Laboratori Nazionali di Frascati

LNF-60/12 (27.4.60)

C. Bernardini, B. Touschek: ON THE QUANTUM LOSSES IN AN ELECTRON SYNCHROTRON.

Laboratori Nazionali di Frascati del CNRN
Servizio Documentazione

Nota interna: n° 34
27 Aprile 1960

C. Bernardini and B. Touschek(x):

ON THE QUANTUM LOSSES IN AN ELECTRON SYNCHROTRON

A diffusion equation, describing the phase motion of electrons in a synchrotron is derived and solved. It is shown that non linear corrections to the Christy[1] formula are small.

1) Introduction.

The present paper deals with the loss of particles from a synchrotron or a storage ring due to the quantum effects of synchrotron radiation. In particular we consider the loss of particles caused by the stochastic excitation of synchrotron oscillations.

According to the work of Christy there is a strong exponential dependence of the lifetime on the driving voltage of the radio frequency, such that in some practical cases exponents of the order 10 or 20 may be expected; on the other

---

(x) Istituto Nazionale di Fisica Nucleare, Sezione di Roma.

hand - and we have to rely on description of Christy's unpublished work - it appears that in the derivation of the final results certain approximations had to be made. It is clear, that in view of the largeness of the exponent small corrections of this exponent may result in considerable corrections of the lifetime. Recently Sands[2] has compared Christy's theory with experiment. The agreement reached is very satisfactory, but the driving voltage had to be 'renormalized' by 10%. This immediately raises the problem: is this renormalisation to be explained in terms of non linear corrections to the exponent? The answer to this question given in the present paper is: no. Indeed the exponent coincides exactly also in a non linear theory with the exponent calculated by Christy.

Section 1 of the present paper derives the synchrotron equations in Hamiltonian form, the influence of damping and stochastic perturbations being treated as external (non Hamiltonian) forces. In section 3 the Fokker Planck method is applied in order to obtain a diffusion equation in the phase space of the system. In section 4 we introduce action and angle variables. Those permit one to write the diffusion equation in a particularly simple form and to introduce a canonical distribution in phase space. In section 5 we give an accurate solution of the Fokker Planck equation. Instead of solving the timedependent problem, we make the system stationary by introducing a source density of particles which compensates exactly for the loss of particles at the stability limit. The lifetime is then determined as a function of this source density and given in the form of a double integral.

In section 6 we discuss these limiting cases for which the lifetime can be calculated in closed form. For big exponents one finds that the lifetime is independent of the particular distribution of the sources in phase space and is given by the Christy formula multiplied by a correction factor of the order of 0.8: this correction factor is expressed in terms of the action integral at the stability limit. For moderately large exponents the Christy formula is no longer valid. In this case it is still possible to give an expression for the lifetime for a constant source distribution. A closed expression is given in this case and the nonlinear terms are treated as a perturbation. This gives a generalisation of the Christy formula, which is also valid for very small exponents, but which is limited to constant source densities. Finally the case of a δ - source is treated.

In section 7 we give a brief resummè of the factors which influence the exponent.

Owing to the dispersion of the literature of the subject it seems very hard for the student to find his way through the wondrous maze of cross references. We therefore apologize if we have not done full justice to other workers in the field. In view of this fact we do not want to consider the present paper as a very original contribution, but rather as a work of clarification. With this scope in mind the sections 2, 7 and 8 have been introduced, though they could have been dispensed with by a reference to the existing literature.

2) The Synchrotron Equations.

The phase oscillations in a synchrotron are governed by the equation[(1)]

$$(1) \qquad \ddot{\varphi} + \rho_s \dot{\varphi} + \Omega_o^2 (\sin \varphi - \sin \varphi_s) = \dot{\mathcal{E}}(t)$$

