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$+ e^- + \bar{v}$ DECAY.

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1.- INTRODUCTION

Feynmann and Gell-Mann(1) introduced the hypothesis of conserved current to explain the absence of renormalization effects in the V part of the β decay. In their scheme the weak vector current is identifield with the (+) component of the isotopic spin current $J_K^{(+)}$. One of the suggested tests of the theory is an accurate measurement of the decay rate for the leptonic decay of the pion:

$$\pi^- \longrightarrow \pi^0 + e^- + \bar{\nu} \qquad (1)$$

In fact, neglecting electromagnetic corrections, the corresponding matrix element is given by:

$$i g \sqrt{2} (\bar{e} \gamma^* [1 + i\gamma^5] \nu)(\pi^0 | J_K^{(+)} | \pi^-) \qquad (2)$$

and we have a simple connection between the relevant matrix element of the vector current and the electromagnetic form factor of the pion F_π:

$$(\pi^0 | J_K^{(+)} | \pi^-) = (\pi_K^0 + \pi_K^-) \, F_\pi(k^2) \qquad (3)$$

where $k^2 = (\pi^- - \pi^\circ)^2$ is the momentum transfer to the lepton poir. In the actual process (1) this momentum transfer is very small, so that one can safely put $F_\pi = 1$. In this work we propose to evaluate the radiative corrections (to order e^2) to process (1). This would be important for a comparison of an accurate experimental result and the prediction of the Feynmann an Gell-Mann Theory.

Since it is difficult to introduce the pion form factor in a gange-invariant way for vertices with virtual pion lines, we will use a local Lagrangian and a Feynmann cutoff in the calculation of radiative corrections. The results will not depend critically on this cutoff, since the divergence will be found to be only logarithmic.

2.- FORMULATION

In the following we shall use the notations and the conventions of the textbook of Bogoliubov and Shirkov (2). Let p_1, p_2, p_3, p_4 be the momenta and m_1, m_2, m_3, $m_4 = 0$ the masses of the π^-, e^-, π°, $\bar{\nu}$. We put also:

$$\cosh\theta = \frac{(p_1 p_2)}{m_1 m_2} \tag{4}$$

All the calculations will be performed in the center of mass system.

The Lagrangian responsible for the process is:

$$\mathcal{L}_1 = \sqrt{2}\, g \{\bar{\pi}\, \partial_k \pi^\circ - \pi^\circ \partial_k \bar{\pi}\}(\bar{e}\gamma^k [1+i\gamma^5]\nu) + h.c. \tag{5}$$

Following the principle of the minimal electromagnetic interaction, the Lagrangian that takes into account the electromagnetic interactions as well, is obtained from the complete Lagrangian without them:

$$\mathcal{L}_0 = \mathcal{L}^{(e)}_{free} + \mathcal{L}^{(\nu)}_{free} + \mathcal{L}^{(\pi)}_{free} + \mathcal{L}^{(\gamma)}_{free} + \mathcal{L}_1 \tag{6}$$

by the substitution:

$$\partial_k \phi(x) \longrightarrow (\partial_k - ieA_k(x))\,\phi(x) \tag{7}$$

where A(x) and ϕ(x) are respectively the field operators of the photon and of the generic charged particle that appears in the process (1).

In this way we obtain: for the complete interaction Lagrangian L:

$$\mathcal{L} = \mathcal{L}_1 + e : \bar{e}\hat{A}e : + ie : [\partial^k \pi^* \pi - \pi^* \partial^k \pi] : A_k - e^2 : \pi^* \pi A_k A^k : + [\sqrt{2}\, g (\bar{e}\gamma^k[1+i\gamma^5]\nu)\, ie A_k \pi^* \pi^0 + h.c.] \tag{8}$$

The last term is a new direct interaction between the five particles π^-, π^0 , e^-, v and γ .

3. - FEYNMANN DIAGRAMS.

Using the Lagrangian (8) we have seven diagrams corresponding to the process (1) up to the order g^2 e^2 (fig.1).

