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SPACE CHARGE EFFECTS IN e^-e^- AND e^+e^- COLLIDING OR CROSSING BEAM RINGS

F. Amman - D. Ritson[(x)]

1) Introduction

The interaction between two proton beams in storage rings has been studied, but as the Lioville's theorem does not allow the achievement of beam densities higher than the tranfer densities, it need not to be considered as a strongly limiting factor.

In electron electron or electron positron storage rings, the strong damping due to radiation losses causes the beam to shrink to a cross section which can be as small as 10^{-4} cm^2, and accordingly current densities up to $10^3 \div 10^4$ A/cm^2 are achievable by conventional means.

Our results, which apply to the case of electron electron or electron positron rings, show that if two beams collide, the less intense beam A will be stable in the presence of beam B up to some limiting effective density of beam B and will then break up into a diffuse halo around beam B.

The calculations separate therefore into an investigation of the point at which the beam A will become unstable and break up, and an investigation as to the point at which the diffuse halo will become stable.

The two calculations agree, and therefore lead to confidence that our description is correct. The calculations for the diffuse halo are made in the approximation that the field due to a ribbon of charge is constant, except close to the ribbon plane, and changes sign as this plane is crossed.

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We have also calculated whether arranging for the two beams to intersect at an angle in the vertical plane can change the allowed effective space charge density. The results give substantially the same limits as those obtained for the head on collision of two beams. While the calculations and effects are complicated and tedious, the simple result is obtained that the effective charge density achievable with the condition that the two beams still interact is given by:

$$1) \qquad \frac{N_B}{wh} \sim \frac{k}{p} \frac{Q\,\delta Q}{2\,F\,r_e} \left(\frac{\gamma}{R}\right)$$

for head on collision and by:

$$2) \qquad \frac{N_B}{w\,l\,\delta} \sim \frac{k}{p} \frac{Q\,\delta Q}{2\,F\,r_e} \left(\frac{\gamma}{R}\right)$$

for crossing beams, where:

N_B is the number of charges in the more intense beam B;

k is the number of bunches per turn;

p is the number of interaction regions per turn;

w, h and l are the radial, vertical and azimuthal dimensions of the beam;

δ is the angle of the direction of the bunch axis of one of the beams with the median plane of the ring (see fig. 1); if δ is different for the two beams, in equ. 2) the higher value of the two δ's must be used;

Q is the number of betatron wavelengths around the ring, for the vertical betatron mode;

δQ is the distance to the closest proper resonance;

γ is the relativistic factor $\frac{E}{mc^2}$;

$F = \beta \frac{Q}{R}$ is the "beat factor" in the interaction region (β is the amplitude factor);

R is the mean radius of the machine;

$r_e = \frac{e^2}{mc^2}$ is the classical electron radius.

As γ/R is proportional to the mean magnetic field along the ring, this result shows that the effective charge density is almost a constant irrespective of machine size or design, limited to values of the order of 10^{12} particles/cm^2, as compared to $10^{14} \div 10^{16}$ particles/cm^2 which could be quite easily achieved in absence of the space charge limitation.

Our calculations for the space charge limited density agree with the calculations done for the Stanford electron electron storage ring[1].

The notation used in the following is that of Green and Courant[2] and the method of solution follows their treatment.

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1 - W.C. Barber et al.: An experiment on the limits of quantum electrodynamics - Internal Report of Stanford University HEPL 170 (June 1959)

2 - G.K. Green and E.D. Courant: The proton synchrotron Handbuch der Physik, Bd. XLIV, pag. 218-340

2) Head-on collision. Instability limit

We consider the motion of an electron or a positron in beam A interacting head on with a more intense electron beam B. Beam B has k bunches, a total number of electrons N_B, its length is l, width w, height h, and the distribution of charges is assumed to be uniform.

