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A Generalization of the Foldy-Wouthuysen Transformation.

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Summary. — A method for constructing a unitary operator which transforms the Hamiltonian of a Dirac electron in the presence of an electric field into an even Hamiltonian is proposed and discussed. The transformation function and the transformed Hamiltonian are expressed through an operator G which satisfies an operator equation; when the electric field is absent (free particle) the Foldy-Wouthuysen transformation is re-derived; when the electric field is present and the equation for G is solved in series of m^{-1} the Pauli, Darwin, Foldy, Wouthuysen non relativistic Hamiltonian is re-obtained; but in addition a method of solution is now possible which converges rapidly and is *not* restricted to the non relativistic case; the expansion parameter in this method of solution is $(1/mc^2)(e\hbar/mc)E$ for an uniform electric field and $Ze^2/\hbar c$ for a Coulomb potential. Though the treatment in this paper considers in detail only the case of an electrostatic potential, the inclusion of any other term in the Hamiltonian is easy and, in particular, a magnetic field may be trivially included.

1. – Introduction.

In a well known paper (1) Foldy and Wouthuysen have shown how it is possible, for the case of a free particle, to obtain a representation of the Dirac Hamiltonian which does not contain odd matrices. Subsequently Case (2) has, among other things, shown that the F. W. transformation can be extended to the case in which a time independent magnetic field is present; in the sense

(1) L. Foldy and S. Wouthuysen: *Phys. Rev.*, **78**, 29 (1950).
(2) K. M. Case: *Phys. Rev.*, **95**, 1323 (1953).

that, also in such case, an unitary transformation can be found which transforms exactly the Dirac Hamiltonian into a Hamiltonian free from odd matrices.

The fact that the Hamiltonian does not contain odd matrices implies that two component wave functions are sufficient for the description of a Dirac particle and in particular makes it possible to discuss the transition from the relativistic to the non-relativistic case; this has been fully discussed by FOLDY and WOUTHUYSEN, who have shown, in addition, how some «paradoxes» arising in the Dirac theory can be solved: for instance the fact that the velocity of a free Dirac particle is always the velocity of light, or that, in the Dirac theory, only the projection of the spin in the direction of the momentum is a constant of the motion for a free particle.

More recently ERIKSEN [3] has considered a more general problem: is it possible to find an unitary transformation leading to an even Hamiltonian also in cases more general than that of a free particle or of a particle in a magnetic field? ERIKSEN has shown that this question may be answered in the affirmative; indeed he has been able to find an expression for the operator U inducing the transformation. While Eriksen's treatment is very appropriate to discuss existence problems, it is not easy to find in general from the expression of U given by ERIKSEN the transformed Hamiltonian, in those cases where the initial Hamiltonian contains both odd and even terms in addition to βmc^2.

The purpose of the present paper is to consider again the same problem (that is to find in general an unitary transformation leading to an even Hamiltonian) using a method different from that followed by ERIKSEN. We shall show that it is possible to give an expression for the transformed Hamiltonian, in terms of an operator G, which satisfies an operator equation; this operator equation cannot, in general, be solved exactly, but approximation methods can be invented to obtain solutions in many cases of interest. We shall confine here for simplicity, to an Hamiltonian containing only an electrostatic potential; this case is already sufficiently general to illustrate the method; it is easy to consider the presence in the Hamiltonian of other terms, and in particular the addition of a magnetic field is trivial.

It should be remarked at this point that the problem of constructing an unitary transformation leading to an even Hamiltonian in the presence of an electric and magnetic field has been already treated by FOLDY and WOUTHUYSEN in their original paper. The way followed by FOLDY and WOUTHUYSEN consisted in removing the odd parts of the Hamiltonian through a sequence of unitary transformations, each of which eliminates the odd terms in the Hamiltonian to one higher order in an expansion parameter which is chosen as m^{-1}, m being the electron mass.

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(3) E. ERIKSEN: *Phys. Rev.*, **111**, 1011 (1958).

It will appear that in the particular case in which the solution of the operator equation mentioned above is constructed by successive approximations in m^{-1} the results of FOLDY and WOUTHUYSEN are reproduced; but in addition we shall suggest an other method of solution which does not depend on the expansion in m^{-1} and is not limited to the non-relativistic case.

2. – A group of transformations of the Dirac equation.

