generally give different curves.

We call T the family of all these curves. They have the following three properties:

a) - Let $\overrightarrow{\Delta P}_N$ $(\Delta x_{1N}, \Delta x_{2N}, \ldots \Delta x_{nN})$ be the vector solution of the Newton-Raphson method system:

$$\sum_j \frac{\partial f_i}{\partial x_j} \Delta x_{jN} + f_i = 0 \qquad i = 1, 2, \ldots\ldots, n$$

In each point P which is not a solution and the jacobian:

$$J = \frac{\partial (f_1 \ldots\ldots f_n)}{\partial (x_1 \ldots\ldots x_n)}$$

is not vanishing, $\overrightarrow{\Delta P}_N$ is tangent to the curve of T through P.

Proof.

In order to go from P to P + ΔP along curves (4) in the sense of decreasing t every f_i must decrease by the quantity $f_i\,\delta$ with infinitesimal δ.

Neglecting 2nd order infinitesimals, we can write:

(5) $$\sum_j \frac{\partial f_i}{\partial x_j} \Delta x_j = - f_i \delta \qquad i = 1, 2, \ldots, n$$

for $\delta = 1$ (5) yields the iterative formula of Newton-Raphson method.

Therefore one can write the solution of (5) as:

(6) $$\overrightarrow{\Delta P} = \delta \cdot \overrightarrow{\Delta P}_N$$

As δ tends to zero, ΔP becomes tangent to the curves (4) and the computation to first order becomes exact.

Thus it follows from (6) that $\overrightarrow{\Delta P}_N$ is tangent to the curve of family T through P.

b) - For the points of curves (4) where the t parameter is stationary, we have J = 0.

Viceversa if J is of rank n - 1 t is stationary.

Proof.

If t is stationary we deduce from (4) that the f_i's must also be stationary and therefore:

$$\sum_j \left(\frac{\partial f_i}{\partial x_j}\right) \Delta x_j = 0 \qquad i = 1, 2, \ldots, n \tag{7}$$

and for $\vec{\Delta P} \neq 0$ the above equation is compatible only if $J = 0$.

Viceversa if J is of rank n - 1 there is a $\vec{\Delta x}$ for which the (7) holds. Thus the increments of the f_i, in the direction defined by $\vec{\Delta x}$, are vanishing.

Therefore the

$$\frac{f_p}{f_q} \qquad p, q = 1, 2, \ldots, n \quad p \neq q$$

are constant: then from (4) we deduce that $\vec{\Delta x}$ is tangent to curve t and since the f_i are stationary, we deduce that also the t is stationary.

c) - An infinite number of curves (4) goes through any point solution of system (1).

Proof.

For a point close to the solution (we call X this solution point) we can write

$$f_i = \sum_j \left(\frac{\partial f_i}{\partial x_j}\right)_X \Delta x_j \tag{8}$$

where $\vec{\Delta P}(\Delta x_1, \Delta x_2, \ldots, \Delta x_n)$ represents the displacement from X.

Thus we can say that the r_i values depend on the direction of the vector:

$$\vec{\Delta P} = (\Delta x_1 \ldots \Delta x_n)$$

and that an infinite number of T-family curves passes through X.

2. -

Let us now consider the T family. By means of T we can associate to each point P another point of the space, as follows.

Consider, for every point P, the T - family curve passing through it. Let us follow such a curve, in the sense of decreasing t, until we reach a zero or a point with $J = 0$.

We call such points X (zero) and Y respectively.

We associate to each point P the corresponding point X, or Y.

The set of X points is formed by the solution points of the system of equations, whereas the Y points lie on the surfaces of vanishing jacobian.

Starting then from any point P (where the jacobian J is not vanishing) and following the T-family curve which passes through it, in the Newton-Raphson verse, either we reach an X point, or a Y point.

If we are dealing with real solutions the neighborhood in which we may select the starting trial value is larger than the Newton-Raphson one, as the amplitude of such neighborhood, in our case is only defined by the surfaces of vanishing jacobian and by some curves t.

