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C. PELLEGRINI: A CALCULATION OF RADIATION EFFECTS ON ELECTRON OSCILLATIONS IN A CIRCULAR ACCELERATOR

1) In the magnetic field of a cyclic accelerator the electrons oscillate around an equilibrium orbit.

Three modes of oscillations are possible, which under suitable conditions, can be considered uncoupled: radial betatron oscillations, vertical betatron oscillations and synchrotron oscillations.

The radiofrequency accelerating fields and the radiation losses damp down each oscillation mode. These damping rates have been evaluated by several authors.

Sokolov and Ternov [1] analized the problem determining the quantum states of an electron in the magnetic field of the accelerator and treating the radiation as a perturbation coupling those states.

Robinson[2] and Kolomenskii and Lebedev[3] considered the damping as produced by the classical self-force due to radiation; further they also took into account the random character of photon emission which induces oscillations.

These have been the main lines of approach to the problem followed also by other authors.

While the results obtained for the values of the damping rates are all in agreement for the case of a constant gradient machine, it is not so for an accelerator with a more complicated structure.

There is now in many laboratories a great interest in storage rings requiring quite complicated magnet systems.

It occurs sometimes in these machines that the radial betatron oscillations are antidamped. The possible means suggested by Robinson and Orlov, Tarasov, Khejfets[4] or by Kolomenskii and Lebedev to avoid such a situation differ strongly.

Following Robinson what matters is the average energy radiated per turn which depends on the whole magnet structure.

In the other case, (Kolomenskii and Lebedev) the important element is the average of the local gradient of the magnetic field. All this makes clear that it is necessary to have a new independent evaluation of the damping rates. This is done here using a somewhat different approach to the problem.

The results obtaines will be in agreement with those of Robinson.

The method is the following: the equations of motion are solved neglecting the radiation and the effect of the accelerating cavities (R.F.); the solutions contain, five constants depending on the initial value of the amplitudes and phases of oscillation and of the electron energy; the only effect of the radiation of one photon and of one passage through R.F., or, what is the same, of the emission and absorbtion of one photon; is to change the values of these constants; the overall effect due to the random emission or absorption ofa great number of photons is evaluated and it is shown that it gives rise to a damping.

We remark here that we consider from the very beginning that electrons radiate photons at random and instantaneously. Afterwards we will also assume that the energy spectrum of these photons is correctly given by the classical formula. This is justified in the energy region we are considering [5].

2) Neglecting radiation the motion is described by the classical equations

$$m_0 c \frac{dU_i}{dt} = \frac{e}{c} F_{ik} U_k \qquad (1)$$

where U_i is the electron four-velocity and F_{ik} is in our case the guide magnetic field.

Let us introduce a reference trajectory (R.T.), which can be thought of as the trajectory of an ideal electron of energy E_s moving in the accelerator without gaining or loosing energy. The position of any electron will be referred to R.T. in the following way*:
let s be the arc lenght on R.T.; $P(s)$ one of its points; $\underline{\lambda}_1$, $\underline{\lambda}_2$, $\underline{\lambda}_3$ an orthonormal triad such that, if $\frac{df}{ds} = f'$,

$$\begin{cases} \underline{\lambda}_1 = P' \\ k\underline{\lambda}_2 = \underline{\lambda}_1' \\ -k\underline{\lambda}_1 + H\underline{\lambda}_3 = \underline{\lambda}_2' \\ -H\underline{\lambda}_2 = \lambda_3' \end{cases} \qquad (2)$$

* The use of this frame of reference was suggested to us by C. Bernardini.

with K and H curvature and torsion of R.T.; then the position of an electron will be P + δP where

$$\begin{cases} \delta P(s) = \underline{\lambda}_1 \sigma(s) + \underline{\lambda}_2 x(s) + \underline{\lambda}_3 z(s) \\ \sigma(s=0) = 0 \end{cases} \qquad (3)$$

In the following R.T. will be assumed to be a plane curve (H = 0); F_{ik} to be a static magnetic field $\underline{\chi}$. The R.F. will simply compensate the radiation losses.

Every vector will be decomposed along the :

$$\underline{v} = v_i \underline{\lambda}_i \qquad (i = 1, 2, 3)$$

The value that a physical quantity assumes on R.T. will be labelled by a subscript "s".

δP is assumed to be small of the first order so that quantities like $\underline{\chi}$ can be expanded as a power series in σ, x, z neglecting second order terms.