The problem described in the past section is to find an unitary operator U which eliminates the odd parts of the Dirac Hamiltonian H in the presence of an external time independent electric field described by a potential energy V. With the usual meaning of the simbols:

$$H = \gamma_5(\boldsymbol{\sigma}\cdot\boldsymbol{p}) + \beta m + V \,. \tag{1}$$

We want now to find an unitary U $(UU^+ = 1)$ such that:

$$U^+HU = h_1 + \beta h_2 \,, \tag{2}$$

where h_1 and h_2 are hermitian operators constructed through $\boldsymbol{x}$, $\boldsymbol{p}$ and $\boldsymbol{\sigma}$, but not containing odd matrices. From the unitarity of U the equation (2) may be rewritten:

$$HU = U(h_1 + \beta h_2) \,. \tag{3}$$

The most general operator U is now, as well known, a linear combination of the 16 independent Dirac matrices and may be written as:

$$U = A + B\beta + C\gamma_5 + iD\beta\gamma_5 \,, \tag{4}$$

where A, B, C, D are operators not containing β or γ_5 but depending on $\boldsymbol{x}$, $\boldsymbol{p}$ and $\boldsymbol{\sigma}$.

The equation (3) may be written explicitly, using (4) and (2):

$$(\gamma_5\boldsymbol{\sigma}\cdot\boldsymbol{p} + \beta m + V)(A + B\beta + C\gamma_5 + iD\beta\gamma_5) = \\ = (A + B\beta + C\gamma_5 + iD\beta\gamma_5)(h_1 + \beta h_2) \,. \tag{5}$$

For this equation to be satisfied it is necessary and sufficient that the operators which multiply respectively 1, β, γ_5, $\beta\gamma_5$ on the left and right hand side of (5) are individually equal; we therefore obtain, setting $\boldsymbol{\sigma}\cdot\boldsymbol{p} = \Omega$:

$$VA + mB + \Omega C = Ah_1 + Bh_2\,, \tag{6}$$

$$mA + VB - i\Omega D = Bh_1 + Ah_2\,, \tag{7}$$

$$\Omega A + VC + imD = Ch_1 - iDh_2\,, \tag{8}$$

$$-\Omega B + mC + iVD = -Ch_1 + iDh_2\,. \tag{9}$$

We now sum and subtract the equations (6) and (7); and we do the same with the equations (8) and (9); we introduce also the following combinations of operators:

$$\begin{cases} X = A - B & Y = C + iD & R = V + m \\ \overline{X} = A + B & \overline{Y} = C - iD & R' = V - m \end{cases} \tag{10}$$

and

$$\begin{cases} x = h_1 - h_2\,, \\ y = h_1 + h_2\,. \end{cases} \tag{11}$$

We obtain in this way the two independent sets of operator equations:

$$\begin{cases} R\overline{X} + \Omega\overline{Y} = \overline{X}y \\ \Omega\overline{X} + R'\overline{Y} = \overline{Y}y \end{cases} \tag{12}$$

and

$$\begin{cases} R'X + \Omega Y = Xx\,, \\ \Omega X + RY = Yx\,. \end{cases} \tag{13}$$

In the two systems of equations (12) and (13), R, R', Ω are known operators and X, Y, x and $\overline{X}$, $\overline{Y}$, y are unknown operators. If x and y can be determined, h_1 and h_2 are then determined through (11). Similarly if X, Y, $\overline{X}$, $\overline{Y}$ are determined, the operators A, B, C, D are given by (10) and the expression of U is then known.

It must be remarked at this point that we have not yet considered the restrictions on X, Y, $\overline{X}$, $\overline{Y}$, arising from the unitarity condition. This will be done later (Section **3**); for the moment we concentrate on the equations (12) and (13), independently from the unitarity condition.

Consider the system (13); for the two equations to be compatible it is obviously necessary that:

$$X^{-1}(R'X + \Omega Y) = Y^{-1}(\Omega X + RY) \tag{14}$$

an equality which expresses that the operator x obtained from the first equation must be the same as that from the second. In obtaining the equation (14) we have assumed that the inverses both of X and of Y exist; in the following it will be stipulated that this is the case.

We now introduce the operator

$$G = XY^{-1} . \tag{15}$$

Multiplying both sides of the equation (14) by X from the left and by Y^{-1} from the right we obtain the following operator equation for G

$$G(\Omega G + R) = R'G + \Omega . \tag{16}$$

If a solution G of this equation can be found, x is then given by:

$$x = Y^{-1}(\Omega G + R)Y \tag{17}$$

and a relation between X and Y is established through:

$$X = GY . \tag{18}$$

In a completely similar way we have from the system (12):

$$\bar{G}(\Omega\bar{G} + R') = R\bar{G} + \Omega \tag{19}$$

and

$$y = \bar{Y}^{-1}(\Omega\bar{G} + R')\,\bar{Y} , \tag{20}$$

where now

$$\bar{X} = \bar{G}\bar{Y} . \tag{21}$$

Obviously the equations (17) and (20) show that x and y (and therefore h_1 and h_2) are not uniquely determined, even if the solutions G and $\bar{G}$ of the equations (16) and (19) were unique. This freedom in x, y, which is of course expected, will be discussed in Section **4**.