As the function t(2) is the same as the one defined by the steepest--descent-method, the neighborhoods of convergence of our improved Newton-Raphson method, possibly coincide with the steepest-descend-method ones.

3. -

Let us now explain how one may use in practice the improved Newton-Raphson method.

Let $$f_i = 0 \qquad (i = 1, \ldots, n)$$

be the system of equations to be solved, P_j the generic non solution point obtained at the j^{th} iteration, and ΔP_j the increment computated with the non-modified Newton-Raphson method, by which it should be

$$\vec{P}_{j+1} = \vec{P}_j + \overrightarrow{\Delta P}_j$$

We introduce at every iteration, a check function $t(P_j)$ defined by

$$t(P_j) = \left(\sum \left[f_i^2(P_j) \right] \right)^{1/2}$$

and a reduction factor α, less or equal to 1, so that, taking

$$\vec{P}_{j+1} = \vec{P}_j + \alpha \overrightarrow{\Delta P}_j \tag{9}$$

it follows

$$t(P_{j+1}) < t(P_j) \tag{10}$$

In the relation (9) the largest value of α (≤ 1), compatible with relation (10), is chosen.

The quantity $t(P_j)$ as $j \rightarrow \infty$ vanishes in the case of a real solution, otherwise it tends to a non zero value.

To get a solution in the neighborhood of the initial value another small limiting check number h_o must be introduced on the absolute value $|\overrightarrow{\Delta P_j}|$: as a consequence the calculation will follow formula (9) until both

(10a) $$t(P_{j+1}) < t(P_j)$$

(10b) $$|\overrightarrow{\Delta P_j}| < h_o$$

are verified.

The above method does not follow exactly what we said before.

In order to do that we must proceed by infinitesimal steeps, i.e. keep the h_o parameter infinitesimal. But that is not necessary because in order to reach the solution we do not follow exactly the curve (4) but only make t to decrease.

Such a way makes the neighborhood a little smaller.

4. -

For example, let us now examine the resolution of the system:

(11) $$f_1 = x^2 + y^2 - 4 = 0$$ $$f_2 = y^2 - x - 2y = 0$$

the two real solutions of which are:

(12) $$x_1 = 1.225072 \ldots\ldots$$ $$y_1 = 1.580885 \ldots\ldots$$

(13) $$x_2 = -1.938491 \ldots\ldots$$ $$y_2 = 0.4921913 \ldots,$$

and the vanishing jacobian:

$$J = y^2 - y + 2x - x^2 = 0$$

In fig. 1 the two neighborhoods of convergence to the real solutions are drawn as dashed regions.

The curve AB is a particular t curve tangent to J = 0 and fixes with the J = 0 the limits of the neighborhood of convergence.

Such a curve AB is a maximum slope one. (Numerically proved).

In fig. 2 we illustrate the step-by-step convergence with the improved N. R. method starting from certain trial points with an assigned h_o. As h_o becomes smaller the step-by-step curve tends to the theoretical one (t(P)). The neighborhood of convergence depends on the h_o value when one starts with a point near to the curve AB.

Starting from a point near to a complex solution (as the point P_5) we reach the J = 0 curve.

Let us remark that starting from two initial points separated by the J = 0 (as the points P_3, P_4) one reaches two different solutions. Therefore changing the sign of α after either J = 0 or a resolution is reached (see (9)) and following the same curve t(P) one can compute all the real solutions on a curve t.

Finally let us remark that if one has

$$t\ (y) = 0$$

where Y is a point on J = 0 (see § 2), a multiple solution has been reached.

Such a thing enlarges the usefulness of our improment to Newton--Raphson method because the classical one fails in the former case.

In fig. 3 we draw a block diagram of the improved N. R. method for us with a digital computer.

