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C. Pellegrini: THE TOTAL ENERGY OF A CHARGED PARTICLE IN THE MØLLER UNIFIED THEORY OF GRAVITATION AND ELECTROMAGNETISM.

1) A unified theory of gravitation and electromagnetism was recently proposed by Møller(1).

This theory is a natural exstension of a new formulation of general relativity founded on the use of a tetrad field(2).

Its main feature is that, while the Einstein and Maxwell equations are left unchanged, the energy-momentum complex of the electromagnetic and gravitational field now is

$$\begin{cases} T_i^{\ k} = U_{i\ \ ,l}^{\ kl} \\ U_i^{\ kl} = -U_i^{\ lk} = \dfrac{\sqrt{-g}}{\chi}\left\{ \lambda_{(a)}^{\ k}\, \lambda^{(a)l}{}_{;i} + \left(\delta_i^k \lambda^{(a)l} - \delta_i^l \lambda^{(a)k}\right)\lambda_{(a);s}^{\ s} \right\} \end{cases} \tag{1}$$

the tetrads $\lambda_{(a)i}$ are connected to the g_{ik} and F_{ik}, as given by the fields equations, by the relations

$$g_{ik} = \lambda_{(a)i}\, \lambda^{(a)}_{\ \ k} \tag{2}$$

$$F_{ik} = \alpha\left(\gamma_{ik}{}^{l}{}_{;l} + \gamma_{ikl}\,\phi^{l}\right) \equiv \alpha\,\xi_{ik} \qquad (3)$$

where α is a constant having the dimension of a charge,

$$\gamma_{ikl} = \lambda_{(a)i}\,\lambda^{(a)}{}_{k;l}$$

are the Ricci rotation coefficients and

$$\phi_{l} = \gamma^{i}{}_{li}$$

For a pure gravitational field (4) becomes $\xi_{ik} = 0$.

The identification between $\alpha\,\xi_{ik}$ and F_{ik} is not necessary. It is possible to consider separately gravitation and electromagnetism ; in this case the equations $\xi_{ik} = 0$ must be regarded as supplementary conditions which, together with (2), determine the $\lambda_{(a)i}$ and, from (1), $T_i{}^k$.

The advantage of the new complex T_i^k follows from the fact that U_i^{kl} is a true tensor density.

With this in mind the transformation properties of T_i^k (and of the gravitational part of the energy-momentum complex t_i^k) are easily derived.

Let us put

$$\mathfrak{J}^{k} = \mathfrak{J}^{kl}{}_{,l}$$

where

$$\mathfrak{J}^{kl} = a^{i}\,U_i{}^{kl}$$

and a^i is an arbitrary vector.

$\mathfrak{J}^k$ is a vector density and for a transformation of coordinates $x'^{i} = x'^{i}(x^{i})$ it is

$$\mathfrak{J}'^{s} = J^{-1}\,\frac{\partial x'^{s}}{\partial x^{k}}\,\mathfrak{J}^{k}$$

Now $\mathfrak{J}^k$ can also be written as

$$\mathfrak{J}^{k} = a^{i}{}_{,l}\,U_i{}^{kl} + a^{i}\,T_i{}^{k}$$

and if a'^i is chosen to be a constant vector ε^i

$$\mathfrak{J}'^k = \varepsilon^i T_i'^k$$

so that

$$\varepsilon^i T_i'^s = J^{-1} \frac{\partial x'^s}{\partial x^k} \left\{ a^i_{,\ell} U_i^{k\ell} + a^i T_i^k \right\}$$

Taking into account the transformation properties of a^i and $a^i_{,\ell}$ it follows that

$$\overline{T}_k'^s = J^{-1} \frac{\partial x'^s}{\partial x^t} \frac{\partial x^i}{\partial x'^k} T_i^t + J^{-1} \frac{\partial x'^s}{\partial x^t} \frac{\partial x'^r}{\partial x^\ell} \frac{\partial^2 x^i}{\partial x'^r \partial x'^k} U_i^{t\ell} \quad (4)$$

In the case of a transformations of the type

$$\begin{cases} x'^\alpha = x'^\alpha (x^\beta) \\ x'^4 = x^4 \end{cases} \quad (5)$$

we get from (4)

$$T_4'^k = J^{-1} \frac{\partial x'^k}{\partial x^\ell} T_4^\ell \quad (6)$$

so that T_4^k, the energy density and energy current, transforms like a vector density under transformation (5)[3].