We end this section by rewriting the equations (16) and (19) in a more explicit form: using (10) we have:

$$G\Omega G + GV - VG + 2mG - \Omega = 0 \tag{22}$$

and

$$\bar{G}\Omega\bar{G} + \bar{G}V - V\bar{G} - 2m\bar{G} - \Omega = 0 . \tag{23}$$

The invariance of the above equations with respect to the substitution $V \to V + \text{const.}$ should be noted.

3. – The condition of unitarity.

We now determine the restrictions on A, B, C, D, or equivalently on X, Y, $\bar{X}$, $\bar{Y}$ implied by the unitarity condition. When the expression (4) for U is inserted in the equation $U^+U=1$, four bilinear equations in A, B, C, D result; these four equations, when reexpressed using (10) in terms of X, Y, $\bar{X}$, $\bar{Y}$ are:

$$\bar{X}^+\bar{X}+\bar{Y}^+\bar{Y}=1\,, \tag{24}$$

$$X^+X+Y^+Y=1\,, \tag{25}$$

$$\bar{Y}^+X+\bar{X}^+Y=0\,, \tag{26}$$

$$Y^+\bar{X}+X^+\bar{Y}=0\,. \tag{27}$$

Recalling the equations (18) and (21) we obtain from the equation (26)

$$\bar{Y}^+\bar{G}^+Y+\bar{Y}^+GY=0$$

that is

$$\bar{G}^+=-G\,. \tag{28}$$

It follows that the operator $\bar{G}$ is determined from the operator G by the unitarity condition; it can be easily checked that the relation (28) is compatible with the equation (22) and (23) which have to be satisfied by G and $\bar{G}$. In fact the hermitian conjugate of the equation (23) is:

$$\bar{G}^+\Omega\bar{G}^++V\bar{G}^+-\bar{G}^+V-2m\bar{G}^+-\Omega=0 \tag{29}$$

which is just the equation satisfied by $-G$. In writing (29) use has been made of the hermiticity of Ω and V.

Coming back to the equations (24) to (27) we note that since the equation (27) is simply the hermitian conjugate of the equation (26) it is already satisfied by (28); we therefore have still to consider only the equations (24) and (25). They give respectively:

$$\bar{Y}^+\bar{G}^+\bar{G}\bar{Y}+\bar{Y}^+\bar{Y}=1 \tag{30}$$

and

$$Y^+G^+GY+Y^+Y=1 \tag{31}$$

which may be rewritten, using also the condition (28):

$$\overline{Y}\overline{Y}^{+} = (1 + \overline{G}^{+}\overline{G})^{-1} = (1 + GG^{+})^{-1}, \tag{32}$$

$$YY^{+} = (1 + G^{+}G)^{-1}. \tag{33}$$

The equations (32) and (33) are conditions on $\overline{Y}$ and Y. We notice that they may be in particular satisfied if we choose $\overline{Y}$ and Y to be hermitian operators, respectively equal to [4]:

$$\overline{Y} = -(1 + GG^{+})^{-\frac{1}{2}}, \tag{34}$$

$$Y = (1 + G^{+}G)^{-\frac{1}{2}}. \tag{35}$$

To end this section we wish to check explicitly that the operators x and y defined by (17) and (20) are hermitian as they must be. Let us consider the expression (17) of x; using (33) we may write it as:

$$\begin{aligned} x &= Y^{+}(1 + G^{+}G)(\Omega G + R)\,Y = \\ &= Y^{+}(\Omega G + R + G^{+}G\Omega G + G^{+}GR)\,Y = \\ &= Y^{+}(\Omega G + R + G^{+}\Omega + G^{+}R'G)\,Y = x^{+}. \end{aligned} \tag{36}$$

A similar check can be made for y which can be written:

$$y = \overline{Y}^{+}(\Omega\overline{G} + R' + \overline{G}^{+}R\overline{G} + \overline{G}^{+}\Omega)\overline{Y} = y^{+}. \tag{37}$$

4. – The arbitrariness in x and y for a given G.

The results of the Sections **2** and **3** show that, if an operator G which obeys the equation (22) can be found, the operators x and y or equivalently h_1 and h_2 can be determined. In fact once a G has been found the equations (34) and (35) may be used to determine a particular choice of $\overline{Y}$ and Y; in addition $\overline{G}$ is simply equal to $-G^{+}$, so that all the operators which appear in the expressions (17) and (20) for x and y are available.