Here φ is (minus) the reduced phase of the synchrotron oscillations: $\varphi = -k(\omega - \omega_s)t$, ω is the actual frequency of revolution of the electron, ω_s the synchronous frequency, $k = 1, 2, \ldots\ldots$ is the 'harmonic index' of the radio frequency, which is assumed to vary in time as $V \sin k \omega_s t$. ρ_s is the damping factor of the synchrotron oscillations, defined by

$$(2) \qquad \rho_s = (4 - \alpha)\ P_s/E_s \qquad (\text{secs}^{-1})$$

is the so called momentum compaction factor (in a weak focussing machine $\alpha = 1/1-n$, where $0 < n < 1$ is minus the logarithmic derivative of the field on the equilibrium orbit; generally $\alpha = p\, dr/r\, dp$, where dr is the change of equilibrium radius and dp the change of momentum). P_s is the average power radiated by the synchronous electron of energy E_s. The damping term in (1) is due to the fact that the power radiated depends on the radius, which in turn varies with a change of phase. For P_s one has

$$(3) \qquad P_s = \frac{2}{3} \frac{r_o\ c}{R^2} \left(\frac{E_s}{m\ c^2}\right)^4 m\ c^2 \qquad (\text{eV secs}^{-1})$$

where $r_o = 2.8\ 10^{-13}$ cms is the classical electron radius and m is its mass. R is the equilibrium radius of the electron orbit: $R = c/\omega_s$ for a machine without straight sections. Ω_o is the 'limiting frequency' of the synchrotron oscillations. It is defined by

$$(4) \qquad \Omega_0^2 \, t_s^2 = 2 \pi k \alpha \frac{eV}{E_s}$$

in which $t_s = 2\pi/\omega_s$ is the period of revolution of the synchronous electron. The synchronous phase φ_s is defined by

$$(5) \qquad \sin \varphi_s = \frac{P_s \, t_s}{eV}$$

It is the phase with which the synchronous electron passes the gap of the RF: the synchronous electron has its average energy loss covered by the energy picked up in passing the gap.

Finally $\dot{\mathcal{E}}(t)$ is a driving force due to the stochastic nature of the radiation losses. One has

$$(6) \qquad \dot{\mathcal{E}}(t) = 2 \pi k \alpha \frac{\delta P_s}{t_s \, E_s}$$

in which δP_s is the fluctuation of the radiated power.

For the following it is convenient to put equation (1) into Hamiltonian form. Putting $\psi = \varphi - \varphi_s$ one can write for the Hamiltonian

$$(7) \qquad H = \frac{1}{2} \chi^2 + V(\psi)$$

in which χ figures as momentum variable and ψ as coordinate. The 'potential energy' $V(\psi)$ is defined as

$$(8) \qquad V(\psi) = \Omega_0^2 (2 \cos \varphi_s \sin^2 \frac{\psi}{2} - \sin \varphi_s (\psi - \sin \psi))$$

and normalized in such a way that $V(0) = 0$. The Hamiltonian takes care only of the conservative part of the phase oscilla tions: damping and the stochastic term are considered external forces. It is then easy to verify that instead of the equations (1) one can write

$$(9)\qquad \dot{\psi} = \frac{\partial H}{\partial \chi}, \qquad \dot{\chi} = -\frac{\partial H}{\partial \psi} - \rho_s \chi + \dot{\mathcal{E}}(t)$$

It is well known that, if φ becomes larger than $\pi - \varphi_s$ the phase oscillations become unstable. For this value of φ - which corresponds to $\psi = \pi - 2\varphi_s$, $V(\psi)$ has a maximum and the restoring force $-\partial V/\partial\psi$ therefore vanishes. It follows that the stable region of H is defined by the inequality

$$(10)\qquad 0 \leq H < \overline{H}$$

and that $\overline{H} = V(\pi - 2\varphi_s)$. Using (8) one finds

$$(11)\qquad \overline{H} = \Omega_0^2(2\cos\varphi_s - (\pi - 2\varphi_s)\sin\varphi_s) = \Omega_0^2 f(\varphi_s)$$

in which the last equation defines $f(\varphi_s)$. The useful range of φ_s is $0 \leq \varphi_s \leq \pi/2$. For $\varphi_s = \mathcal{E} << 1$ (high RF voltage) one has $\overline{H} = \Omega_0^2(2 - \varepsilon\pi)$ and in the limiting case of a small RF voltage: $\varphi_s = \pi/2 - \mathcal{E}$: $\overline{H} = \frac{1}{6}\Omega_0^2\,\varepsilon^2$.

3) The Diffusion Equation.