Other notations introduced are

$$n = \frac{1}{k\chi_{33}} \frac{\partial \chi_3}{\partial x} \qquad (4)$$

$$p = \frac{\mathcal{E} - \mathcal{E}_s}{\mathcal{E}_s} \qquad (5)$$

where $\mathcal{E}$ is the electron energy .

Then $\underline{\chi}$ is given by

$$\begin{cases} \chi_1 \equiv 0 \\ \chi_2 = \chi_{33} k n z \\ \chi_3 = \chi_{33} (1 + k n x) \end{cases} \qquad (6)$$

$$|\chi_{33}| = \frac{k v_s}{ec} \mathcal{E}_s \qquad \chi_{33} < 0$$

The linearized equantions of motion are

$$\begin{cases} \sigma' = K x \\ x'' + K^2(1-n)x = -K p \\ z'' + K^2 n z = 0 \\ p = \mathrm{cost} \end{cases} \qquad (7)$$

In writing down (7) terms in $\frac{\delta v}{v} = \frac{v - v_s}{v_s}$ have been neglected.

This is justified because at the energy considered

$$\frac{\delta v}{v} = p \, \frac{m_0^2 c^6}{E_s^2 v^2} \ll p$$

and p is assumed to be of the same order of Kx, Kz.

3) The solutions of the equations (7) for x and z have been widely discussed for the case in which K and n are periodic functions of s (6).

Using the same notations as in reference (6) the solution for the x mode can be written

$$\begin{aligned} x &= x_0 \sqrt{\beta} \cos(\nu\phi + \xi) - p\psi \\ \psi &= \nu^2 \sqrt{\beta} \sum_{0}^{\infty}{}_{k} \frac{a_k}{\nu^2 - k^2} e^{ik\phi} \\ a_k &= \frac{1}{2\pi\nu} \int_0^{C_s} K \sqrt{\beta}\, e^{-ik\phi} ds \end{aligned} \qquad (8)$$

C_s is the length of R.T. and if L is the arc length for one period of the machine than β has the period L .

$$x_\beta = x_0 \sqrt{\beta} \cos(\nu\phi + \xi) = x_0 \sqrt{\beta} \cos\gamma \qquad (9)$$

describes the radial betatron oscillations.

The term $-p\psi$ gives the deviations from R.T. due to the energy difference $E - E_s$.

The solution for the vertical mode is

$$z = z_0\sqrt{\delta}\,\cos(\omega\chi + \zeta) = z_0\sqrt{\delta}\,\cos\eta \qquad (10)$$

in complete analogy with (9).

(8), (10) describe the motion of one electron with respect to R.T. when the radiation and the accelerating cavities are neglected.

When these are included the synchrotron mode of oscillation, i.e. an oscillation for p, comes in and all the constants x_0, ξ, z_0, ζ, p become step, functions of s.

As a matter of fact let us think that, after an interval $\Delta s'$, in which the motion is described by (8), (10) the electrons emit or absorb a photon.

Then, in the next interval $\Delta s''$ the motion is still given by (8), (10) but with different values of the parameters x_0, ξ, z_0, ζ, p.

In the next section first it will be seen how these quantities change at the emission or absorption of a single photon and next the average behaviour corresponding to the whole story will be studied, together with fluctuations around the average behaviour (by means of r.m.s. values).

4) Let us consider an electron emitting or absorbing instantaneonsly a photon of momentum $\underline{q}$ energy cq; as a consequence its position, velocity and energy will change as follows

$$\Delta x = \Delta z = \Delta\sigma = 0 \qquad \Delta\sigma' = 0$$

$$\Delta x' = -c\,\frac{\underline{q}\times\underline{\lambda}_z + q(x' + K\sigma)}{E}$$

$$\Delta z' = -c\,\frac{\underline{q}\times\underline{\lambda}_x - q z'}{E} \qquad \Delta p = \frac{q}{E_s} = \varepsilon \qquad (11)$$

$\mathcal{E} > 0$ for an absorbed photon, $\mathcal{E} < 0$ for an emitted one.

The variations in x', y', σ' are obtained from

$$\begin{cases} \Delta\left(\frac{\mathcal{E}}{c^2}\underline{v}\right) = -y \\ \underline{v} = \underline{v}_s + c\underline{\Delta}_2(x' + k\sigma) + c\underline{\lambda}_3 z' \end{cases}$$

To see the effect on x_0, ξ, z_0, ζ we express these quantities as functions of x, x', z, z';

$$\begin{cases} \beta x_0^2 = (x+p\psi)^2 + \left[\frac{1}{2}\beta'(x+p\psi) - \beta(x'+p\psi')\right]^2 \\ \operatorname{tg}\gamma = \frac{1}{2}\beta' - \beta\,\frac{x'+p\psi'}{x+p\psi} \end{cases} \tag{12}$$

$$\begin{cases} \delta z_0^2 = z^2 + \left(\frac{1}{2}\delta' z - \delta z'\right)^2 \\ \operatorname{tg}\eta = \frac{1}{2}\delta' - \frac{\delta z'}{z} \end{cases} \tag{13}$$