This means that, together with the total energy, also the energy density can be an observable, contrary to what happens using for example the Einstein energy-momentum complex.

Of course the transformations rule (4), (6) are valid in the case of the unified a theory as well as in the case when the $\overline{\mathfrak{F}}_{ik} = 0$ are considered like supplementary conditions.

2) In order that the unification have a physical meaning, and the equation $F_{ik} = \alpha \xi_{ik}$ be preferred to $\xi_{ik} = 0$, it is necessary to determine the value of α.

The way to do this is through the determinations of observables connected with T_i^k.

Between these observable quantities there are certainly the total energy and momentum of a system.

Another possible quantity is given by the energy flux through a surface Σ enclosing a three-dimensional volume V

$$\int_\Sigma S^\alpha d\sigma_\alpha = - \frac{\partial}{\partial t} \int_V h \, dV \tag{7}$$

where

$$h = T_4^{\ 4} / \sqrt{-\gamma}$$

$$S^\alpha = T_4^{\ \alpha} / \sqrt{-\gamma}$$

with

$$\sqrt{-g} = \sqrt{g_{44}} \sqrt{-\gamma}$$

The integrals appearing in (7) have now a well defined meaning because of (6).

The simplest case in which it is possible to apply the theory is that of a charged particle at rest.

We shall evaluate the total energy for this system.

Using isotropic coordinate ds^2 is of the form

$$ds^2 = e^\nu dx_4^2 - e^\mu (dx_1^2 + dx_2^2 + dx_3^2)$$

and the g_{ik} and F_{ik} are given by

$$e^{\nu} = \left(1 + \frac{4\pi e^2 - m^2}{4r^2}\right)^2 \Big/ \left\{\left(1+\frac{m}{2r}\right)^2 - \frac{\pi e^2}{r^2}\right\}^2$$

$$e^{\mu} = \left\{\left(1+\frac{m}{2r}\right)^2 - \frac{\pi e^2}{r^2}\right\}^2$$

$$F_{\alpha 4} = \frac{e}{r^2}\, e^{\frac{1}{2}(\nu-\mu)}\, \frac{x_\alpha}{r}$$

all the other components being zero.

Using the superpotential the total energy can be written as

$$E = \lim_{r\to\infty} \int_S \frac{U_4^{4\lambda}}{\sqrt{-\gamma}}\, d\sigma_\lambda \tag{8}$$

where the integration is performed over a sphere whose radius goes to infinity.

To evaluate (5) only the asymptotic values of the tetrads are necessary, i.e. the values for large r where the g_{ik} and F_{ik} are given by

$$\begin{aligned} g_{\alpha\alpha} &\sim -\left(1+\frac{2m}{r}\right) \\ g_{44} &\sim \left(1-\frac{2m}{r}\right) \\ F_{\alpha 4} &\sim e\, x_\alpha / r^3 \end{aligned} \tag{9}$$