The motion of the representative point of an electron represented by a closed loop in the 'phase-diagram' in the $\chi\psi$ - plane is - according to equation (9) - disturbed by two effects: the damping, which will tend to make the representative point revolve in ever decreasing loops and the stochastic driving force, which will have the opposite effect. To describe the play of these non conservative forces, we introduce a density $W(\chi,\psi;t)$; $W(\chi,\psi;t)d\chi\, d\psi$ is the probability of finding a representative point in $d\chi\, d\psi$ at time t. We want to establish the transport equation for W. To this end we choose a small time interval τ (so that $\partial H/\partial\chi$ can be considered a constant) and observe: all the particles which at time τ have χ in $d\chi$ and ψ in $d\psi$ must at time $t = 0$ have had

$$\psi' = \psi - \frac{\partial H}{\partial \chi}\tau \qquad \chi' = \frac{\partial H}{\partial \psi}\tau + \rho_s \chi\tau - \mathcal{E}(\tau) + \chi$$

and must have been contained in an interval

$$d\chi' d\psi' = d\chi\, d\psi\, (1 + \rho_s \tau) + O(\tau^2)$$

provided that $\mathcal{E}(t)$ is a given function. If $\mathcal{E}(t)$ (which is proportional to the difference between the average and the actual energy loss in the time interval t) is a stochastic quantity, we still can define a probability $P(\mathcal{E}(t))d\mathcal{E}(t)$ of finding $\mathcal{E}(t)$ in the interval $d\mathcal{E}(t)$ after the time t has elapsed and it is clear that we must have

$$(12) \qquad \int P(x)dx = 1 \qquad \int P(x)x\,dx = 0 \qquad \int P(x)x^2 dx = \overline{x^2}$$

the latter because the average fluctuation of energy must be zero. This immediately gives the following transport equation

$$(13) \qquad W(\chi,\psi;\tau) = \int d\mathcal{E}(\tau)\, P(\,(\mathcal{E}(\tau)\,)\, \times$$
$$\times\; W(\chi + \frac{\partial H}{\partial \psi}\tau + \rho_s\chi\tau - \mathcal{E}(\tau),\, \psi - \frac{\partial H}{\partial \chi}\tau;\, 0)(1 + \rho_s\tau)$$

From this a diffusion equation can be obtained by applying the Fokker - Planck method, i. e. by expanding the right hand side in powers of the increments. Neglecting terms of third order this gives

$$(14) \qquad \frac{\partial W}{\partial t} = \frac{\partial H}{\partial \varphi} \frac{\partial W}{\partial \chi} - \frac{\partial H}{\partial \chi} \frac{\partial W}{\partial \varphi} + \rho_s \left(W + \chi \frac{\partial W}{\partial \chi} \right) + \frac{1}{2} \overline{\dot{\varepsilon}^2(t)} \frac{\partial^2 W}{\partial \chi^2}$$

The first two terms are due to the phase motion of the unperturbed system, the second term is due to damping and the third is due to the diffusion caused by the quantum effects of the synchrotron radiation.

4) The Diffusion Equation in Action and Angle Variables.

The action integral $J(H)$ is defined by

$$(15) \qquad J(H) = \oint \chi \, d\varphi$$

extended over a full period $1/\nu(H)$ of the unperturbed motion of the system. H is a function of J alone and not dependent on the angle variable $w = \nu(H)t + w_o$. If $H = H(J)$ is given ν can be obtained by

$$(16) \qquad \nu = \frac{dH}{dJ}$$

Since J and w are canonical variables the volume element in phase space is $dJ\,dw$. We now assume that the density function does not depend on w and that therefore the probability of finding the system with J in dJ is given by

W(J)dJ. Using

$$\frac{\partial}{\partial \chi} = \frac{\partial H}{\partial \chi}\frac{dJ}{dH}\frac{\partial}{\partial J} = \frac{\chi}{\nu}\frac{\partial}{\partial J}\,; \qquad \frac{\partial}{\partial \psi} = \frac{\partial H}{\partial \psi}\frac{1}{\nu}\frac{\partial}{\partial J}$$

and inserting into equation (14) one gets

$$\frac{\partial W}{\partial t} = \varrho_s \left(W + \frac{\chi^2}{\nu}\frac{\partial W}{\partial J}\right) + \frac{1}{2}\,\overline{\dot{\varepsilon}^2(t)}\,\left(\frac{\chi^2}{\nu}\frac{\partial}{\partial J}\frac{1}{\nu}\frac{\partial W}{\partial J} + \frac{1}{\nu}\frac{\partial W}{\partial J}\right)$$

