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C. Pellegrini: THE TOTAL ENERGY-MOMENTUM VECTOR OF THE SCHWARTZSCHILD FIELD.

The difficulties connected with the concept of energy in the general theory of relativity have been widely studied by various authors(1,2,3) during the latest years.

In particular C. Møller(4) concluded that it is impossible to find an energy-momentum complex, $\theta_i^{\,k}$, which satisfies the following conditions:

1) The total energy and momentum $P_i = \int_{V x_4 = cost} \theta_i^{\,4} d^3x$ are time indipendent; the $P_i$ transform as a four-vector under Lorentz transformations and furthermore are unchanged under arbitrary transformations of the coordinates, provided that the new system coincides with the original one at large spatial distances;
2) $\theta_i^{\,k}(x^0)$ in the event point $x^o$ is an affine tensor density that depends algebraically on the metric tensor $g_{iK}$ and on its first and second derivatives in the same point;
3) $\theta_{i\,,k}^{\,k} = 0$
4) $\theta_4^{\,k}$ transforms as a four-vector under the group of pu-

rely spatial transformations

$$\bar{x}^{\alpha} = f^{\alpha}(x^{\beta}) \quad ; \qquad \bar{x}^{4} = x^{4} \tag{1}$$

Greek indices take on the value 1, 2, 3 and Latin indices 1 to 4.

Condition four is necessary if we want $\theta_4^4$ to be a scalar density under trasformations (1) so that it becomes possible to give $\theta_4^4$ the meaning of energy-density[3].

Condition three is necessary to have local energy conservation.

The conditions, 2, 3, 4 determine the Møller energy-momentum complex $T_i^K$ uniquely[5], but this doesn't satisfy[4] 1, i.e. it doesn't transform like a four-vector under Lorentz transformations.

Furthermore assumptions 1, 2, 3, lead uniquely to the Einstein[6] energy-momentum complex but this doesn't satisfy 4.

We want to show a possible way out of this situation.

We take assumptions 2, 3, 4 to be valid, so that we get the Møller energy-momentum complex. The case is considered of a single mass giving rise to a Schwartzschild gravitational field.

Now we assume that the definition $P_i = \int_{x_4=const} T_i^4 d^3x$ is valid only for the rest system of the mass, while for a system where the mass is moving with uniform velocity $u_i$, we have

$$P_i = \int_{\sigma} \frac{T_i^k}{\sqrt{-g}} u_k \, d\sigma \tag{2}$$

where $g = \| g_{ik} \|$ and σ is a three - dimensional surface perpendicular to $u_K$.

In the rest system (2) reduces to

$$\overset{o}{P}_i = \int T_i^{\,4} d^3x = \delta_i^{\,4} M c \tag{3}$$

where M is the Newtonian mass(3).

Further more $P_i^o$ satisfies the requirements of being time-independent and of being unchanged under arbitrary transformations(3) which leave the coordinates system asimptotically invariant.

Since $P_i$ transforms like a four-vector under Lorentz transformations we have now that, at least for the case of the Schwartzschild field and for masses moving with uniform velocity all the conditions 1, ....., 4 are satisfied.

As a matter of fact since $P_i$ is a linear combination of $P_i^o$ with coefficients functions of $\beta$, the relative velocity of the two systems, as long as $\beta$ is constant $P_i$ has all the properties of $P_i^o$ and so satisfies completely 1.

We will show now that our $P_i$ transforms like a four-vector. The square of the linear element in the rest system is taken to be of the form

$$ds_o^2 = a(r) \left\{ \sum_{a} (dx_o^{\alpha})^2 \right\} - b(r) (dx_o^4)^2 \tag{4}$$

The Lorentz trasformation

$$x^1 = \frac{x_o^1 + \beta x_o^4}{\sqrt{1-\beta^2}}$$

$$x^2 = x_o^2 \quad ; \quad x^3 = x_o^3 \tag{5}$$

$$x^4 = \frac{x_o^4 + \beta x_o^1}{\sqrt{1-\beta^2}}$$

is performed.