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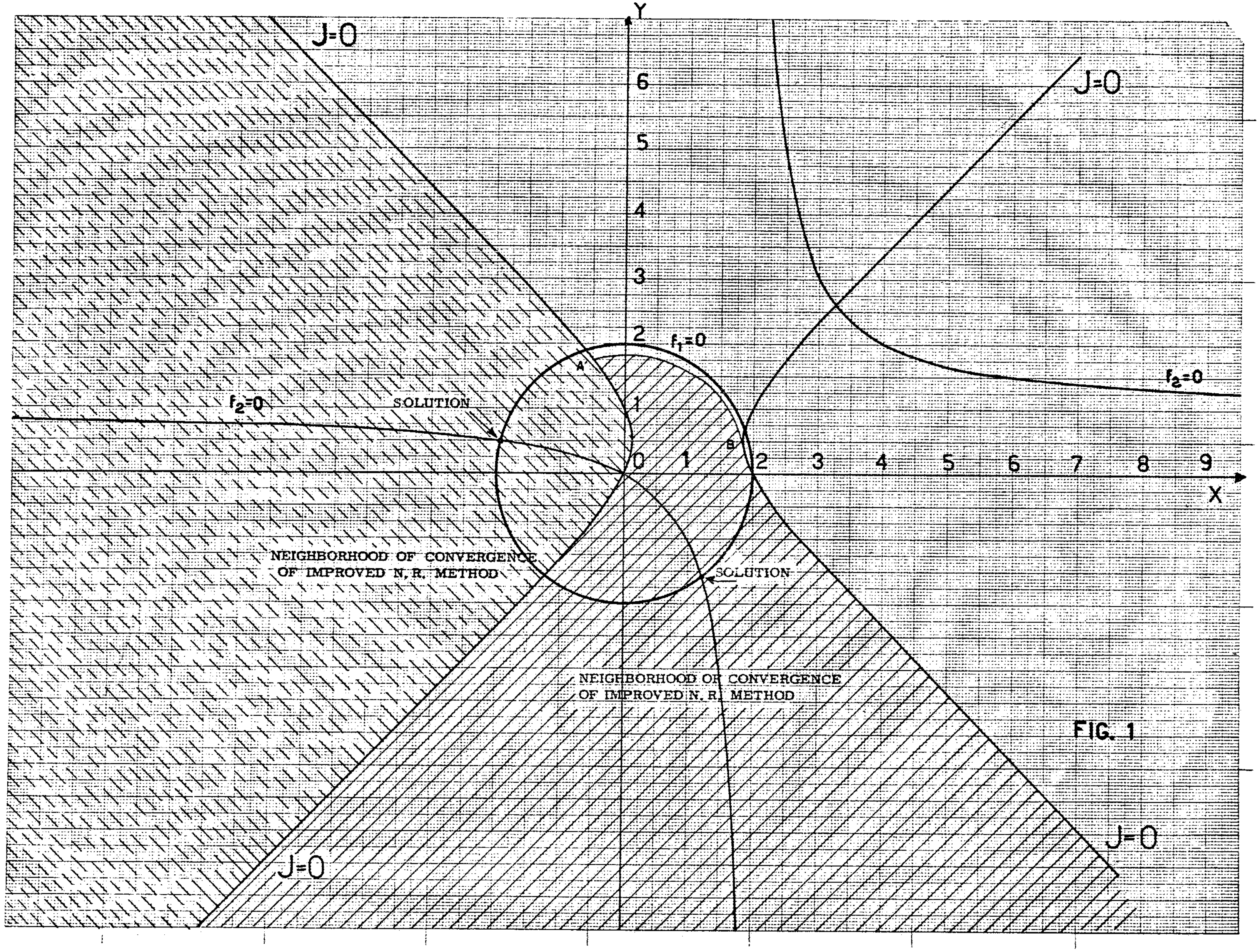
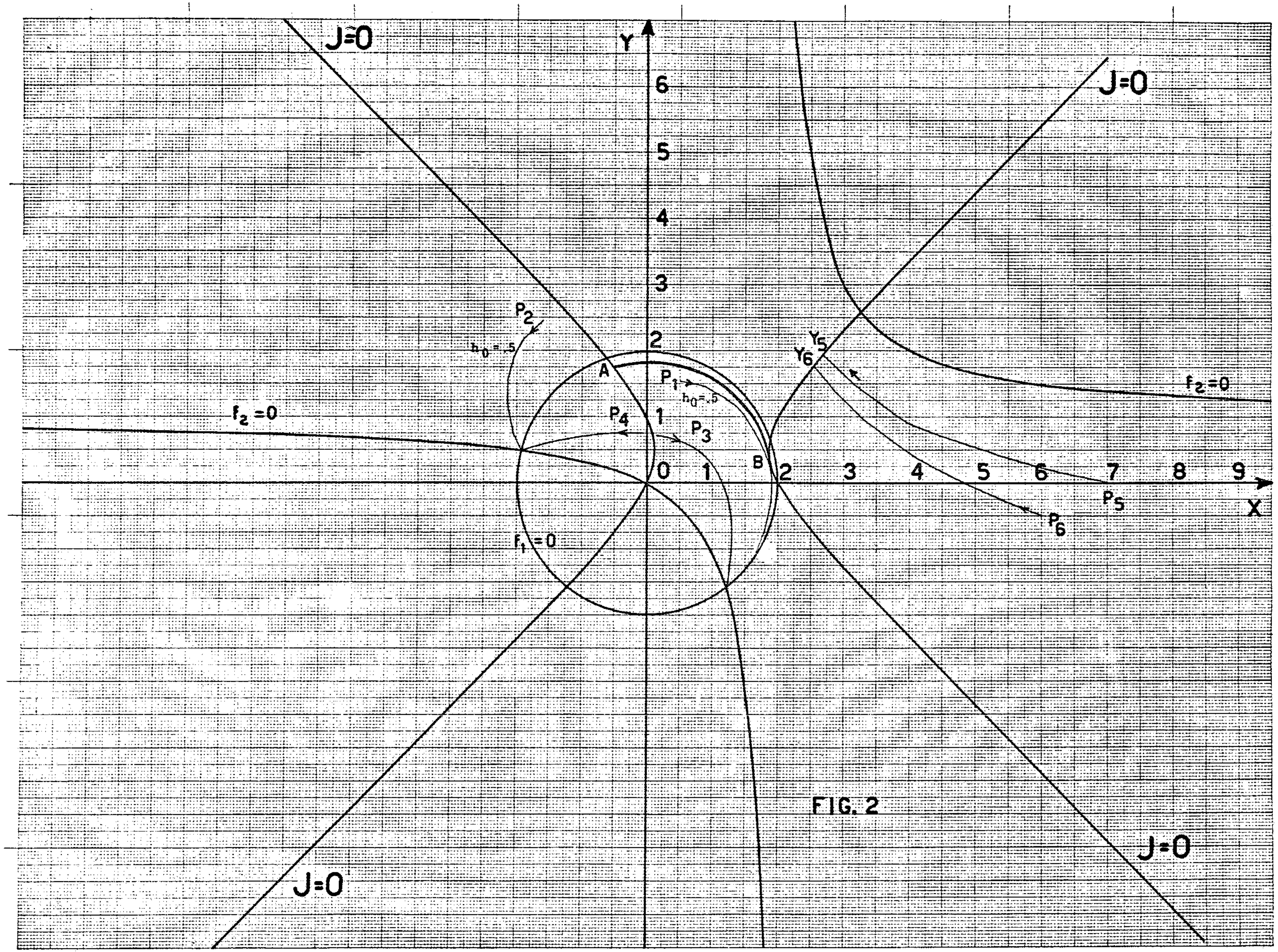