Now, the changes caused in the function W(J,t) by the stochastic and damping terms are slow changes in the sense that they will become appreciable only after a timeinterval,which is long compared to the period of the phase motion. It is therefore legitimate to replace χ^2 by its time average. Because of (15) one has

$$(17) \qquad \overline{\chi^2} = \nu \int_0^{1/\nu} \chi^2 dt = \nu \int_0^{1/\nu} \chi \dot{\psi}\, dt = \nu J$$

so that the diffusion equation for the action variable J becomes

$$(18) \qquad \frac{\partial W}{\partial t} = \varrho_s \left(\frac{\partial}{\partial J}(JW) + \frac{\overline{H}}{\delta}\frac{\partial}{\partial J}\frac{J}{\nu}\frac{\partial W}{\partial J}\right)$$

Here δ is defined by

$$(19) \qquad \frac{1}{2}\,\overline{\dot{\varepsilon}^2(t)} = \frac{\overline{H}\varrho_s}{\delta}$$

and the definition is chosen in such a way that δ becomes identical with the exponent in Christy's formula for the lifetime of the beam in the synchrotron. δ will be determined in section .

5) Solution of the Diffusion Equation.

The diffusion equation (19) has to be solved subject to the boundary conditions

$$(20) \qquad \lim_{J \to 0} JW = \lim_{J \to 0} \frac{J}{\nu} \frac{\partial W}{\partial J} = 0 \; ; \qquad W(\bar{J}) = 0$$

the first two of which express that there are no sources of particles at $J = 0$, the last that there is an 'absorbing wall' at $J = \bar{J}$. This absorbing wall corresponds to the fact that once the limit of stability is reached the particles diffuse rapidly to $J = \infty$.

There are two distinct ways of attempting to solve an equation of diffusion in the presence of an absorbing wall. The one consists in fixing an initial distribution and in trying to determine the time development of this distribution. A simplified version of this method consists in imposing an exponential timedevelopment and in determining the exponent as a solution of the eigenvalue equation imposed by the boundary conditions. In the second method the procedure is the following: Equation (18) does not admit stationary solutions since particles are continually lost at the boundary $J = \bar{J}$. We can, however enforce a stationary behaviour by adding a source-density in such a way that the losses at the boundary are exactly compensated at each instant of time. The relation between the second method and the simplified first is the following: the latter is obtained by choosing the source distribution proportional to $W(J)$ - the solution of the stationary equation.

We shall here adopt the second method: firstly because it admits an exact solution of the diffusion equation, secondly, because it is more general than the simplified method one and thirdly because the physical situation most closely described by this method is one encountered in storage rings, which recently have gained so much interest.

We assume that per unit of time $\dot{N}_o(J)$ electrons with action less than J are injected into the stable part of the phase diagram. The total number injected is $\dot{N}_o(\bar{J}) = \bar{\dot{N}}_o$. We interpret W(J)dJ as the number of particles in dJ. W(J) will in general be a functional of N(J), but we shall be able to show that this functional is degenerate in most cases of practical importance and that W(J) depends only on $\bar{N}_o$. With

$$(21) \qquad N = \int_0^{\bar{J}} dJ \; W(J)$$

the total number of circulating particles, we can then define a lifetime

$$(22) \qquad \tau = N/\dot{\bar{N}}_o$$

which in general will again be a functional of $\dot{N}_o(J)$, but will only depend on $\bar{N}$ in most practical cases.