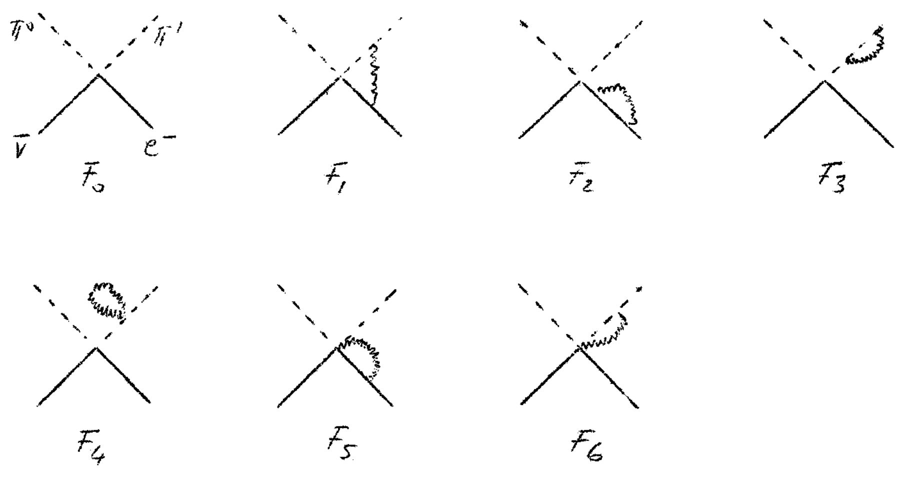

FIG. 1

Besides them we have also to consider three diagrams (fig. 2) relative to the same process with bremsstrahlung:

$$\pi^- \longrightarrow \pi^0 + e^- + \nu + \gamma \qquad (9)$$

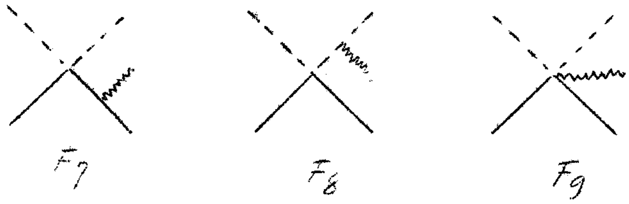

FIG. 2

They give to the transition probability a contribution of the order $g^2\ e^2$, depending on the experimental situazion (see par. 5).

4. - TRANSITION PROBABILITIES (Virtual photons).

We put:

$$(e^-, \pi^0, \bar{\nu} \,|\, S-1 \,|\, \pi^-) = \delta(p_1 - p_2 - p_3 - p_4)\, F \qquad (10)$$

and we have:

$$F = \sum_{1h}^{6} F_h \qquad (11)$$

where the F_h's refer to the various diagrams of fig. 1. Calling dP_1 the corresponding differential transition probability, summed over the final spins, we have:

$$dP_1 = \frac{1}{2\pi} \sum_{spin} |F|^2 \delta(p_1 - p_2 - p_3 - p_4)\, d\underline{p}_1\, d\underline{p}_2\, d\underline{p}_3 \qquad (12)$$

where to the order $g^2 e^2$ we have:

$$|F|^2 = |F_0|^2 + 2\,\mathcal{R} \sum_{1h}^{6} F_0^* F_h \qquad (13)$$

4.1 - Calculation of F_0

$$F_0 = \frac{i_0 \sqrt{2}}{2(2\pi)^2 \sqrt{p_{10} p_{30}}} \left(\bar{v}_2^+ [\hat{p}_1 + \hat{p}_3][1 + i\gamma_5]\, v_4^+ \right) \qquad (14)$$

$$\sum_{spin} |F_0|^2 = \frac{g^2}{(2\pi)^4 p_{10} p_{20} p_{30} p_{40}} \Big\{ 8(p_1 p_2)(p_1 p_4) - 4m_2^2 (p_1 p_4) + [m_2^2 - 4m_1^2](p_2 p_4) \Big\} \tag{15}$$

4.2 - Calculation of $(F_1 + F_5 + F_6)$

$$F_1 + F_5 + F_6 = -\frac{ge^2}{(2\pi)^6} \frac{1}{\sqrt{2p_{10}p_{30}}} \bar{v}_2^+ \int dk \Big\{ \Big[(2\hat{p}_1 - \hat{k})(k^2 - 2(p_2 k)) + (4m_2 - 2\hat{p}_2 + 2\hat{k})(k^2 - 2(p_1 k)) + (2\hat{p}_1 - \hat{k})(m_2 + \hat{p}_2 - \hat{k})(\hat{p}_1 + \hat{p}_3 + \hat{k}) \Big] \cdot k^{-2} [k^2 - 2(p_1 k)]^{-1} \tag{16}$$

This integral shows ultraviolet and infrared divergences; consequently it has been evaluated introducing a Feynmann cutoff ($-\frac{\lambda^2}{k^2-\lambda^2}$) and a fictitions photon mass . The same will be made, where necessary, for the other diagrams.