If we put

$$\begin{aligned} g_{ik} &= \eta_{ik} + y_{ik} \\ \lambda_{(a)i} &= \eta_{(a)i} + \mu_{(a)i} \end{aligned} \tag{10}$$

with y_{ik} and $\mu_{(a)k}$ small of the first order (2) and (3) become

$$y_{ik} = \mu_{(i)k} + \mu_{(k)i}$$
$$\frac{1}{\alpha} F_{ik} = \Box \nu_{ik} - \frac{1}{2}\left(y^{l}{}_{i,k} - y_{k}{}^{l}{}_{,i}\right)_{,l} \qquad (11)$$

where

$$\nu_{ik} = \frac{1}{2}\left(\mu_{(i)k} - \mu_{(k)i}\right)$$

With the values of g_{ik} given by (9) the quantity

$$\left(y^{l}{}_{i,k} - y_{k}{}^{l}{}_{,i}\right)_{,l}$$

vanishes and (11) become

$$2\mu_{(k)k} = y_{kk}$$
$$\Box \nu_{\alpha\beta} = 0 \qquad (12)$$
$$\Box \nu_{\lambda 4} = -\Delta \nu_{\lambda 4} = \frac{e}{\alpha}\,\frac{x_{\lambda}}{r^{3}}$$

which gives us

$$\mu_{(k)k} = -\frac{m}{r}$$
$$\nu_{\alpha\beta} = 0 \qquad (13)$$
$$\nu_{\lambda 4} = \frac{1}{2}\,\frac{e}{\alpha}\,\frac{x_{\lambda}}{r}$$

Now it is easy to see that in these conditions $U_4^{4\lambda}$ turns out to be indipendent of $\nu_{\lambda 4}$ and $\mu_{(\lambda)4}$ and it has exactly the same value which holds when $F_{ik} = 0$, i.e. for a pure gravitational field:

$$U_4^{4\lambda} = \frac{2m}{\chi}\,\frac{x_{\lambda}}{r^{3}} \qquad (14)$$

The situation would be quite different if $\nu_{\lambda\rho} \neq 0$, $\nu_{\lambda 4} = 0$ for in this case

$$U_4^{\;4\lambda} = \frac{2m}{\chi}\frac{x_\lambda}{r^3} - \frac{1}{\chi}\nu_{\gamma\lambda,\gamma} \qquad (15)$$

One could think to modify (3) and to substitute it with

$$\begin{aligned} F^*_{ik} &= \alpha\, \xi_{ik} \\ F^*_{ik} &= \frac{1}{2}\varepsilon_{iklm}F^{lm} \end{aligned} \qquad (3')$$

In this case (15) would hold instead of (14).

Anyway it is possible to see that, since from (12) and (3) it follows

$$\nu_{\lambda\rho} = \frac{1}{2}\frac{e}{\alpha}\varepsilon_{\lambda\rho\gamma}\frac{x_\gamma}{r}$$

$$\nu_{\gamma\lambda,\gamma} = \frac{1}{2}\frac{e}{\alpha}\varepsilon_{\gamma\lambda\rho}\left(\frac{\delta_{\gamma\rho}}{r} - \frac{x_\gamma x_\rho}{r^3}\right) \equiv 0$$

we got again the result E = m which is a consequence of (14).

In general it is possible to notice that from (11) or (12) follows that given an electromagnetic field going to infinity like $1/r^2$ the ν_{ik} will not necessarely vanish for $r \longrightarrow \infty$ so that the terms $\nu_{ik,l}$ appearing in $U_i^{\;kl}$ will give a divergent contribution if they do not vanish.

This suggest that to get information on α from the total energy one should study the case of an electrostatic field going to infinity like $1/r^3$.

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Byobliography

(1) - C. Møller: Conservation laws and absolute parallelism in general relativity. To be published in Annals of Physics.

(2) A tetrad is defined as a complex of four ortho-normal unit vectors $\lambda_{(a)}{}^{i}$ where the index in bracket labels the vectors. Using the relations $\lambda^{(a)i} = \eta^{ab} \lambda_{(b)}{}^{i}$, $\eta^{ab} = \begin{pmatrix} -1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix}$ it is possible to obtain (2).

For the use of tetrads in general relativity see also F.A.E. Pirani - Bull. Acad. Polon. Sci. 5, 143 (1957) and J.L. Synge: Relativity: General theory, North Holland (1960).

(3) The transformation law (6) was already obtained by C. Møller (reference (1)) indipendently from (4).