COMITATO NAZIONALE PER L'ENERGIA NUCLEARE
Laboratori Nazionali di Frascati

LNF - 67/1
13 Gennaio 1967

M. Greco and G. Rossi :

A NOTE ON THE INFRARED DIVERGENCE. -

(Nota interna: n. 343)

Servizio Documentazione
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1 - INTRODUCTION -

It is well known that no physical process involving the creation and destruction of charged particles can take place without the emission of soft photons. This fact together with the observation that the number of soft photons emitted in a reaction can never really be measured, leads one to introduce a new definition of the final states. This definition is closer to physical reality than the use of states which are diagonal in the number oh photons.

We shall show that with this redefinition of the final state it is possible to determine a matrix element M for the process, which is

1) finite and does not therefore present an infrared divergence,
2) directly comparable to the experimental cross section which results to be proportional to $|M|^2$.
3) and separable: M = AM', where A depends on the cut off for soft photons and M = AM', where A depends on the cut off for soft photons and M' is finite.

The present method therefore leads to a compensation of the infrared divergence for the matrix element itself and does not require - as in the usual method - the compensation of the terms corresponding to real and virtual photons in the expression for the cross section. This simplification is due to the fact that the Bloch-Nordsieck theorem allows one to predict the phases of the emitted photons.

The new final states are not eigenstates of the energy: this corresponds exactly to the experimental situation, which does not allow a check of energy conservation to any desired accuracy.

2 - REALISTIC DEFINITION OF FINAL STATES -

We consider a process

$$A + B \rightarrow C + D + \ldots\ldots \tag{1}$$

in which at least part of the particles A, B, C, D, ... involved are charged. What one actually observes is the process

$$A + B \rightarrow C + D + \ldots\ldots + \Gamma \tag{2}$$

where Γ stands for any number of photons, limited only by the condition that the total 4-momentum of the emitted photons should be contained in that part of the forward light cone, which is defined by the energy and momentum resolution of the experimental apparatus. In the following we shall assume that there is energy resolution only - and no momentum resolution and we assume that $\Delta\omega \ll E$, where for example E is the centre of mass energy of an incident particle.

The final state of reaction (1) will be called $|f)$. For describing reaction (2) we introduce a statevector $|f')$ defined as

$$|f') = e^{i\Lambda_c} |f) \tag{3}$$

with

$$\Lambda_c = \int d^4 x \, j_\mu(x) A_\mu(x) \tag{4}$$

where $A_\mu(x)$ is the 4-potential of the electromagnetic field in the interaction representation and $j_\mu(x)$ is a c-number current defined as

$$j_\mu(x) = (2\pi)^{-4} \int d^4 K j_\mu(K) e^{iKx} \tag{5}$$

and

$$\mu(k) = ie(2\pi)^{-3/2} \sum_i \varepsilon_i P_{i\mu} (P_i k)^{-1} \quad \text{for } K_o \leq \Delta\omega \text{ and zero otherwice.} \tag{6}$$

The signature ε_i is +1 for positive outgoing and negative incoming particles and -1 otherwise and p_i with $p_{i,o} > 0$ is the energy momentum vector of the ith particle. We use the metrics $g_{oo} = 1$, $g_{11} = g_{22} = g_{33} = -1$. The current defined by equation (6) is the Fourier component of the classical current accompanying the destruction of the particles A, B and the creation of the particles C, D... The cut off introduced in this equation is not completely realistic, since it concerns the energy of the single photons, rather than the total energy carried away by the electromagnetic field. We shall discuss this point in section 5.

The operator $e^{i\Lambda_c}$ defined in equation (3) is unitary. The new states $|f')$ are therefore normalized provided that the old states $|f)$ were normalized. In the new frame of reference the matrix-element for the transition $i \to f$ is given by

(7) $$\bar{M} = (f'|S|i) = (f|e^{-i\Lambda_c}\, T\,(e^{iL})|\,i)$$

L is the action integral describing the interaction that leads to process (1).