By standard methods one obtains:

$$F_1 + F_5 + F_6 = -\frac{ge^2}{(2\pi)^6} \frac{1}{\sqrt{2p_{10}p_{30}}} \bar{v}_2^+ \Big\{ 2 I_1 \hat{p}_1 [\hat{p}_2 + m_2][\hat{p}_1 + \hat{p}_3] - \hat{I}_2 [5m_2 \hat{p}_1 + m_2 \hat{p}_3 + \hat{p}_2 \hat{p}_3 + \hat{p}_2 \hat{p}_1 - 2\hat{p}_1 \hat{p}_2] - 2[4(p_1 p_2) - m_2 \hat{p}_2] \hat{I}_2 - 2\hat{p}_1 \hat{I}_2 [\hat{p}_1 + \hat{p}_3 + 2m_2] + I_3 [4\hat{p}_1 + 3m_2] - 2\hat{I}_4 \Big\} (1 + i\gamma^5) v_4^+ \tag{17}$$

where

$$\begin{aligned} I_1 &= \frac{i\pi^2}{2} J_1 \\ \hat{I}_2 &= \frac{i\pi^2}{2} \sum_h g^{hh} \gamma^h J_2^h \\ I_3 &= \frac{i\pi^2}{2} \sum_h g^{hh} J_3^{hh} \\ \hat{I}_4 &= \frac{i\pi^2}{2} \sum_{hk} g^{hh} g^{kk} \gamma^h (\hat{p}_2 - \hat{p}_1) \gamma^k J_3^{hk} \end{aligned} \tag{18}$$

J_1, J_2^h and J_3^{hk} have been defined and evaluated from Behrends, Finkelstein and Sirlin[3] (formulae (7a) and following).

Substituting the expressions for the j's, we obtain:

$$F_1 + F_5 + F_6 = -\frac{ge^2}{(2\pi)^6}\frac{1}{\sqrt{2p_{10}p_{30}}}\frac{i\pi^2}{2}\bar{v}_2^{+}\left(Am_2 + 2B\hat{p}_1\right)\left(1+i\gamma^5\right)v_4^{+} \tag{19}$$

where:

$$\begin{aligned} A &= -4I_1(p_1p_2) - 2I_{21}\left(m_1^2 - (p_1p_2)\right) + 2I_{22}\left[4(p_1p_2) - m_2^2 - 2m_1^2\right] + \\ &\quad + 3I_3 - 2I_{42} \\ B &= 4I_1(p_1p_2) - 2I_{21}\left[3(p_1p_2) + m_1^2\right] - I_{22}\left[3m_2^2 + 4(p_1p_2)\right] + \\ &\quad + 2I_3 - I_{41} \end{aligned} \tag{20}$$

I_{21}, I_{22}, I_{41} and I_{42} are defined by the positions:

$$\begin{aligned} \hat{I}_2 &= I_{21}\hat{p}_1 + I_{22}\hat{p}_2 \\ \hat{I}_4 &= I_{41}\hat{p}_1 + I_{42}\hat{p}_2 \end{aligned} \tag{21}$$

Finally the contribution of $(F_1 + F_5 + F_6)$ to the transition probability is:

$$\sum_{spin} 2\,\mathcal{R}\left[(F_1 + F_5 + F_6)\cdot F_0^{*}\right] = -\frac{g^2}{2(2\pi)^4}\frac{\alpha}{\pi}\left[8B(p_1p_2)(p_1p_4) + \right. \\ \left. + 2(A-B)m_2^2(p_1p_4) - (Am_2^2 + 4Bm_1^2)(p_2p_4)\right] \tag{22}$$

and the fractional correction $\delta^{(1,5,6)}$ is:

$$\delta^{(1,5,6)} = \frac{\sum_{spin} 2\,\mathcal{R}\left[(F_1+F_5+F_6)F_0^{*}\right]}{\sum_{spin}|F_0|^2} =$$

$$= -\frac{1}{2}\frac{\alpha}{\pi}\frac{8B(p_1p_2)(p_1p_4) + 2(A-B)m_2^2(p_1p_4) - (Am_2^2 + 4Bm_1^2)(p_2p_4)}{8(p_1p_2)(p_1p_4) - 4m_2^2(p_1p_4) + (m_2^2 - 4m_1^2)(p_2p_4)} \tag{23}$$

4.3 - Calculation of F_2, F_3 and F_4.