After the introduction of the source density $\dot{N}_o'(J)dJ$ (where the ' indicates differentiation with respect to J) we can put $\partial W/\partial t = 0$ and obtain from equation (18)

$$(23) \qquad 0 = \varrho_s \left(JW + \frac{\bar{H}}{\delta} \left(\frac{J}{\nu} W' \right) \right)' + \dot{N}_o'$$

We can integrate this equation from 0 to J and obtain by using the boundary conditions (20)

$$(24) \qquad 0 = \varrho_s \left(JW + \frac{\bar{H}}{\delta} \left(\frac{J}{\nu} W' \right) \right) + \dot{N}_o$$

Observing that from equation (16) it follows that

$$(25) \qquad \int_0^{\bar{J}} \nu \, dJ = H$$

we find that the exact solution of (24) which satisfies the boundary conditions (20) is given by

$$(26) \qquad W = \frac{\delta}{\varrho_s \bar{\bar{H}}} e^{-\delta H/\bar{\bar{H}}} \int_J^{\bar{J}} \nu \, dJ \; e^{\delta H/\bar{\bar{H}}} \frac{\dot{N}_o(J)}{J}$$

from which with the help of (22) we deduce for the lifetime τ:

$$(27) \qquad \tau = \frac{\delta}{\varrho_s \bar{\bar{H}}} \int_0^{\bar{J}} dJ \; e^{-\delta H/\bar{\bar{H}}} \int_J^{\bar{J}} \nu \, dJ \; e^{\delta H/\bar{\bar{H}}} \frac{\dot{N}_o(J)}{\bar{\dot{N}}_o \; J}$$

The 'inner integral' has an increment $\nu \, dJ$, which suggests to introduce H as a new variable of integration. Putting $x = H/\bar{\bar{H}}$ one then obtains instead of (27):

$$(28) \qquad \tau = \frac{\delta \bar{\bar{H}}}{\varrho_s \nu(0) \bar{J}} \int_0^1 dx \; e^{-x\delta} \frac{\nu(0)}{\nu} \int_x^1 e^{x\delta} \frac{\dot{N}_o(J) \; \bar{J}}{\bar{\dot{N}}_o \; J} \, dx$$

An evaluation of these integrale will be carried out in the next section.

6.) Discussion of Limiting Cases for which the Lifetime can be evaluated in Closed Form.

In most practical designs one has $\delta \gg 1$. In this case the actual form of $\dot{N}(J)$ becomes irrelevant for τ as long as $N(J)$ is restricted to a class of functions, which vary only little in an interval ΔJ corresponding to an interval $\Delta H = \bar{\bar{H}}/\delta$. An inspection of equation (28) shows that in this case only the vicinity of the upper limit contributes to the inner integral. In this region we can replace $\dot{N}_o(J)/J$ by $\bar{\dot{N}}/\bar{J}$. Since in the outer integral the maximum contribution comes from the vicinity of the lower limit, we can replace the lower limit of the lower integral by 0. In this case we obtain from (28)

$$(29) \qquad \tau = \frac{e^{\delta}}{\rho_s \delta} \frac{\bar{H}}{\nu(0)\,\bar{J}}$$

and this differs from the Christy formula only by the factor

$$(30) \quad \eta(\varphi_s) = \frac{\bar{H}(\varphi_s)}{\nu(0,\varphi_s)\,\bar{J}(\varphi_s)} \quad ; \quad \nu(0,\varphi_s) = \frac{\Omega_o}{2\pi}\sqrt{\cos\varphi_s}$$

This factor can be determined in closed form for the limiting cases $\varphi_s = 0$ and $\varphi_s = \frac{\pi}{2}$ and is then respectively $\frac{\pi}{4}$ and $\frac{5\pi}{18}$, it can be shown that $5\pi/18 \leq \eta \leq \pi/4$. It is seen that the correction to the Christy formula deriving from an accurate consideration of the 'non linear' region of the phase diagram is quite trivial.

The Christy formula as well as its modification (29) is only valid, if $\delta >> 1$. To get a closed expression the validity of which does not depend on this assumption, we have to introduce some simplifications.

The first consists in assuming that

$$(31) \qquad \dot{N}_o(J) = \bar{N}_o J/\bar{J}$$

i.e. that the density of the injected electrons in action space is a constant. This will - in general - give shorter lifetimes (by a factor 2 if there is no damping) than for example a pointsource placed at $J = 0$. The lifetimes will even be shorter than that obtained for the 'natural' injection in which $\dot{N}_o'(J) = \text{const}\ W(J)$. In a sense we can consider the lifetimes so obtained as a lower limit, since it is hard to imagine an injection method in which the number of injected electrons increases on approaching the limit of stability.