The question which we now ask is: having found an operator G which satisfies the equation (22) which is the arbitrariness in h_1 and h_2? It is easy to

(4) The choice of the $-$ sign in eq. (34) has been made simply to re-obtain the same result of ref. (2) for U in the case of a free particle, but has no particular meaning; compare Sect. 4.

answer this question simply by looking at the expressions which give x and y; it is convenient for this purpose to use the expressions (36) and (37). It is then apparent that for a given G the arbitrariness which remains in x is simply the arbitrariness in the choice of Y; the unitarity condition only fixes (compare the equation (33)) the product YY^+ but not separately Y and Y^+; if therefore Y is a particular operator satisfying the equation (33) (*e.g.* the operator (35)) the most general operator which still satisfies the equation (33) is:

$$Y' = YT\,, \tag{38}$$

where T is an arbitrary unitary operator; it follows from the equation (36) that x is correspondingly transformed into:

$$x' = T^+ x T\,. \tag{39}$$

For a given G this is, therefore, the arbitrariness in x. A similar arbitrariness exists for y; in fact if $\overline{Y}$ is a particular solution of the equation (32) (*e.g.* the operator (34)) another solution is $\overline{Y}' = \overline{Y}S$, where S is again an unitary operator independent from T. From the equation (37) it then follows that:

$$y' = S^+ y S\,. \tag{40}$$

The origin of the arbitrariness in x and y which we have just illustrated (we shall call it «normal») is clear: at the beginning of the Section **2** we did ask to find an unitary operator U which transforms a given Hamiltonian into a Hamiltonian free from odd operators, that is of the form $h_1 + \beta h_2$; but we have not put restrictions on the form of h_1 and h_2. This implies that, once a particular operator $h_1 + \beta h_2$ has been found, the possibility remains of obtaining other even hamiltonians through unitary transformations produced by unitary operators of the form $L + \beta M$, where L and M are arbitrary operators constructed through $\boldsymbol{x}$, $\boldsymbol{p}$ and $\boldsymbol{\sigma}$. It is apparent that the normal arbitrariness illustrated above is just the arbitrariness which is present in such unitary trasnformations; it is easy to find the connection between the operators L and M and S and T previously introduced; we do not need however to write down the explicit formulas.

In addition to the normal arbitrariness in x and y there is another kind of arbitrariness; this stems from the fact that in general different solutions for the equation (22) for G can exist; it is not yet clear to us what is the extent and the meaning of this arbitrariness; some remarks concerning this point will be made in Section **9**.

5. – The operator equation for G.

We now discuss the equation (22) for G; we shall from now on use respectively mc^2 and $\hbar/mc$ as units for energy and length. In these units the equation for G is simply written:

$$GΩG + GV - VG + 2G - Ω = 0\,, \tag{22'}$$

where now all the quantities which appear in the equation are dimensionless.

a) Free particle. As a first example we consider the case of a free particle ($V = 0$), simply to check that we obtain the same results as FOLDY and WOUTHUYSEN. The operator G for this case will be called G_0; we have:

$$G_0ΩG_0 + 2G_0 - Ω = 0\,. \tag{41}$$

Obviously G_0 is a function of the operator $Ω$ only, so that $Ω$ and G_0 commute and the solution of the equation (41) proceeds as if G_0 were a C number. We have two solutions:

$$G_0^{(1)} = \frac{-1+\sqrt{1+Ω^2}}{Ω} = Ω\,\frac{\left(-1+\sqrt{1+p^2}\right)}{p^2} = Ω\,\frac{1}{1+\sqrt{1+p^2}}\,, \tag{42}$$

$$G_0^{(2)} = \frac{Ω\left(-1-\sqrt{1+p^2}\right)}{p^2} = -\frac{Ω}{\sqrt{p^2+1}-1}\,, \tag{43}$$

where the properties $Ω^2 = p_x^2 + p_y^2 + p_z^2 = p^2$ and $Ω^{-1} = Ωp^{-2}$ have been used. Recalling the expressions (17) and (20) for x and y, noting that in this case Y^{-1} or $\overline{Y}^{-1}$ may be freely transferred to the right, using for x and y our system of units and taking the solution $G_0^{(1)}$ we get

$$x = \sqrt{1+p^2}\,,$$
$$y = -\sqrt{1+p^2}\,,$$

where the equation (28) has been used.

In the transformed hamiltonian one has accordingly:

$$h_1 = \tfrac{1}{2}(x+y) = 0\,, \qquad h_2 = \tfrac{1}{2}(y-x) = -\sqrt{1+p^2}\,,$$

in agreement with the result of FOLDY and WOUTHUYSEN.

Had we chosen the solution $G_0^{(2)}$ we would have obtained:

$$h_1 = 0\,, \qquad h_2 = +\sqrt{1+p^2}\,,$$

which differs from the previous result only for having exchanged the positive with the negative energy states.