The impulse I given to an electron (or positron) passing through one of the p interaction regions with vertical displacement z is:

$$3)\qquad I = \pm \frac{4 \pi N_B e^2}{w h k c} z \qquad \text{for } |z| \lessgtr h/2$$

As a result the region behaves like a thin lens with a transfer matrix T_i:

$$4)\qquad T_i = \begin{vmatrix} 1 & 0 \\ \pm A & 1 \end{vmatrix}$$

with:

$$5)\qquad A = \frac{4 \pi N_B r_e}{\gamma whk}$$

The unperturbed transfer matrix from one collision region to the next, with the condition that the number of periods of the magnetic structure be a multiple of the number of collision regions p, is:

$$6)\qquad T_m = \mathbb{1} \cos \frac{\mu}{p} + \begin{vmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{vmatrix} \sin \frac{\mu}{p}$$

where μ is the phase shift of the betatron oscillations around the ring.

The resulting transfer matrix is the product of 4) and 6); the phase shift between two collision regions becomes:

$$7)\qquad \cos \frac{\mu + \Delta\mu}{p} = \cos \frac{\mu}{p} \pm \frac{A\beta}{2} \operatorname{sen} \frac{\mu}{p}$$

if $\frac{\Delta\mu}{p}$ is small and $\frac{\mu + \Delta\mu}{p}$ is not close to a multiple of π, we can rewrite the 7):

$$8)\qquad \frac{\Delta\mu}{p} \simeq \mp \frac{A\beta}{2}$$

If the next integral or half integral resonance is δQ removed, we have for the allowed change in phase shift:

9) $$\Delta\mu < 2\pi\,\delta Q$$

or

10) $$\mp p\,\frac{A\beta}{2} < 2\pi\,\delta Q$$

and, with the 5), remembering that $\beta = F\,\frac{R}{Q}$,

11) $$\frac{N}{w\,h} \lessapprox \mp\,\frac{2\,k}{p}\,\frac{Q\,\delta Q}{2\,F\,r_e}\left(\frac{\gamma}{R}\right)$$

The - or + sign in the 11) means that in the electron electron case the closest lower resonance must be taken into account (δQ must be negative), while in the positron electron case is the closest higher resonance which must be considered.

If $\frac{\mu + \Delta\mu}{p}$ is very close to a multiple of π (which is, for instance, the case when $p = 1$), the approximation in 8) is wrong by a factor of 2; we can take it into account redefining p as a parameter which depends on the number of collision regions per turn, which is always bigger than 2, and is approximately equal to the number of collision regions only when they are more than 2 per turn.

3) Head on collision. Stable diffuse orbits.

Consider the motion of an electron (or positron) in beam A executing on orbit which does not intersect beam B. We shall approximate the impulse per unit length I given to the electron (or positron) in traversing the interaction region as:

$$12) \qquad I = \pm \frac{4 \pi N_B e^2}{w k l c} \frac{z}{|z|} \qquad \text{for } |z| \gtrsim h/2$$

The resulting equation for the vertical motion is:

$$13) \qquad \frac{d^2 z}{ds^2} + k(s) z = \pm \frac{4 \pi N_B r_e}{\gamma k w l} \frac{z}{|z|} L(s)$$

s is the distance along the equilibrium orbit, k(s) represents the focussing properties of the machine, L(s) is equal to 1 over the p regions of interaction and to 0 outside , and the length of interaction regions is 1/2.

This equation may be transformed in the standard fashion in terms of the variables:

$$14) \qquad \eta = \beta^{-1/2} z \qquad \varphi = \int \frac{ds}{Q}$$

into:

$$15) \qquad \frac{d^2 \eta}{d\varphi^2} + Q^2 \eta = \pm Q^2 \beta^{3/2} \frac{4 \pi r_e}{\gamma k w l} \frac{\eta}{|\eta|} L(\varphi)$$

where $L(\varphi)$ is equal to one over the p interaction regions and to zero outside.

Because of the non-linearity, the 15) may have periodic solutions (closed orbits) with periods $2\pi q$ in φ , where q is an integer.

Let us Fourier expand the perturbation term over q turns:

$$16)\qquad Q^2 \beta^{3/2} \frac{4\pi r_e N_B}{\gamma k w h l} \frac{\eta}{|\eta|} L\left(\frac{\psi}{q}\right) \cong$$

$$\cong 2 \frac{Q \beta^{1/2} N_B r_e}{\gamma k w q} \sum_{1}^{p}{}_{i} \left[\frac{\eta_i}{|\eta_i|} \frac{1}{2} + \frac{\eta_i}{|\eta_i|} \sum_{1}^{S}{}_{r} \cos r \frac{\psi - \psi_i}{q} \right]$$

where ψ_i are the coordinates of the p interaction regions; $\frac{\eta_i}{|\eta_i|}$ gives the sign of the perturbation in the interaction regions.