The matrix-element $\bar{M}$ can be formally written as

(8) $$\bar{M} = \langle f|(e^{-i\Lambda_c} - \mathbb{1})S|i\rangle + \langle f|S|i\rangle = M_c + M_{el}$$

Of course such a definition has no physical meaning on the other hand mathematically M_c and M_{el} are clearly defined by equation (8).

We emphasize that M is formally the matrix-element of the operator $S' = e^{-i\Lambda_c} S$ between the states $\langle f|$ and $|i\rangle$ which do not include outgoing soft photons: in other words the infinite number of soft photons created by S in the final state, is destroyed by $e^{-i\Lambda_c}$.

3 - FINITENESS OF $\bar{M}$ -

We are now going to show that M is finite, i.e. that, apart from ultraviolet divergences, it does not present any infrared divergence. More precisely we shall prove that the expression

$$|\bar{M}|^2 = |M_c|^2 + |M_{el}|^2 + 2\,\mathrm{Re}\,(M_c M_{el}^*)$$

is finite.

For this purpose we shall use the well-known theorem (1, 2) which states that the infrared divergences in the cross section concerning the bremsstrahlung graphs are exactly compensated to any order in the fine-structure constant, by analogous divergences in the corresponding graphs with virtual photons. Then, as:

$$M_{el} = \langle f|S|i\rangle$$

it will be sufficient to show that, in the limit $\omega_j \to 0$ (being ω_j the energy of the j-th bremsstrahlung photon), the cross section for the m-th bremsstrahlung calculated with the usual methods of the electrodynamics, coincides with the equal-order term of

$$|M_c|^2 + 2\,\mathrm{Re}\,(M_c M_{el}^{*})$$

From equation (7), with our definitions, we have:

$$\sum_{n+t-s=2m=\mathrm{cost}} \frac{(-i)^t}{t!} \Lambda_c^t \frac{(i)^n}{n!} T(L^n) =$$

(9)

$$= \frac{1}{m!}\left\{\frac{1}{2}\int d^4k\, \delta(k^2)\theta(k_o) j_\mu(k) j_\mu^{*}(k)\right\}^m \langle f | \frac{(i)^s}{s!} T(L^s) | i \rangle$$

Now we can easily find term of order α^{m+s} in $|M_c|^2 + 2\,\mathrm{Re}(M_c M_{el}^{*})$:

(10) $$\left[|M_c|^2 + 2\,\mathrm{Re}(M_c M_{el}^{*})\right]_{(m+s)} = \frac{1}{m!}\left\{\int \frac{d^3k}{2k_o} j_\mu(k) j_\mu^{*}(k)\right\}^m \left| \langle f | \frac{(i)^s}{s!} T(L^s) | i \rangle \right|^2$$

The definition of $j_\mu(k)$ (equation (5) and (6)) sets the upper limits in the integrals which appear in equation (10).

The analogous term calculated by the usual states is

(11) $$\frac{1}{m!}\left\{\int \frac{d^3k}{2k_o} j'_\mu(k) j'^{*}_\mu(k)\right\}^m \left| \langle f | \frac{(i)^s}{s!} T(L^s) | i \rangle \right|^2$$

where now, adopting the notation already introduced, $j'_\mu(k)$ is defined by

(12) $$j'_\mu(k) = ie(2\pi)^{-3/2} \sum_i i\, P_{i\mu} (P_i k)^{-1}$$

without any restriction imposed on the values which k can assume. The upper limits in the integrals are fixed by the global condition

$$\sum_{k=1}^{m} \omega_k \leq \Delta\omega$$

Although equations (10) and (11) would not seen to be exactly equal, we must recall that the description of the final states so far given in our formulism, is not completely realistic (compare section 2) and therefore actually we ought to cut also the integrals in (10) according to the above global condition.

As we shall discuss latter (section 5) the difference between equations (9) and (10) is given only by a normalization factor very near to 1.

In this way we have shown the finiteness of $|\overline{M}|^2$ and consequently of $\overline{M}$.