It is also straightforward to deduce an expression for U, in this case; the general expression of U, in terms of G, will be given in Section **8** (formula (61)). Specializing this expression to the present case, using the solution $G^{(1)}$ and the formulas (34) and (35) of Section **3** we get:

$$U = G_0^{(1)} \frac{1}{\sqrt{1+G_0^{(1)2}}} + \beta\gamma_5 \frac{1}{\sqrt{1+G_0^{(1)2}}}\,. \tag{44}$$

Putting

$$\cos\frac{\varphi}{2} = G_0^{(1)} \frac{1}{\sqrt{1+G_0^{(1)2}}}\,,$$

and consequently $\operatorname{tg}\varphi = -\Omega$ the expression (44) may be rewritten:

$$U = \cos\frac{\varphi}{2} + \beta\gamma_5 \sin\frac{\varphi}{2} = \exp\left[\beta\gamma_5\frac{\varphi}{2}\right],$$

in agreement with ref. (2).

b) It does not seem possible to find a formal solution of the equation (22′) in an arbitrary field. It is possible to find a solution in the particular case of an uniform electric field (compare the Section **9**); or it is possible to try to construct the solution by successive approximations in V (we recall that V is now adimensional, being equal to the potential energy, divided by mc^2). This will be done in the next Section.

6. – The perturbative solution of the operator equation for G.

The terms containing V in the equation (22′) are usually small with respect to the others; in fact the quantity V in (22′) appears usually multiplied by a very small coefficient whene verything is made adimensional. For instance in the case of an uniform electric field (putting $\tilde{\boldsymbol{r}} = (mc/\hbar)\boldsymbol{r}$)

$$V = -\frac{e}{mc^2}\boldsymbol{E}\cdot\boldsymbol{r} \equiv -\tilde{\boldsymbol{E}}\cdot\boldsymbol{r}\frac{mc}{\hbar} \equiv -\tilde{\boldsymbol{E}}\cdot\tilde{\boldsymbol{r}}\,, \tag{45}$$

with $\tilde{E} = (e\hbar/mc)(1/mc^2)E$, a quantity which is exceedingly small ($< 10^{-12}$) even for the strongest macroscopic electric fields obtainable.

For the case of a Coulomb field:

$$V = \frac{Ze^2}{mc^2}\frac{mc}{\hbar}\frac{1}{\tilde{r}} = \frac{Ze^2}{\hbar c}\frac{1}{\tilde{r}}, \tag{46}$$

which is reasonably small for small Z.

It appears therefore reasonable to try to get the solution of the equation (22′) by perturbation theory, regarding V as small; this is what will be done in this Section.

We write:

$$G = G_0 + G_1 + G_2 + \dots, \tag{47}$$

where the lower index characterizes the order in V; and we proceed to determine the succeeding operators of the series by substituting the expansion (47) in the equation (22′) and equating to zero terms of any given order in V. In this way we obtain, of course, for G_0 the equation (41); for G_1 we have:

$$G_1\Omega G_0 + G_0\Omega G_1 + 2G_1 = VG_0 - G_0V. \tag{48}$$

For G_2 we obtain:

$$G_2\Omega G_0 + G_0\Omega G_2 + 2G_2 = VG_1 - G_1V - G_1\Omega G_1. \tag{49}$$

For the successive terms we have similar equations where in each case terms of any order are determined from the knowledge of terms of lower order.

We now choose for G_0 the solution $G_0^{(1)}$ given by (42) and we confine here to the determination of G_1. From (48), using a representation in which σ_z and $\boldsymbol{p}$ are diagonal we have:

$$\langle \boldsymbol{p}'\lambda' | G_1 | \boldsymbol{p}''\lambda''\rangle = \boldsymbol{\sigma}_{\lambda'\lambda''} \cdot \left\{\frac{\langle \boldsymbol{p}' | V | \boldsymbol{p}''\rangle \boldsymbol{p}''(\sqrt{1+p''^2}+1)^{-1} - \boldsymbol{p}'(\sqrt{1+p'^2}+1)^{-1}\langle \boldsymbol{p}' | V | \boldsymbol{p}''\rangle}{\sqrt{1+p'^2}+\sqrt{1+p''^2}}\right\}. \tag{50}$$

It is possible to write G_1 as follows:

$$G_1 = \int_{-\infty}^{0} \mathrm{d}s \exp\left[s\sqrt{1+p^2}\right]\left[V, \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{1+\sqrt{1+p^2}}\right]\exp\left[s\sqrt{1+p^2}\right], \tag{51}$$

where the brackets in the integrand indicate the commutator.

It is important to recognize that, on account of the smallness of V pointed out at the beginning of this section, the term G_1 is sufficient for all practical purposes, at least in the case of macroscopic fields. Of course the convergence of the whole procedure has to be examined; some remarks on this point will be made in Section **9**.

It is also important to stress that the expansion discussed in this Section is *not* a non-relativistic approximation; it is simply an expansion in terms of a well defined parameter; in other words the formula (51), in particular, is completely general as far as the velocity of our electron is concerned.

We can now exhibit the connection between our procedure and the non-relativistic expansion in m^{-1} discussed by FOLDY and WOUTHUYSEN which we have mentioned in the introduction; this is very simple.