Substituting (11) in (12), (13) the new values $\bar{x}_0$ and $\bar{z}_0$ of x_0 and z_0 are obtained:

$$\begin{cases} \bar{x}_0^2 = x_0^2 + 2x_0\Delta x_0 + \Delta x_0^2 \\ \Delta x_0 = \frac{\mathcal{E}}{\sqrt{\beta}}\left\{\psi\cos\gamma + \frac{1}{\sqrt{\beta}}\left(\frac{1}{2}\psi\beta' - \beta\psi'\right)\sin\gamma\right\} - \Delta x'\sqrt{\beta}\sin\gamma \\ \Delta x_0^2 = \frac{\mathcal{E}^2}{\beta}\left\{\psi^2 + \left(\frac{1}{2}\beta'\psi - \beta\psi'\right)^2\right\} + \beta\Delta x'^2 \end{cases} \tag{14}$$

$$\begin{cases} \bar{z}_0^2 = z_0^2 + 2z_0\Delta z_0 + \Delta z_0^2 \\ \Delta z_0 = -\Delta z'\sqrt{\delta}\sin\eta \\ \Delta z_0^2 = \delta\Delta z'^2 \end{cases} \tag{15}$$

Now consider the case in which a photon of energy $\mathcal{E}_i$ is radiates at $s=s_i$; this formally corresponds to a rate of change of the energy $\mathcal{E}_i\,\delta(s - s_i)$. The direction of these photons makes with the electron velocity an angle of order $m_0 c^2/E$. This angle can be obviously neglected because the photons that matter are very soft.

It follows from (11) that the corresponding variation in x', z', σ' is zero (to the order $\frac{mc^2}{E}$). Further we assume that only one R.F. cavity is present in the machine in the position $s = C_s$ and that it exchanges with the electrons a photon of energy $\mathcal{E}_\gamma$ directed as $\underline{\lambda}_1$.

The corresponding variations in p, x', z', are

$$\begin{cases} \Delta p = \mathcal{E}_\gamma \, \delta(s+\sigma-\ell C_s) \\ \Delta x' = -\mathcal{E}_\gamma \, \delta(x'+k\sigma) \, \delta(s+\sigma-\ell C_s) \\ \Delta z' = -\mathcal{E}_\gamma \, z' \, \delta(s+\sigma-\ell C_s) \end{cases} \tag{16}$$

where l is an integer number.
Substituting in (14), (15), and summing the variations from s=0 to s= NC_s we can get the damping rates, but before doing this it is necessary to put $\mathcal{E}_i$ and $\mathcal{E}_\gamma$ in a different form.

Actually, since we are interested in average values $\mathcal{E}_i$ can be substituted with $w \, \Delta s_i$ where w in the radiated power per unit arc length and Δs_i is the interval between the emission of the (i-1)- th and i- th photon.
w can be obtained from the fourth component of the self-force Fi acting on the electron (7):

$$w = w_s \{ 1 + 2nKx + 2p \} \tag{17}$$

$$w_s = \frac{2}{3} e^2 E_s^2 K^2 \tag{18}$$

We assume for $\mathcal{E}_\gamma$

$$\mathcal{E}_\gamma = \frac{eV_0}{E_s} \cos\varphi = \frac{eV_0}{E_s} \cos\varphi_s + f^2 \sigma(\ell C_s) \tag{19}$$

φ_s is the synchronous phase and

$$f^2 = \frac{2\pi K \, eV_0}{C_s \, E_s} \sin\varphi_s$$

where K is the R.F. harmonic.
We assume further that

$$\frac{eV_0}{C_s E_s} \cos\varphi_s = \langle w_s \rangle$$

and $\langle w_s \rangle$ is the average value of w_s on R.T.
We are now able to calculate the damping rates beginning with the synchronous oscillations: the variation of p per turn is given by

$$D_p = \frac{1}{C} \left\{ \mathcal{E}_\gamma \delta(s+\sigma-C_s) - \sum_i \mathcal{E}_i \delta(s-s_i) \right\} =$$

$$= \langle w_s \rangle \frac{C_s}{C} + f^2 \frac{\sigma(C_s)}{C_s} - \frac{1}{C} \int_0^C w_s (1 - 2nK\Psi p + 2p) \, ds$$