The expansion 16) is approximate, and is valid in the assumption that

$$17)\qquad r \frac{l}{4 R q} \ll 1$$

Accordingly it is limited to a certain harmonic term of order S; it can be seen a posteriori that this approximation is usually quite good.

The behaviour of the periodic solutions of 15) is dominated by the terms of 16) for which

$$18)\qquad \frac{r}{q} \simeq Q$$

Taking only this term we have:

$$19)\qquad \eta = \pm \frac{2 Q \beta^{1/2} N_B r_e}{\gamma k w q} \sum_{1}^{p}{}_{i} \frac{\cos \frac{r}{q} (\psi - \psi_i)}{Q^2 - (r/q)^2} \frac{\eta_i}{|\eta_i|}$$

The solution, when $\langle \eta \rangle = 0$, is consistent for:

$$20)\qquad \begin{cases} Q > r/q & \text{for electron electron interaction} \\ Q < r/q & \text{for positron electron interaction.} \end{cases}$$

We find again that in the electron electron case the relevant resonance is the closest lower one, while in the positron electron case is the closest higher one.

The numerical value of the sum in 19) for the j^{th} interaction region depends on p and on the closed orbit; let us introduce its maximum value which is given by:

$$21) \quad \sum_{i}^{p} \frac{\eta_i}{|\eta_i|} \cos \frac{r}{q} (\varphi_j - \varphi_i) \leq qp$$

and substituting:

$$22) \quad \eta = \beta^{-1/2} z$$

$$23) \quad \frac{r}{q} = Q + \delta Q$$

equ. 19), for the j^{th} interaction region, gives

$$24) \quad z_j = \mp \frac{\beta N_B \; r_e \; p}{\gamma \; w \; \delta Q k}$$

The condition both for the validity of the potential used and for all the orbits of beam A to lie outside beam B is:

$$25) \quad z_j \geq h$$

which finally gives the density limitation:

$$26) \quad \frac{N_B}{wh} \cong \mp \frac{2 \; k}{p} \; \frac{Q \;\; \delta Q}{2F \;\; r_e} \left(\frac{\gamma}{R} \right)$$

The - sign is for electron electron (δQ must be negative for the solution to be consistent) and the + sign is for positron electron.

δQ should be the distance from the closest resonance of every order; in fact it can be easily seen that for $q \gtrsim 4$ the treatment given here begins to fail; δQ, for higher values of q, decreases, but at the same time the value of the sum given by the 21) decreases. Within the limits of this approximation we think that the 26) can be taken as a thumb rule for resonances up to third or fourth order; the higher order resonances, except in special cases, should not give a worse limit.

4) Crossing beams

Let us consider a crossing beam ring; in the interaction region the equilibrium orbits $\bar{Z}_A$ and $\bar{Z}_B$ of the two beams, considered separately, cross at an angle 2α in a vertical plane; this crossing angle is assumed to be given by suitable perturbations in the guide field of harmonic order higher than the Q of the two beams, so that a change of about one unit in Q leaves α substantially unchanged.

Fig. 1 represents a possible situation when the density in both A and B beams is high enough to effect the other one. The bunch axes are tilted by angles ϑ_A and ϑ_B with respect to the direction of $\bar{Z}_A$ and $\bar{Z}_B$, but the velocities of the particles are always directed like $\bar{Z}_A$ and $\bar{Z}_B$.

Defining the vertical displacement Z_A with respect to the orbit $\bar{Z}_A$, a p rticle of beam A, in traversing the interaction region, receives an impulse I given by:

$$27)\qquad I = \pm \left\{ \frac{2\pi N_B e^2}{wkc} \; \frac{2x-1}{l} + \frac{4\pi N_B e^2}{wkc} \; \frac{Z_A}{l\,\delta_B} \right\} \qquad \text{for} \quad -x < \frac{Z_A}{\delta_B} < 1 - x$$

and

$$28)\qquad I = \pm \frac{2\pi N_B e^2}{wkc} \; \frac{Z_A}{|Z_A|} \qquad \text{for} \quad \frac{Z_A}{\delta_B} < -x \quad \text{or} \quad \frac{Z_A}{\delta_B} > 1-x$$

where: x is the distance of the particle from the leading edge of the bunch;

$\delta_B = \alpha + \vartheta_B$ is the angle of the axis of beam B with the median plane of the ring.