The non-relativistic limit of Pauli, Darwin, Foldy and Wouthuysen is simply obtained from our formulas by expanding G in series of m^{-1} and limiting ourselves to the terms of order m^{-2}.

We recall that our p stays for p/m and our V is V/m; it is then straightforward to show that to take into account all the terms up to the order m^{-2} it is sufficient to confine ourselves to G_0 and G_1; in other words G_2 is already of order m^{-4} and it does not contribute to terms of order m^{-2} in the expressions of h_1 and h_2.

To be more definite we note below the relevant terms in the expansion in m^{-1} of the various quantities which are necessary for the determination of h_1 and h_2 including terms of order m^{-2}. In order to exhibit the mass explicitely we shall write the formulas below in conventional units. We have:

$$G_0^{(1)} = \frac{1}{2}\frac{\Omega}{m} + 0\left(\frac{1}{m^3}\right), \tag{52}$$

$$G_1 = \int_{-\infty}^{0} \exp\,[2s]\,\mathrm{d}s \left(\frac{V\Omega}{2m^2} - \frac{\Omega V}{2m^2}\right) + 0\left(\frac{1}{m^4}\right) = -\frac{i\hbar e}{4m^2}\,(\boldsymbol{\sigma}\cdot\boldsymbol{E}) + 0\left(\frac{1}{m^4}\right) \tag{53}$$

$$[e\boldsymbol{E} = -\operatorname{grad} V],$$

$$G_2 = 0\left(\frac{1}{m^4}\right). \tag{54}$$

The expressions (17) and (20) of x and y may now be written:

$$x = m\,\Upsilon^{-1}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\,[G_0^{(1)} + G_1] + \frac{V}{m} + 1\right)\Upsilon, \tag{55}$$

$$y = m\,\overline{\Upsilon}^{-1}\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{m}\,[-G_0^{(1)} + G_1] + \frac{V}{m} - 1\right)\overline{\Upsilon}, \tag{56}$$

where use has been made of the relation $\bar{G}^+ = -G$ and of the antihermiticity of G_1. It appears from (55), (56) that $\bar{Y}^{-1}$, Y, Y^{-1} and $\bar{Y}$ have to be calculated including terms of order m^{-3}.

We simply give here the expression of Y:

$$Y = \frac{1}{\sqrt{1+G^+G}} \cong \frac{1}{\sqrt{1+G_0^2+G_0G_1-G_1G_0}} = \\ = 1 - \frac{1}{8}\frac{p^2}{m^2} + \frac{1}{16}\frac{e\hbar^2}{m^3}\operatorname{div}\boldsymbol{E} + \frac{e\hbar}{8m^3}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\wedge\boldsymbol{p}) . \tag{57}$$

To obtain Y^{-1} simply change the sign of all the ferms in Y except the first: $-\bar{Y}$ differs from Y for a change in sign in the terms of order m^{-3} and $-\bar{Y}^{-1}$ differs in the same way from Y^{-1}.

Though we have written down the Y's correct to the order m^{-3}, the terms of order m^{-3} disappear in the calculation of x and y because $\operatorname{div}\boldsymbol{E}$ and $\boldsymbol{\sigma}\cdot(\boldsymbol{E}\wedge\boldsymbol{p})$ both commute with V. Hence we obtain:

$$x = m\left(1+\frac{p^2}{8m^2}\right)\left(\frac{p^2}{2m^2} - \frac{i\hbar e}{4m^3}(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{E}) + \frac{V}{m} + 1\right)\left(1-\frac{p^2}{8m^2}\right),$$

$$y = m\left(1+\frac{p^2}{8m^2}\right)\left(-\frac{p^2}{2m^2} - \frac{i\hbar e}{4m^3}\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{E})}{m^3} + \frac{V}{m} - 1\right)\left(1-\frac{p^2}{8m^2}\right).$$

We then get easily:

$$\left\{\begin{aligned} h_1 &= -\frac{e\hbar^2}{8m^2}\operatorname{div}\boldsymbol{E} - \frac{e\hbar}{4m^2}\boldsymbol{\sigma}\cdot(\boldsymbol{E}\wedge\boldsymbol{p}) + V, \\ h_2 &= -\frac{p^2}{2m} - m, \end{aligned}\right. \tag{58}$$

which is the same result of Foldy and Wouthuysen, for a pure electric field.

The results of this Section may be summarized as follows: we have made an expansion of G in terms of the adimensional quantity V/m, a quantity which is usually very small. In this expansion, of which (42) and (51) give the zero and first order terms, no non-relativistic approximation is made; when each term of the expansion is further expanded in series of m^{-1}, the non-relativistic Foldy-Wouthuysen expansion is reobtained.

7. – The evaluation of G_1 for an uniform electric field.

The sense in which our series expansion is a generalization of the Foldy-Wouthuysen one may be fully appreciated if the example of an uniform electric field is considered. In such case G_1 may be easily determined exactly and h_1, h_2, may therefore be given exactly to the first order in $\tilde{E}$.