In the rest sistem

$$d\sigma_0 = \sqrt{\gamma_0}\; d^3x_0 \tag{6}$$

$$\gamma = \| g_{\alpha\beta} \| \tag{7}$$

$$\overset{0}{u}_k \equiv (0,0,0,\sqrt{b}) \tag{8}$$

Now $$b\gamma = -g$$

so that

$$\overset{0}{P}_i = \int \overset{0}{T}{}_i^{\,4}\, d^3x_0 \tag{9}$$ .

In the moving system

$$\begin{cases} u_1 = -\dfrac{\beta}{\sqrt{1-\beta^2}}\sqrt{b} \\ u_2 = u_3 = 0 \\ u_4 = \sqrt{b}/\sqrt{1-\beta^2} \end{cases} \tag{10}$$

The law of transformation for $T_i^K$ is

$$\frac{T_i^{\,k}}{\sqrt{-g}} = \frac{\overset{0}{T}{}_l^{\,m}}{\sqrt{-g_0}}\; \frac{\partial x_0^l}{\partial x^i}\; \frac{\partial x^k}{\partial x_0^m}$$

so that

$$\begin{cases} T_i^{\,1}/\sqrt{-g} = \dfrac{1}{\sqrt{-g_0}}\; \dfrac{\partial x_0^l}{\partial x^i}\sqrt{1-\beta^2}\left\{\overset{0}{T}{}_l^{\,1} + \beta\,\overset{0}{T}{}_l^{\,4}\right\} \\ T_i^{\,2}/\sqrt{-g} = \dfrac{\partial x_0^l}{\partial x^i}\; \overset{0}{T}{}_i^{\,2}/\sqrt{-g_0} \\ T_i^{\,3}/\sqrt{-g} = \dfrac{\partial x_0^l}{\partial x^i}\; \overset{0}{T}{}_i^{\,3}/\sqrt{-g_0} \\ T_i^{\,4}/\sqrt{-g} = \dfrac{1}{\sqrt{-g}}\; \dfrac{\partial x_0^l}{\partial x^i}\sqrt{1-\beta^2}\left\{\beta\,\overset{0}{T}{}_l^{\,1} + \overset{0}{T}{}_l^{\,4}\right\} \end{cases} \tag{11}$$

It follows

$$P_i = \int \frac{T_i^{\,k}}{\sqrt{-g}}\,\mu_k\, d\sigma = \int \frac{\sqrt{1-\beta^2}}{\sqrt{-g_0}}\,\frac{\partial x_0^{\,l}}{\partial x^i}\Big\{-\big(\overset{0}{T}{}_l^{\,1} + \beta\,\overset{0}{T}{}_l^{\,4}\big)\frac{\beta\sqrt{b}}{\sqrt{1-\beta^2}} + \big(\beta\,\overset{0}{T}{}_l^{\,1} + \overset{0}{T}{}_l^{\,4}\big)\frac{\sqrt{b}}{\sqrt{1-\beta^2}}\Big\}\sqrt{\gamma_0}\,d^3x_0 = \tag{12}$$

$$= \frac{\partial x_0^{\,l}}{\partial x^i}\int \overset{0}{T}{}_l^{\,4}\,d^3x_0 = \frac{\partial x_0^{\,l}}{\partial x^i}\,\overset{0}{P}_l$$

We turn now to our definition of $P_i$ and try to point out some of the problems arising when we consider the case of an accelerated motion of the mass.

In this case $T_i^K$ should separate into the near field and the radiation field and anly the near field contributes to our integral. Further $P_i$ will not be time - independent. It is clear that the problem becomes now strictly connected with the gravitational radiation one.

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Bibliography

(1) J.N. Goldeberg Phys. Rev. 111, 315, (1958)

(2) P.G. Bergmann Phys. Rev. 112, 287 (1958)

(3) C. Møller - Annals of Physics 4, 347 (1958)

(4) C. Møller - Annals of Physics 12, 118 (1961)

(5) M. Magnusson Mat. Phys. Medd. Dan. Vid. Selsk. 32, n.6 (1960).

(6) A.I. Einstein - Berlin Ber. 778 (1915)
Ann. Physik 49, 769 (1916)
Berlin Ber 448 (1918)