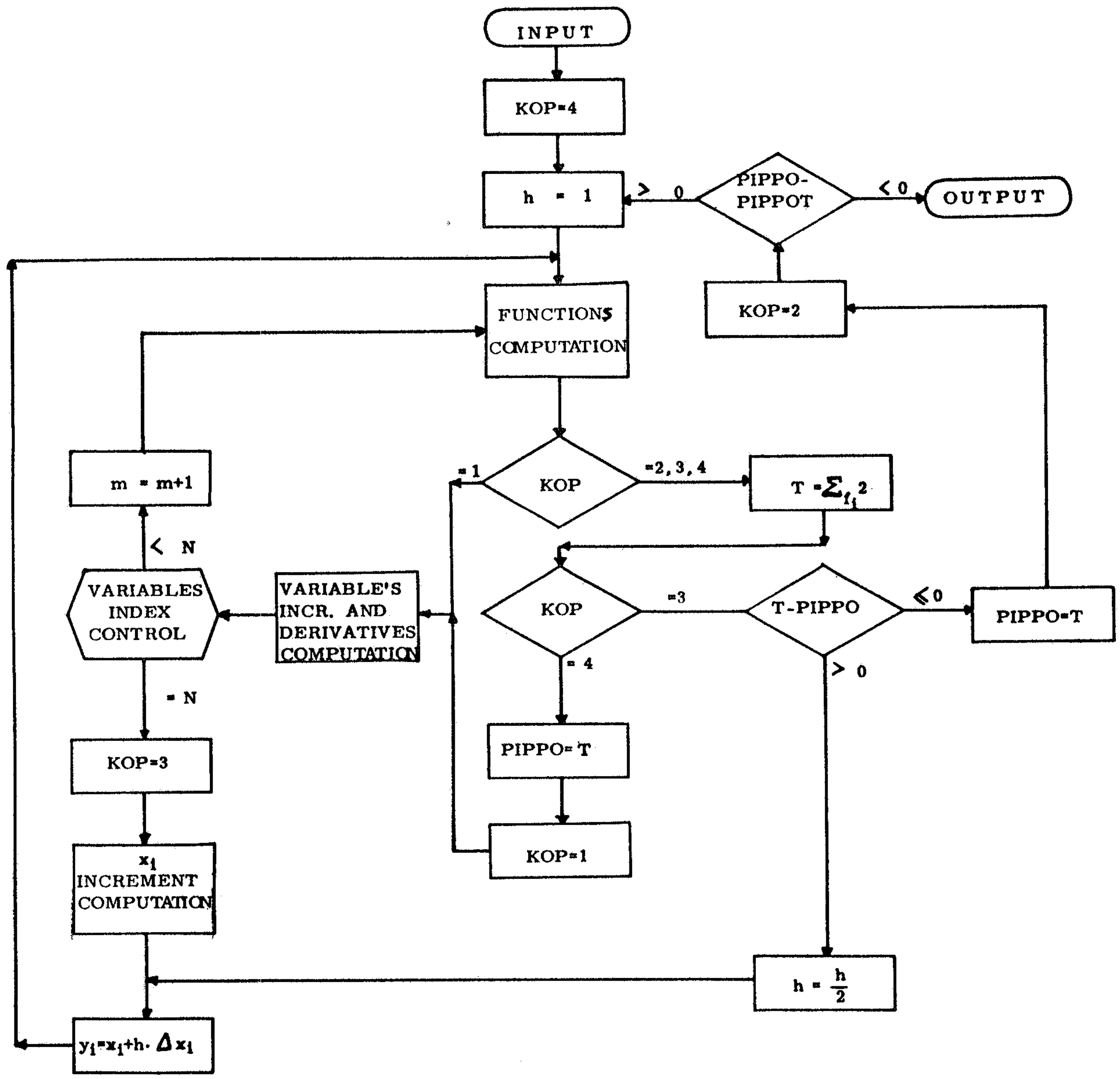
Y
X
0
1
2
3
4
5
6
7
8
9
J=0
J=0
J=0
J=0
$f_1=0$
$f_2=0$
$f_2=0$
A
B
SOLUTION
SOLUTION
NEIGHBORHOOD OF CONVERGENCE
OF IMPROVED N. R. METHOD
NEIGHBORHOOD OF CONVERGENCE
OF IMPROVED N. R. METHOD

FIG. 1



Y
X
J=0
J=0
J=0
J=0
$f_2=0$
$f_2=0$
$f_1=0$
A
B
P_1
P_2
P_3
P_4
P_5
P_6
Y_5
Y_6
$h_0=.5$
$h_0=-.5$
0
1
2
3
4
5
6
7
8
9

FIG. 2



m = variables index
N = number of equations
PIPPOT = assigned minimum value for PIPPO

FIG. 3