We have:

$$G_1 = -\frac{i}{2}\,\frac{(\widetilde{\boldsymbol{E}}\cdot\boldsymbol{\sigma})\sqrt{1+p^2}\,(1+\sqrt{1+p^2}) - (\widetilde{\boldsymbol{E}}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\boldsymbol{p})}{(1+p^2)(1+\sqrt{1+p^2})^2}, \tag{59}$$

where V has been again written, like in (45), as:

$$V = -\widetilde{\boldsymbol{E}}\cdot\widetilde{\boldsymbol{r}}\,.$$

The calculation of h_1 and h_2 gives:

$$h_1 = V + \frac{m}{2}\,\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\wedge\widetilde{\boldsymbol{E}}}{\sqrt{1+p^2}\,(1+\sqrt{1+p^2})}, \tag{60_1}$$

$$h_2 = -m\sqrt{1+p^2}\,. \tag{60_2}$$

The formulas are correct to the first order in the expansion parameter $\widetilde{E}$ and to all orders in m^{-1}; from the formulas (60), the non-relativistic limit for the present case is obtained simply neglecting p^2 with respect to the unity in the second term on the right hand side of (60_1).

8. – Comparison with the results of Eriksen [5].

These results of the past Sections show that, if a G exists, the transformed hamiltonian is even; therefore for the transformed hamiltonian $h_1+\beta h_2$, β is a constant of the motion. This implies that $U\beta U^+$ is a constant of the motion for the initial hamiltonian H. Since the eigenvalues of β and $U\beta U^+$ must be the same, that is ± 1, ERIKSEN has observed that it should be possible to identify $U\beta U^+$ with an operator commuting with H and having only the two eigenvalues ± 1; he has called this operator λ and has remarked that a possible explicit expression for λ may be: $H/(H^2)^{\frac{1}{2}}$.

ERIKSEN then proceeds to determine U in terms of λ from the equation $U\beta U^+ = \lambda$ to which he adds the further restriction $U\beta = \beta U^+$ which is seen to be compatible with the previous one. As we have said in the introduction ERIKSEN has found an expression for U in terms of λ; this expression is useful for discussing existence problems; our method is perhaps more appropriate to determine the transformed hamiltonian explicitly, as we did already mention in the introduction. Obviously however a relation exists between the two procedures; we shall briefly show below which is the connection.

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[5] Our U^+ is Eriksen's U.

From our formulas it is easy to obtain an expression for U; it is

$$(61)\qquad U = \tfrac{1}{2}[(G+\gamma_5)(1-\beta)Y + (-G^+ + \gamma_5)(1+\beta)\overline{Y}]\,.$$

The connection between our G and Eriksen's λ [6] can be established constructing $U\beta U^+$ and identifying this quantity with λ: Explicitly:

$$(62)\qquad U\beta U^+ = \frac{1}{1+GG^+} - \frac{1}{1+G^+G} + \beta\left(1 - \frac{1}{1+GG^+} - \frac{1}{1+G^+G}\right) -$$

$$-\gamma_5\left(G^+\frac{1}{1+GG^+} + G\frac{1}{1+G^+G}\right) + \beta\gamma_5\left(G^+\frac{1}{1+GG^+} - G\frac{1}{1+G^+G}\right) = \lambda\,.$$

Equation (62) shows that Eriksen's λ can be expressed through our G; however Eriksen's condition $U\beta = \beta U^+$ does not correspond to our particular hermitian choice of Y and $\overline{Y}$, but corresponds instead to the choice: $GY = Y^+G^+$ $YG = G^+Y^+$, $Y = -\overline{Y}^+$; these equations and the equations (32), (33) can be satisfied with $Y = G^{-1}(GG^+/(1+GG^+))^{\frac{1}{2}}$.

It should be noticed that the equation (62) is important for establishing the existence of a G once the existence of a λ is established; in fact if we write λ as $\lambda_1+\beta\lambda_2+\gamma_5\lambda_3+\beta\gamma_5\lambda_4$ G can be explicitly expressed in terms of λ_1, λ_2, λ_3, λ_4, by:

$$G = (-\lambda_3 + \lambda_4)(1 - \lambda_1 - \lambda_2)^{-1}\,.$$

This observation implies that the result of ERIKSEN showing the existence of an U entails necessarily the existence of a G, solution of the equation (22′).

9. – Some final problems.

This Section contains a few comments on some questions which arise naturally from the previous treatment; they are: *a*) how many solutions of the operator equation (22′) for G do exist? *b*) Which is the relation between the class of transformed (even) hamiltonians obtained using a particular solution and the class obtained using another solution? *c*) Does the sum of the perturbative series of Section 6 in V/m give a solution of the equation for G and which is the connection between this perturbative expansion and the general solution?