These diagrams describe self-energy effects; after mass and wave function renormalisation the diagram F_4 does not give any contribution, while the diagrams F_2 and F_3 give the following fractional corrections to the transition probability: $\delta^{(2)} = -\frac{\alpha}{\pi}\left(\frac{1}{2}\ln\frac{\lambda}{m_2} + \ln\frac{\lambda_m}{m_2} + \frac{9}{8}\right)$

$$\delta^{(3)} = -\frac{\alpha}{\pi}\left(-\ln\frac{\lambda}{m_1} + \ln\frac{\lambda_m}{m_1} + 1\right) \tag{24}$$

5. - TRANSITION PROBABILITIES (real photons).

As is well known, when treating approximations to the order g^2e^2, one must consider the corrections due to real photons of energy inferior to a maximum value $\mathcal{E}$ depending upon the experimental resolution, as well as the corrections due to virtual photons. In our case we have assumed $\mathcal{E} << m_2$. We put:

$$(e^-, \pi^0, \bar{\nu}, \gamma \,|(S-I)|\, \pi^-) = \delta(p_1 - p_2 - p_3 - p_4 - k)\, G \tag{25}$$

Where K is the momentum of the emitted photon, and $G = F_7 + F_8 + F_9$ (see fig. 2).

Calling dP_2 the corresponding differential transition probability, summed over the final spins and polarizations, and integrated over $\underline{K}$ with $|\underline{K}| \leq \mathcal{E}$, one obtains:

$$dP_2 = \frac{1}{2\pi}\sum_{spin}\sum_{pol}\int_{|\underline{K}|\leq\mathcal{E}} d\underline{k}\,|G|^2\delta(p_1-p_2-p_3-p_4-k)\,d\underline{p}_2\,d\underline{p}_3\,d\underline{p}_4 \tag{26}$$

Because $\mathcal{E} << m_2$, we can neglect the diagram F_9, wich gives corrections proportional to $\mathcal{E}$ and $\mathcal{E}^2$; for the correction relative to $(F_7 + F_8)$ one obtains by standard methods:

$$dP_2 = \delta^{(7,8)}\frac{1}{2\pi}\sum_{spin}|F_0|^2\delta(p_1-p_2-p_3-p_4)\,d\underline{p}_2\,d\underline{p}_3\,d\underline{p}_4 \tag{27}$$

where $\delta^{(7,8)}$, that is consequently the fractional correction due to the bremsstrahlung, is given by:

$$\delta^{(7,8)} = -\frac{\alpha}{\pi}\Big[2\ln\frac{2\mathcal{E}}{\lambda_m}(1-\theta\coth\theta) - \theta\,\mathrm{cth}\,\theta - 1 + (p_1 p_2)\int_{-1}^{+1} dz\,\frac{1}{p_z^2}\,\frac{1}{2v_z}\ln\frac{1+v_z}{1-v_z}\Big] \tag{28}$$

where $p_z = \frac{1}{2}\left[p_1(1+z) + p_2(1-z)\right]$ and $v_z = \frac{|\underline{p}_z|}{p_{z0}}$

6. - TOTAL CORRECTION AND APPROXIMATIONS.

The total percentual correction is given by:

$$\delta = \delta^{(4,5,6)} + \delta^{(2)} + \delta^{(3)} + \delta^{(7,8)} \tag{29}$$

where the $\delta^{(\ )}$'s are given from (23), (24) and (28). As it must be δ does not contain λ_m.