The energy loss due to radiation must be calculated on the actual electron path of length C.
We neglect the contribution to w of the betatron oscillation writing simply in (17) $x = -\psi p$

Now $C = C_s + \Delta C$, $\Delta C = \int_0^{C_s} d\sigma = \int_0^{C_s} \sigma' ds = C_s \langle K\psi \rangle p$

Introducing the momentum compaction $\alpha = \frac{1}{p}\frac{\Delta C}{C}$

$$C = C_s(1 + \alpha p)$$

and

$$\mathcal{D}p = \langle w_s \rangle (1 - \alpha p) + f^2 \frac{\sigma(C_s)}{C_s} - \langle w_s \rangle (1 - \alpha p) -$$
$$- \frac{1}{C}\int_0^C p\, w_s (2 - 2nK\psi) ds = f^2 \frac{\sigma(C_s)}{C_s} - p\left\langle \frac{dw}{dp} \right\rangle$$

where the relation

$$\frac{dw}{dp} = w_s(2 - 2nK\psi) \tag{20}$$

is used, and in the integral p is considered almost constant. Then the second variation of p is

$$\mathcal{D}^2 p = f^2 \frac{\sigma(2C_s) - \sigma(C_s)}{C_s} - \mathcal{D}p\left\langle \frac{dw}{dp} \right\rangle =$$
$$= -f^2 \langle K\psi \rangle p - \mathcal{D}p \left\langle \frac{dw}{dp} \right\rangle \tag{21}$$

From (21) it is seen that the mean value of p in one turn oscillate with frequency $w_s^2 = f^2\alpha$ and that the damping rate of the synchronous oscillations is

$$\frac{1}{\tau_s} = -\frac{1}{2}\left\langle \frac{dw}{dp} \right\rangle \tag{22}$$

Let us consider now the vertical oscillations.
From (15), (19)

$$\frac{\langle \Delta z_0 \rangle}{z_0} = \frac{1}{N}\sum_e \langle w_s \rangle \delta(s + - lC_s)\sqrt{\delta}\, \mathrm{sen}\,\eta \left\{ \frac{1}{2}\frac{\delta'}{\sqrt{\delta}}\cos\eta - \frac{1}{\sqrt{\delta}}\mathrm{sen}\,\eta \right\}$$

were the average has been taken over a number N of turns such that many betatron oscillations occur.

Now

$$\frac{\langle \Delta z_0 \rangle}{\langle z_0 \rangle} = -\langle w_s \rangle \frac{1}{l}\sum_e \left\{ \mathrm{sen}^2\eta - \frac{1}{2}\delta'\sqrt{\delta}\,\mathrm{sen}\,\eta\cos\eta \right\} \delta(s - lC_s)$$

As the betatron frequency w is generally not an integer multiple of $2\pi/C_s$ the phase in the point C_s takes on all the possible values. It follows

$$\frac{1}{\tau_{\beta v}} = \frac{\langle \Delta z_o \rangle}{z_o} = -\frac{1}{2}\langle w_s \rangle \tag{23}$$

where the average $\langle \Delta z_o \rangle$ is now taken from $s=0$ to $s=C$.

The term Δz_o^2 gives rise to an additive damping

$$\frac{1}{\tau'_{\beta v}} = \frac{\langle \Delta z_o^2 \rangle}{z_o^2} = \frac{1}{2}\langle w_s \rangle^2 C_s \left\{1+\delta'^2\right\}\Big|_{s=C_s} \tag{24}$$

The ratio $\frac{\tau_{\beta v}}{\tau'_{\beta v}}$ is of the order $\langle w_s \rangle$ C_s, i.e. of the total energy radiated per turn divided by E_s.

This is always very small, so that $\frac{1}{\tau'_{\beta v}}$ can be neglected a with respect to $\frac{1}{\tau_{\beta v}}$.

To evaluate the damping for the radial betatron oscillations w is written in the form

$$w = w_s + \frac{\partial w}{\partial x_o} x_o + \frac{\partial w}{\partial p} p = w_s + \frac{\partial w}{\partial x_\beta} x_\beta + \frac{\partial w}{\partial p} p$$