We are not able, at present, to answer these quenstions: we confine to

(6) When speaking of a λ from now on we do not imply that this must be the same as Eriksen's particular choice.

illustrate them on the simple example of the uniform electric field where the operator equation for G reduces to a differential equation; we have already examined the case of the uniform field in the first perturbative approximation in Section **7**.

It is easy to see that, since for an uniform field

$$GV - VG = - i\boldsymbol{E}\cdot \operatorname{grad}_p G$$

the equation (22′) becomes a differential equation.

If we write:

$$G = \frac{\boldsymbol{\sigma}(\boldsymbol{\epsilon}\mathscr{A} - \boldsymbol{p})}{\mathscr{B} - 1}\,,$$

where $\boldsymbol{\epsilon}$ is a unit vector in the direction of B and $\mathscr{A}$, $\mathscr{B}$ are some functions of p_x, p_y, p_z, it is easy to see that $\mathscr{A}$ and $\mathscr{B}$ satisfy a pair of coupled differential equations. If we further introduce the function Z of p_x, p_y, p_z through:

$$\mathscr{A} = \frac{p^2 + 1 - Z^2}{2(\boldsymbol{p}\cdot\boldsymbol{\epsilon} - Z)} \qquad \text{and} \qquad \mathscr{B} = Z - \frac{p^2 + 1 - Z^2}{2(\boldsymbol{p}\cdot\boldsymbol{\epsilon} - Z)}\,,$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$, the problem reduces to find a function Z which satisfies the following ordinary differential equation:

$$Z^2 - i\widetilde{E}\,\frac{\mathrm{d}Z}{\mathrm{d}p_x} - p^2 - 1 + i\widetilde{E} = 0\,. \tag{63}$$

Here the direction of $\boldsymbol{\epsilon}$ has been assumed to be that of the x axis. The equation (63) is a Riccati equation with complex coefficients. It is therefore clear not only that a solution exists, but also that a manifold of them exist depending on an arbitrary complex constant C.

The questions which have been raised at the beginning of this Section reduce therefore in the present case to: 1) which relation exists between the transformed hamiltonian corresponding to different choices of the above mentioned complex constant? 2) It is possible to find two solutions of the equation (63) corresponding to two values of C, which are analytical in E near $E = 0$ and may be constructed by the perturbative approach? Notice that also here, as in general, there are two different perturbative series, one starting with $Z_0 = \sqrt{p^2+1}$ the other with $Z_0 = -\sqrt{p^2+1}$.

Now, in the present case, it would be clearly possible, using known results in the theory of differential equations, to give a detailed answer to the above questions 1) and 2). Since, however, the example of the uniform field was discussed here only with the purpose of making more concrete the general questions introduced at the beginning of this section, we shall stop our analysis here, adding only two remarks: the first is that, both in general and in the particular case of the uniform field, the simplest answer to our questions would be that the perturbation solutions converge at least asymptotically and at least for a certain range of values of the expansion parameter to a solution of the equation for G, while all the other solutions have some kind of irregularity which makes them not usable for the construction of the transformed hamiltonian; that this is true has been, however, not proved. The second remark is that, in the case of the uniform field at least an asymptotic convergence of the perturbative series exists almost certainly as one may conjecture from general theorems [7] on non-linear differential equations belonging to a class very near to that of equation (63).

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[7] W. WASOV: *Asymptotic properties of non-linear analytic differential equations. Proc. of the Varenna Course 1954 Sponsored by C.I.M.E.* I thank prof. CONTI of the department of Mathematics for having indicated this reference to me.

RIASSUNTO

Viene proposto e discusso un metodo di costruzione di un operatore unitario che trasforma l'Hamiltoniana di Dirac in presenza di un campo elettrico in un'Hamiltoniana priva di matrici dispari. La funzione di trasformazione e l'Hamiltoniana trasformata sono espresse mediante un operatore G che soddisfa un'equazione operatoriale; quando il campo elettrico è assente (particella libera) si ricava di nuovo la trasformazione di Foldy-Wouthuysen; quando il campo elettrico è presente e l'equazione per G è risolta in serie di m^{-1} si riottiene l'Hamiltoniana non relativistica di Pauli, Darwin, Foldy, Wouthuysen; ma in aggiunta un nuovo metodo di soluzione è possibile che converge rapidamente e non è limitato al caso non relativistico; il parametro di sviluppo in questo metodo è $(1/mc^2)(e\hbar/mc)E$ per un campo elettrico uniforme e $Ze^2/\hbar c$ per un potenziale coulombiano. Sebbene in questo lavoro si consideri soltanto il caso di un potenziale elettrostatico è facile includere un qualsiasi altro termine nell'Hamiltoniana ed in particolare l'introduzione di un campo magnetico non presenta alcuna difficoltà.