If the density of beam A is very small, then $\vartheta_B \simeq 0$ and $\delta_B \simeq \alpha$, as the tilt angle ϑ_B is due to the effect of beam A; the beam A will change its Q (decreasing it in the electron electron case, increasing it in the positron electron case), and the direction of the bunch axis will be tilted by an angle ϑ_A with respect to the $\bar{Z}_A$ direction, because of the interaction with beam B and according to its effective charge density.

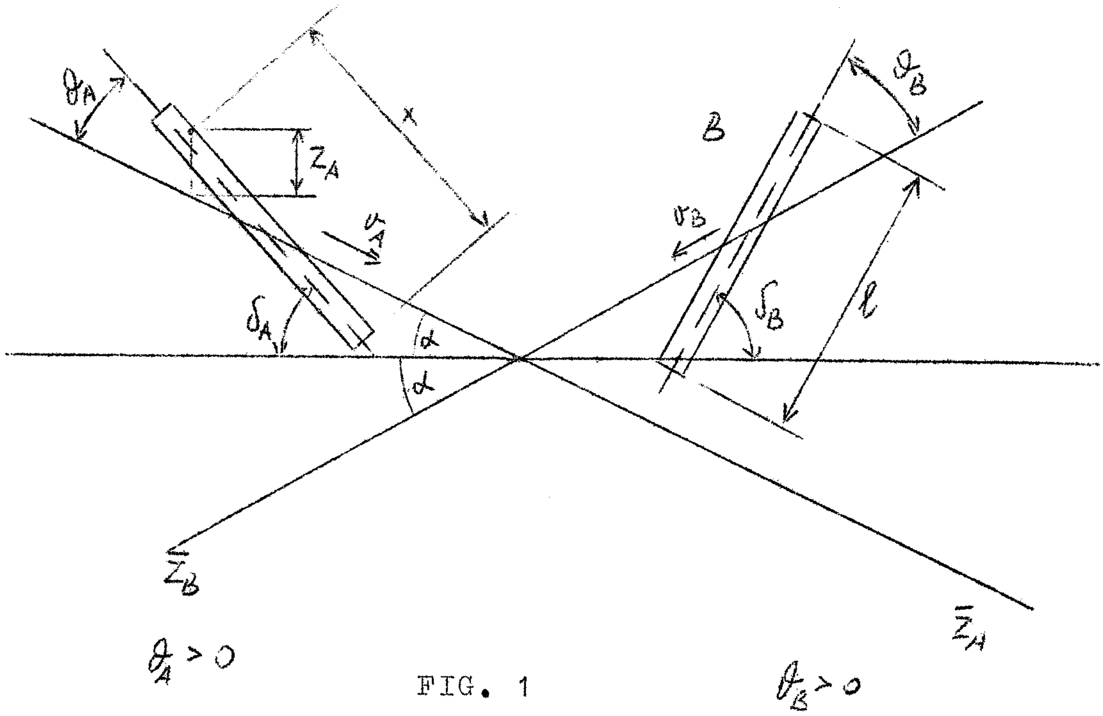

FIG. 1

Interaction region in a crossing beam ring

The sign of the tilt angle depends on the closest integral resonance: in the electron electron case ϑ_A will be negative (see fig. 1 for the definition of the sign of the angles) if the closest integral resonance is lower than the $\overline{Q}$, and positive if the closest integral resonance is higher than the $\overline{Q}$; the opposite is true for the positron electron case.

$\overline{Q}$ is the number of betatron wavelengths per turn due to both the main field focussing and the lens effect of the interaction region.

The density of beam B which gives rise to the instability limit for crossing beams comes out to be the same as the one calculated for colliding beams, provided we substitute the height of the beam with the equivalent height $l\delta_B$:

$$29) \qquad \frac{N_B}{l\delta_B w} \leq \mp \frac{2\ k}{p} \frac{Q\ \delta Q}{2F\ r_e} \left(\frac{\gamma}{R}\right)$$

All the considerations made for 11) are valid for equ. 29).