In this demonstration we have not taken into consideration the emission of photons by propagators: indeed, it has been shown (3, 4), that these terms do not lead to infrared divergences.

4 - SEPARABILITY OF $\bar{M}$ -

From equation (9) we easily obtain for the matrix - element of process (2), to the order e^{2m+s}

$$\bar{M}^{(2m+s)} = \frac{1}{m!}\left[\frac{\beta}{2}\int_{\lambda}^{\Delta\omega}\frac{dk}{k}\right]^{m} \langle f \,|\, \frac{(i)^{s}}{s!}\, T(L^{s}) \,|\, i\rangle_{\lambda} \tag{13}$$

where

$$\beta = \frac{1}{2}\int d\Omega_k \, |k|^2 \, j_\mu(k)\, j_\mu^{*}(k)$$

and the auxiliary quantity λ, which has been introduced consistently in the two factors of equation (13) as the lower limit to the permitted energies of real and virtual photons, has to become equal to zero.

Adding over m in the equation (13), we have:

$$\bar{M}^{(s)} = \sum_m \bar{M}^{(2m+s)} = \sum_m \frac{1}{m!}\left(\frac{\beta}{2}\right)^m \left(\lg\frac{\Delta\omega}{\lambda}\right)^m \langle f \,|\, \frac{(i)^{s}}{s!}\, T(L^{s}) \,|\, i\rangle_{\lambda}$$

and then

$$\bar{M}^{(s)} = \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \langle f \,|\, \frac{(i)^{s}}{s!}\, T(L^{s}) \,|\, i\rangle_{\lambda} \tag{14}$$

As equation (14) holds for any s, by adding over s we finally have:

$$\bar{M} = \langle f \,|\, e^{-i\Lambda_c} S \,|\, i\rangle = \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \langle f | S | i\rangle_{\lambda} \tag{15}$$

But, as we have shown in the preceeding section $\bar{M}$ is finite, the apparent divergence in λ of the equation (15) when $\lambda \to 0$, will be cancelled by $\langle f|S|i\rangle_{\lambda}$ which consequently must go to zero when $\lambda \to 0$.

We can therefore write:

$$\bar{M} = \left(\frac{\Delta\omega}{E}\right)^{\beta/2} M_E \tag{16}$$

The equation (16) in which the auxiliary quantity λ does not appear any longer, defines formaley M_E as a finite matrix element (whithout any infrared divergence) for the perturbative series of process (1). Of course M_E may have ultraviolet divergences that, after renormalization, will give correction factors to the lowest order matrix element for process (1).

The cross section for process (2), will be proportional to $|\overline{M}|^2$

$$d\sigma \propto |\overline{M}|^2 = \left(\frac{\Delta\omega}{E}\right)^{\beta} |M_E|^2 = \left(\frac{\Delta\omega}{E}\right)^{\beta} d\sigma_E \tag{17}$$

Actually the proportionality factor in equation (17) can be completely determined if we take exactly account of the relation

$$\sum_{k=1}^{\infty} \omega_k \leq \Delta\omega \tag{18}$$

in defining final states.

5 - CORRECT EXPRESSION OF FINAL STATES IN AN EXPERIMENT WITH ENERGY RESOLUTION $\Delta\omega$.

In order to determine the normalization resulting from (18) we shall adopt the following definition for the final states

$$|f''\rangle = \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\infty}^{\infty} e^{i\tau(H_s - x)}\, d\tau : e^{i\Lambda_c} : |f\rangle \tag{19}$$

where H_s represents the Hamiltonian of the electromagnetic field and where $j_\mu(k)$ is now defined by equation (12).

The vectors $|f''\rangle$ in (19) represent the correct final states: in fact besides the detected particles, they contain also an indefinite number of photons with a total energy not greater than $\Delta\omega$.