In the very good approximation $m_2 << m_1$, $\delta^{(4,5,6)}$ and $\delta^{(7,8)}$ may be written as follows:

$$\begin{aligned}\delta^{(4,5,6)} &= -\frac{\alpha}{\pi}\left[\theta\,\mathrm{cth}\,\theta\left(\theta - 2\ln\frac{\lambda_m}{m_2} - 5\right) - \ln\frac{m_1}{m_2} + \frac{5}{2}\ln\frac{m_1}{\lambda} - \frac{13}{8}\right]\\ \delta^{(7,8)} &= -\frac{\alpha}{\pi}\left[2\ln\frac{2\varepsilon}{\lambda_m}\left(1-\theta\,\mathrm{cth}\,\theta\right) + \theta\,\mathrm{cth}\,\theta - 1\right]\end{aligned} \tag{30}$$

Then:

$$\begin{aligned}\delta = -\frac{\alpha}{\pi}\Big[&\theta^2\,\mathrm{cth}\,\theta - 4\theta\,\mathrm{cth}\,\theta - \frac{1}{2}\ln\frac{m_1}{m_2} - 3\ln\frac{\lambda}{m_1} - \frac{1}{2} +\\ &+ \ln\frac{2\varepsilon}{m_1} + \ln\frac{2\varepsilon}{m_2}\left(1 - 2\theta\,\mathrm{cth}\,\theta\right)\Big]\end{aligned} \tag{31}$$

7. - INTEGRAL TRANSITION PROBABILITY.

We call P_o and P the integral transition probabilities to the order $g^2\ e^2$, and $\delta_p = (P - P_o)/P_o$ the corresponding fractional correction.

By definition:

$$\begin{aligned}P_o &= \frac{1}{2\pi}\int\sum_{spin}|F_o|^2\,\delta(p_1-p_2-p_3-p_4)\,dp_2\,dp_3\,dp_4\\ P &= \frac{1}{2\pi}\int\sum_{spin}|F_o|^2(1+\delta)\,\delta(p_1-p_2-p_3-p_4)\,dp_2\,dp_3\,dp_4\end{aligned} \tag{32}$$

Performing the integrations considering also the pion's recoil, and neglecting only terms like m_2^2 in comparison with m_1^2, one obtains:

$$\begin{aligned}P_o = \frac{2g^2}{\pi^3 m_1}\Big[&\frac{1}{5}(\Delta^2-m_2^2)^{\frac{5}{2}}(m_1-2\Delta) + \frac{1}{3}(\Delta^2-m_2^2)^{\frac{3}{2}}\left(\frac{5}{4}\Delta^3 - \frac{1}{2}m_1\Delta^2 - \right.\\ &\left.- \frac{13}{8}m_2^2\Delta + m_1 m_2^2\right) + \frac{1}{8}m_2^2\Delta^2(\Delta-2m_1)(\Delta^2-m_2^2)^{\frac{1}{2}}\Big]\end{aligned} \tag{33}$$

$$\begin{aligned}\delta_p = -\frac{\alpha}{\pi}\Big[&\left(\ln\frac{2\Delta}{m_2}\right)^2 - \frac{167}{30}\ln\frac{2\Delta}{m_2} + \frac{6229}{1800} + \frac{3}{2}\ln\frac{m_2}{m_1} -\\ &- 3\ln\frac{\lambda}{m_1} + 2\ln\frac{2\varepsilon}{m_2}\left(\frac{107}{60} - \ln\frac{2\Delta}{m_2}\right)\Big]\end{aligned} \tag{34}$$

where $\Delta = \frac{m_2^2 - m_3^2}{2m_1}$

Finally one obtains numerically:

$$P_0 = 1,105 \sec^{-1} \tag{35}$$

$$\delta_p = 0,027 + 0,005 \ln \frac{2\mathcal{E}}{m_2} + 0,007 \ln \frac{\lambda}{m_1} \tag{36}$$

8. - DISCUSSION.

The value (35) has been obtained using the following numerical values:

$$\begin{aligned} m_1 &= (139,59 \pm 0,05) \text{ MeV } (4) \\ m_3 &= (135,00 \pm 0,05) \text{ MeV } (4) \\ g m^2_{proton} &= (1,204 \pm 0,001) \cdot 10^{-5} \ (5) \end{aligned} \tag{37}$$

From the errors on m_1, m_3 and g follows an error of 1,6% on P_0, essentially determinated by the incertitude on the masses; hence this error is about one half of the correction (36).

In (36) the cutoff for short wavelenghts is not cancelled; however, the result depends only logaritmically on this cutoff.

Assuming for example $\mathcal{E} = 1/20\ m_2$ and $\lambda = 10$ m, we find:

$$\delta_p \simeq 0,032 \tag{38}$$

Finally for the lifetime τ we get:

$$\tau = 0,904\ (1 + 0,032)\ (1 \pm 0,016) \text{ sec} = (0,933 \pm 0,015) \text{sec}. \tag{39}$$

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