With a method similar to the one used in section 3 we can find the tilt angle ϑ_A for the case of one interaction region per turn, in the electron electron case, for a weak focussing ring; we get:

$$30) \qquad \vartheta_A = -4\ F \frac{R}{\gamma} \frac{N_B}{wl} \frac{r_e}{k} \left\{ \frac{1}{2\overline{Q}^2} - \frac{1}{1-\overline{Q}^2} \right\}$$

As an example, with $Q \simeq 0.8$, the lens effect of the interaction region tends to drive $\overline{Q}$ close to 0.5, which is the closest lower half integral resonance; combining equ. 29) with equ. 30), at the space charge limit ($\overline{Q} = 0.5$), we get for ϑ_A:

$$31) \qquad \vartheta_A \simeq -\frac{\delta_B}{2}$$

In this case then, when the density of beam B is high enough to drive beam A into the half integral resonance, the tilt angle is negative and is given by 31).

As the tilt angle depends on $\bar{Q}$, it is not easy to write down a general formula. However it can be calculated by introducing the impulse given by 27) in the equations of motion and solving it for each particular case.

What can be said is that if the crossing regions are more than one per turn they can be arranged to make θ small.

The tilt angle ϑ_A changes the length L of the interaction region along the direction $\bar{Z}_B$, from the initial value L_o, for $\vartheta_A = 0$.

32) $$L_o = \frac{h}{2\alpha}$$

to

33) $$L' = \frac{h}{2\alpha} + \frac{1\ \vartheta_A}{2\ \alpha}$$

If we drop the assumption $N_A << N_B$, and we consider beams roughly equal in density, then if the effective densities are close to the limit given by 29) the equilibrium will be unstable; it seems then that the crossing angle 2α must be controlled to keep it big enough to avoid these unstable situations.

At the beginning of the present section we made the assumption that α remains constant when Q changes of the order of one unit; if the crossing angle is obtained with perturbations of low harmonic order, close to Q, the treatment must be modified, as α will also depend on $\bar{Q}$.

A final remark must be made about the crossing beam case; every beam beam interaction effect has been calculated with the approximation that the impulse given by beam B to a particle of beam A outside of beam B, does not depend on the distance from the axis of beam B. This approximation is correct when the distance in a vertical plane from the axis is smaller than the radial width of the beam, which is not always the case for a crossing beam ring.

We nevertheless used this approximation, which probably gives a pessimistic value for the space charge limited density, as it permits us more readily to evaluate the effects.

5) Results

The space charge limit sets in at a charge density given by 11), 26) and 29). With the densities given by these relations there should not be any collision between beams A and B; it seems therefore more correct to take as a space charge limited density, allowing for the interaction between the two beams, a value at least a factor of two lower, we get then the values given in section 1.

The interaction rates per interaction region, $\dot{n}$, are given, for colliding beams, by:

34) $$\dot{n} = N_A \frac{N_B}{wh} \frac{f}{k} \sigma \qquad \text{events/sec}$$

and, for crossing beams, by

35) $$\dot{n} = N_A \frac{N_B}{wl\delta} \frac{f}{k} \sigma \qquad \text{events/sec}$$

where f is the revolution frequency, and σ is the cross section in cm^2.

Introducing the space charge limited densities for beam B with the numerical values:

$Q\,\delta Q \simeq 0.1$ (weak focussing)
$\gamma/R \simeq 4 \quad cm^{-1}$
$F \simeq 1$

we find:

36) $$\dot{n} \simeq \frac{4 \times 10^{30}}{p} I_A \sigma \qquad \text{events/sec}$$

where I_A is the current, in Ampere, of the weaker beam A and p the number of the interaction regions per turn.

This result seems to set an effective upper limit for the cross sections that can be investigated with colliding or crossing beam rings.

In the case of colliding beams, where the cross section of the beams is determined by the radiation effects, one should provide some means to increase this cross section if he wants to use for I_A a value bigger than the one which is given by the space charge limited density times the unperturbed cross section of the beam (which is of the order of 1 mA for a 750 MeV weak focussing ring).

No problems, arise in crossing beam rings, as in this case the effective beam cross section can be varied changing the crossing angle.