The states (19) are not normalized: in order ot achieve that, we calculate their square modulus. We have:

$$N = \frac{1}{4\pi^2} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\Delta\omega}^{\Delta\omega} dx' \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \langle f| :e^{-i\Lambda_c}: e^{-i\tau(H_s - x)} e^{i\tau'(H_s - x')} :$$

$$:e^{i\Lambda_c}:|f\rangle = \frac{1}{4\pi^2} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\Delta\omega}^{\Delta\omega} dx' \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, e^{-i(\tau x + \tau' x')} \cdot \tag{20}$$

$$\cdot \exp\left\{ \int d^4k\, \delta(k^2)\theta(k_0) j_\mu(k) j_\mu^*(k) \times e^{i(\tau - \tau')k_0} \right\} = \frac{1}{\gamma^\beta} \left(\frac{\Delta\omega}{\lambda}\right)^\beta \frac{1}{\Gamma(1+\beta)}$$

where $\gamma = e^C = 1.781$ is the Euler constant, β and λ were already defined above.

Now with the normalized states

$$(21) \qquad | f''\rangle = \frac{1}{\sqrt{N}} \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\infty}^{+\infty} d\tau \cdot e^{i\tau(H_s - x)} : e^{i\Lambda_c} : | f\rangle$$

it is possible to define a new matrix element M:

$$(22) \qquad M = \frac{1}{\sqrt{N}} \frac{1}{2\pi} \int_{-\Delta\omega}^{\Delta\omega} dx \int_{-\infty}^{+\infty} d\tau \, \langle f | : e^{-i\Lambda_c} : e^{-i\tau(H_s - x)} \, S | i \rangle$$

and M is such that $|M|^2$ gives us directly the cross ection for process (2).

By proceeding as in sections (3) and (4), we find for M the expression:

$$(23) \qquad M = \frac{1}{\sqrt{N}} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta} \frac{1}{\gamma^{\beta}} \frac{1}{\Gamma(1+\beta)} \langle f | S | i \rangle_{\lambda}$$

and taking into account equation (20 we have

$$M = \frac{1}{\gamma^{\beta/2}} \left(\frac{\Delta\omega}{\lambda}\right)^{\beta/2} \frac{1}{\sqrt{\Gamma(1+\beta)}} \langle f | S | i \rangle_{\lambda}$$

By the same reasoning which led to equation (16), we can write

$$(25) \qquad M = \frac{1}{\gamma^{\beta/2}} \frac{1}{\sqrt{\Gamma(1+\beta)}} \left(\frac{\Delta\omega}{E}\right)^{\beta/2} M_E$$

so that we finally obtain for the cross section

$$(26) \qquad d\sigma = \frac{1}{\gamma^{\beta}} \frac{1}{\Gamma(1+\beta)} \left(\frac{\Delta\omega}{E}\right)^{\beta} d\sigma_E$$

where M_E and $d\sigma_E$ are defined by equation (17).

Equation (26) is in agreement with the results obtained by E. Etim, G. Pancheri and B. Touschek[(5)].

As we can see comparing equations (25) and (16), the only difference between them is the factor

$$C = \frac{1}{\gamma^{\beta/2}} \frac{1}{\sqrt{\Gamma(1+\beta)}}$$

which for small values of β is (near to) $1 - \pi^2\beta^2/24$, so it is very close to 1, as said before.

It case easily be shown that $(1/C^2)-1$ represents the probability that many photons, each with energy $< \Delta\omega$ combine to give a total energy loss $> \Delta\omega$.

Equation (25) obtained for the case of an experiment in which the momentum resolution is zero and only the energy resolution, $\Delta\omega \neq 0$ can be generalized if we replace the condition (18) by a more general condition which limits in some way also photons momenta: consequently equation (15) too must be generalized. A method for doing this has been discussed in reference 5.

ACKNOWLEDGMENTS -

It is a pleasure to thank Prof. B. Touschek for having suggested this work as well as for his constant assistance.

SUMMARY -

In this paper we show how the infrared divergence can be eliminate in the matrix element, provided that physically true final states for an experiment involving creation and distruction of charged particles are used.

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