The second simplification results from using for the energy dependence of the frequency $\nu(H)$ the expression

$$(32) \qquad \nu(H) = \nu(0)\,(1 - H/\bar{H})^{\xi}$$

with

$$(33) \qquad \xi = 1 - \bar{H}/\bar{J}\nu(0)$$

The justification for this approximation is the following: we know of $\nu(H)$ that it must tend to zero if H appreaches the stability limit $\bar{H}$. This is due to the fact that in principle a particle may rest for ever in a position of unstable equilibrium. An analysis of the action integral also shows, that for $x = (\bar{H} - H)/\bar{H} \longrightarrow 0$ ν must behave as

$$\nu = \frac{A}{B - \log x} \qquad A, B = \text{const.}$$

This is due to the fact that near the stability limit the kinetic energy tends to zero as $(\psi_0 - \psi)$ $(\psi_0 = \pi - 2\varphi_s)$ and not as in the stable case as $(\psi_0 - \psi)^{1/2}$. Remembering that

$$\log x = \lim_{\alpha \to 0} \frac{1}{\alpha}\,(1 - x^{-\alpha})$$

we see that unless the exponent α is too big the log can be approximated by a power law.

With these simplifying assumptions we can evaluate the integrals in (28) in closed form provided that we expand in powers in ξ. If this expansion is broken off with the linear term we get

$$(34) \qquad \tau = \frac{1}{\rho_s \delta} \left\{ (1 - \xi)(e^{\delta} - 1) + \xi (\overline{E}_i(\delta) - \log \gamma \delta) - \delta \right\}$$

where

$$(35) \qquad \overline{E}_i(\delta) - \log \gamma \delta = \sum_{n=1}^{\infty} \frac{\delta^n}{n!\, n}$$

and

$$\gamma = e^C = 1.781$$

The function $\overline{E}_i(\delta)$ is tabulated in Jahnke and Emde: 'Tables of Functions'.

The expression (34) should be valid for arbitrary δ. For small δ one has

$$E_i(\delta) - \ln \gamma \delta = \delta + \frac{\delta^2}{4}$$

and therefore

$$(36) \qquad \lim_{\delta \to 0} \tau = \frac{\delta}{2\rho_s} \left(1 - \frac{\xi}{2}\right)$$

An inspection of the diffusion equation shows that $\frac{\delta \Delta H}{\overline{H} \rho_s}$ is the time needed for the diffusion process to establish a change ΔH of energy (in the linear region). Theleading term in (36) can therefore be interpreted as half the time necessary for the diffusion to lead from 0 to $\overline{H}$ - the term proportional to ξ representing the correction for the nonlinearity of the problem. The factor 2 can also be easily understood since owing to our assumption concerning the distribution of the injected electrons their average energy will be $\overline{H}/2$.

If $\delta >> 1$ one can use the semiconvergent expansion of the $\overline{E}_i$ - function. This gives

$$(37) \qquad \tau = \frac{e^{\delta}}{\delta \rho_s} \left\{ (1 - \xi) + \frac{\xi}{\delta} + \frac{\xi}{\delta^2} + \dots \right\}$$

Here the first term in the curly brackets gives the Christy formula and it is seen that the remaining terms representing the contribution of the nonlinearity of the restoring force are quite negligible. Equation (37) will be a good approximation as long as non exponential terms can be neglected: it should be good within 5% as long as $\delta > 3$.

(36) as well as (37) show, that the corrections due to the approach of the limit of stability are proportional to a small parameter ξ and not very important. It is therefore logical to neglect the ξ - contribution altogether and to write for τ :

$$(35) \qquad \tau = \frac{1}{\rho_s \delta} \left(e^{\delta} - 1 - \delta \right)$$

a formula which should be right within 10% in the whole range $0 \leq \delta \leq$. It should be noted again that in the case of small δ this equation describes a particular mode of injection, in which the injected particles are placed with equal probability into the whole stable region of phase space. For large values of δ the validity of (35) is independent of this restriction.