Then from (14) it follows

$$\langle \Delta x_o \rangle = -\frac{1}{ec}\sum_i \frac{\Delta_s}{\sqrt{\beta}}\left[w_s + \frac{\partial w}{\partial x_\beta} x_\beta + \frac{\partial w}{\partial p} p\right]_{s=s_i} \times$$
$$\times \left[\psi \cos\chi + \frac{1}{\sqrt{\beta}}\left(\frac{1}{2}\beta'\psi - \beta\psi'\right)\sin\chi\right]_{s=s_i}$$
$$+ \frac{1}{ec}\sum_h \langle w_s \rangle C_s \left\{\frac{1}{\sqrt{\beta}}\psi\cos\chi + \frac{1}{\beta}\left(\frac{1}{2}\beta'\psi - \beta\psi'\right)\sin\chi + \right.$$
$$\left. + x_o\sqrt{\beta}\sin\chi\left[\frac{1}{2}\frac{\beta'}{\sqrt{\beta}}\cos\chi - \frac{1}{\sqrt{\beta}}\sin\chi - p\psi' + k\sigma\right]\right\}$$

The average is performed on a number N of turns such that many betatron oscillations occur while the synchrotron oscillations can be neglected. This is possible because $f^2\alpha \ll \nu$. Taking into account the periodicity of the various terms one gets the result

$$\frac{1}{\tau_{\beta r}} = -\frac{1}{2}\left\langle \frac{\partial w}{\partial x_\beta}\psi \right\rangle - \frac{1}{2}\langle w_s \rangle \tag{25}$$

Also in this case term $\beta \Delta x'^2$ in (14) gives a negligible contribution to $\frac{1}{\tau_{\beta r}}$.

The evaluation of $\langle \Delta x_o^2 \rangle$ leads to

$$\langle \Delta x_o^2 \rangle = \frac{55}{24\sqrt{3}} r_o c \Lambda \gamma^5 F_1 + \langle w_s^2 \rangle F_2 \tag{26}$$

The term $\frac{55}{24\sqrt{3}} r_o c \Lambda \gamma^5$ results when the average of ε_i^2 over the classical spectrum in performed (8) o and Λ are the classical radius and the Compton wave length of the electron; $\gamma = \frac{E}{m_o c^2}$,

$$F_1 = \left\langle \left[\psi^2 + \left(\tfrac{1}{2}\beta'\psi - \beta\psi'\right)^2 \right] \frac{K^3}{\beta} \right\rangle \tag{27}$$

$$F_2 = \frac{1}{\beta}\left[\psi^2 + \left(\tfrac{1}{2}\beta'\psi - \beta\psi'\right)^2 \right] \Big/_{s=0} \tag{28}$$

In (26)the F_1 term reperesents the contribution due to radiation and the F_2 one is due to R.F.

From (23),... , (26) the equations for the rate of change of the amplitudes of the betatron oscillations are obtained.

$$\begin{cases} \dfrac{d\langle z_o^2 \rangle}{d\ell} = \dfrac{2\langle z_o^2 \rangle}{\tau_{\beta v}} \\[2ex] \dfrac{d\langle x_o^2 \rangle}{d\ell} = \dfrac{2\langle x_o^2 \rangle}{\tau_{\beta r}} + \dfrac{55}{24\sqrt{3}} r_o \Lambda \gamma^5 F_1 + \langle w_s^2 \rangle F_2 \end{cases} \tag{29}$$

where ℓ is the number of turns multiplied by C_s.

5) - In this section we want to discuss some of the resul ts obtaind.

Notice first that $\frac{1}{\tau_{\beta r}}$ can be put in the form given by Robinson (2). In fact from

$$\begin{cases} \dfrac{\partial w}{\partial x_\beta}\psi = 2nK\psi w_s \\[2ex] \dfrac{dw}{dp} = w_s(2 - 2nK\psi) \end{cases}$$

it follows

$$\frac{1}{\tau_{\beta r}} = \frac{1}{2}\left\langle \frac{dw}{dp} \right\rangle - \frac{3}{2}\langle w_s \rangle \tag{30}$$

Another point is that all the averages appearing in (22), ... , (26) must be evaluated on the actual electron path of length C, but one can to evaluate them on R.T. provided the result is multiplied by (1- p). The term αp can be neglected to a first approximation except when the derivative with respect to p of the average has to be calculated, as is the case for $\frac{1}{\tau_{\beta r}}$.

All the values obtained for the damping rates are in agreement with Robinson's results.

This is also true for F_1, when to a first approximation it is assumed (6) that

$$\psi \simeq a_0 \sqrt{\beta} \; ; \qquad a_0 \sim \left(\frac{R}{\nu^3}\right)^{1/2}$$

$$\beta \sim \frac{R}{\nu} \; ; \qquad \alpha \sim \frac{1}{\nu^2}$$

where R is the average radius of the machine.

Substituting in (27) We get

$$F_1 = R^2 \alpha^2$$

that is the same formula given by Robinson.

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