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C. Pellegrini: CONNECTIONS BETWEEN SOME GEOMETRICAL AND MECHANICAL PROPERTIES IN GENERAL RELATIVITY.

Using the tetrad technique(1) it is possible to establish a connection between some properties of the Riemann curvature

$$K = \frac{R_{ikem}\, u^i u^e v^k v^m}{g_{ikem}\, u^i u^e v^k v^m} \qquad (1)$$

relative to the directions defined by the unit vectors v^i, u^K, and some physical properties of the gravitational and matters fields. This connection furnishes some methods to measure certain properties of space-time as they are determined by the matter present in it.

In (1) R_{ikem} is the Riemann tensor and g_{ikem} is given by

$$g_{ikem} = g_{ie}\, g_{km} - g_{im}\, g_{ek} \qquad (2)$$

The metric $ds^2 = g_{iK} dx^i dx^K$ with signature (+++, -) will be used.

Now, if we change the direction of u^K by an infinitesimal rotation, leaving v^i unchanged, we obtain

$$\delta K = \frac{\partial K}{\partial u^l}\delta u^l = \frac{R_{ikem}v^k v^m - K g_{ikem} v^k v^m}{g_{ikem} u^i u^e v^k v^m} u^l \delta u^i \qquad (3)$$

The condition that $\delta K = 0$ is

$$(R_{ikem}v^k v^m - K g_{ikem} v^k v^m) u^l = 0 \qquad (4)$$

When written in the form $P_{ie}\, u^e = 0$, a solution of (4) apart from the trival one $v^e = 0$, exists only if

$$\| P_{ie} \| = 0$$

Now we are interested in two cases: when v^i is identified with a tangent to a geodesic or when with a vector $\lambda_{(a)}{}^i$ of a tetrad. In the first case let Γ be the geodesic, s the parameter along it such that $v^i = \dfrac{dx^i}{ds}$, $v^i v_i = -1$. Also u^K and δu^k are chosen orthogonal to v^i.

Since (4) is a tensor equation a particular coordinate system can be used to simplify the calculation. If a system in which $v^i = \delta^i_4$, so that $u^4 = \delta u^4 = 0$, is considered, (4) reads

$$(R_{\nu k \mu m} v^k v^m - K g_{\nu k \mu m} v^k v^m) \mu^\mu = 0 \qquad (5)$$

or

$$P_{\nu\mu} u^\mu = 0 \qquad (5')$$

The equation $\| P_{\nu\mu} \| = 0$ is of third degree in K. For each solution a direction u^K orthogonal to v^K is found, such that $\delta K = 0$. If coincident solutions occur a degeneracy arises, otherwise the three vectors are orthogonal to each other.

The physical meanings of these directions becomes apparent when a two parameter family of geodesic $\Gamma(t)$, to which Γ belongs, is considered, If $x^i = x^i(s, t)$ is the equation of a generical geodesic of the family and

$$\eta^i = \frac{\partial x^i}{\partial t}\delta t \quad ; \qquad \eta^i v_i = 0$$

then the equation of geodesic deviation is given by

$$\frac{\delta^2\eta^i}{\delta s^2} + R^i_{\ k\ell m} v^k v^m \eta^\ell = 0 \tag{6}$$

Introducing a tetrad $\lambda^{(a)i}$, with $\lambda^{(4)i} = v^i$, propagated by parallel transport along the geodesic, (6) can be written as

$$\frac{\delta^2\eta^{(\alpha)}}{\delta s^2} + K^{(\alpha)}_{\ (\gamma)}\,\eta^{(\gamma)} = 0 \tag{7}$$

where

$$K^{(\alpha)}_{\ (\gamma)} = R^{(\alpha)}_{\ (4\gamma 4)} \tag{8}$$

In the physical interpretation $\eta^{(\alpha)}$ in (7) stands for the relative coordinates of two particles moving along $\Gamma(t)$ and $\Gamma(t + \delta t)$ and accordingly $-K^{(\alpha)}(\gamma)$ is their relative acceleration due to the gravitational field[2].

When eq. (5) is satisfied (6) and (7) become

$$\frac{\delta^2\eta^{(\alpha)}}{\delta s^2} = -k\, g_{\alpha\mu}\,\eta^\mu \tag{9}$$

$$\frac{\delta^2\eta^{(\alpha)}}{\delta s^2} = -k\,\eta^{(\alpha)} \tag{10}$$

So it is seen that in the direction defined by (5) the relative acceleration has the same direction as the vector joining the two particles and is proportional to the Riemann curvature.(x)

In the second case we take $v^{(K)} = \lambda^{(a)K}$. Summing (1) over (a) we obtain

$$\bar{K} = \frac{R_{ikem}\, u^i u^e \lambda^{(a)k} \lambda^m_{(a)}}{g_{ikem}\, u^i u^e \lambda^{(a)k} \lambda^m_{(a)}} = \frac{1}{3}\, \frac{R_{ie}\, u^i u^e}{g_{ie}\, u^i u^e} \qquad (11)$$

for the average curvature relative to the direction u^e.

Then we have that $\delta \bar{K} = 0$ if

$$(R_{ie} - \bar{K} g_{ie})\, u^e = 0 \qquad (12)$$

For every solution of $\| R_{ie} - \bar{K} g_{ie} \| = 0$ a direction is obtained in which $\delta \bar{K} = 0$. Now from the Einstein equation $R_{ie} = - \lambda\, (T_{ie} - 1/2\, g_{ie}\, T)$, so that

$$\bar{K} = \frac{1}{3} \chi\, \varepsilon_{(u)} (T_{ik}\, u^i u^k) + \frac{\chi}{6} T \qquad (13)$$

As seen from (13) the quantity $T_{iK}\, u^i\, u^K$ is stationary when u^i is defined by (12). This means that if u^K is considered as a vector $\lambda^{(a)K}$ of a new tetrad, the intrinsic component $T^{(aa)}$ of the matter energy momentum has a stationary value with respect to infinitesimal rotations of the tetrad.

Conversely to determine the stationary value of $T^{(aa)}$ under local Lorentz transformations is equivalent to finding the principal directions of R_{iK}.

(x) - It is possible to use the (10) to perform a physical measurement of K for our three directions in every point of space-time.

Bibliography

(1) A tetrad is defined as a complex of four ortho-normal unit vectors $\lambda(a)^i$ where the index in bracket labels the vectors. Latin and Greek indices stand for 1, ..., 4 and 1, 2, 3 respectively.
According to the definition $\lambda^{(a)i} = \varepsilon_a \lambda_{(a)}{}^i$, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $\varepsilon_4 = -1$, it is possible to obtain the relations $\lambda^i_{(a)} \lambda^{(a)j} = g^{ij}$; $\lambda^i_{(a)} \lambda^{(b)}_i = \delta^{(b)}_{(a)}$
For the use of tetrad in general relativity see Pirani F.A.E. - Bull. Acad. Polon. Sc. 5 (1957) 143
Synge J.L. - Relativity: General theory, North Holland (1960).

(2) Pirani F.A.E. - Bull. Acad. Polon. Sc. 5 (1957) 143.