To get an opinion about the influence of the distribution $\dot{N}_o(J)$ we can integrate (28), assuming a δ- function source at the origin

$$(36) \qquad \dot{N}_o(J) = \dot{N}_o = \text{const}$$

and neglecting the corrections due to the linearity of the problem: these corrections should be smaller for a δ source than if the assumption (31) is made, since the nonlinearities will act only at the limit of the stable region, which in the present case contributes less. Putting therefore

(37) $$J \simeq \frac{H}{\vartheta(0)}$$

we can again integrate (27) exactly and get

(38) $$\tau = \frac{1}{\rho_s}\left\{ \overline{E}_i(\delta) - \log \gamma\delta \right\}$$

Unsing the semiconvergent series for $\overline{E}_i(\delta)$ in the case of $\delta >> 1$ we again get the Christy formula. In the limit of small δ, however (38) gives

(39) $$\tau = \frac{1}{\rho_s \delta} \qquad \delta << 1$$

and this bears out the comment to the approximation (36).

7) Determination of the Exponent δ (3)

The exponent δ is defined by the equation (19). From equation (6) it follows that

(40) $$\varepsilon(\tau) = 2\pi k \alpha \frac{\delta E}{t_s\ E_s}$$

where δE is the fluctuation of the energy radiated in the time interval $0 \leq t \leq \tau$. Now, assume that in this interval $n(k)dk$ photons of energy k were radiated and that the average energy loss is given by $\overline{n}(k)dk$. Then obviously

(41) $$\int k\overline{n}(k)dk = P_s\tau$$

and further

(42) $$\delta E = \int dk\ k\ (n(k) - \overline{n}(k))$$

The act of emission of a given photon in the interval dk is a completely statistical event and it was shown by Bloch and Nordsieck[4], that the probability P(N) of the emission of N quanta in a situation where in the average $\overline{N}$ quanta are emitted is given by the Poisson distribution

$$(43) \qquad P(N) = \frac{\overline{N}^N}{N!} e^{-\overline{N}}$$

From this it follows immediately that

$$(44) \qquad \overline{(n(k) - \overline{n}(k))\ (n(k') - \overline{n}(k'))} = \delta(k - k')\ \overline{n}(k)$$

where the bar indicates the average value. It follows that for $\overline{\delta E^2}$ one has

$$(45) \qquad \overline{\delta E^2} = \langle k \rangle P_s \tau$$

where $\langle k \rangle$ is - because of (41) - defined as

$$(46) \qquad \langle k \rangle = \int dk\ k^2\ \overline{n}(k) \Big/ \int dk\ k\ \overline{n}(k)$$

Inserting from (45) into equation (19) and using (4), (5) and (11) we obtain

$$(47) \qquad \delta = \frac{4 - \alpha}{\pi.\ k\ \alpha} \ \frac{eV}{\langle k \rangle}\ f(\varphi_s)$$

It remains to determine $\langle k \rangle$. The spectrum $\overline{n}(k)$ has been calculated by various authors[5]: $\hbar^2 n(\hbar\omega)\,\omega\, d\omega$ represents the energy radiated into the frequency interval $d\omega$ in the unit of time. Its determination is a purely classical problem and the results is

$$(48) \qquad n(k)dk \propto dy \int_y^\infty K_{5/3}(x)$$

where $K_\nu(x)$ is defined as

$$(49) \qquad K_\nu(x) = \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{\sin \nu\pi}$$

and $I_\nu(z) = \exp(-i\nu\pi/2)J_\nu(iz)$ and J_ν is a Bessel function (Compare Magnus Oberhettinger, p. 28). y is defined by $y = k/k_o$ and k_o is given by

$$(50) \qquad k_o = \frac{3}{2}\ 137\ \left(\frac{r_o}{R}\right)\ \gamma^3\ mc^2 = \frac{3}{2}\,(\hbar\,\omega)\,\gamma^3$$

The integrations required by (46) can be easily carried out (Magnus Oberhettinger, p. 48) and one gets

$$(51) \qquad \langle k\rangle = \frac{55}{24\sqrt{3}}\ k_o$$

8) Remark on Straight Sections.

Finally, we want to remark about straight sections that their effect can be easily calculated by remembering that electrons radiate only when in magnetic sectors; moreover the rotational frequencies (and the RF one) have to be computed on the whole closed path including field-free sections.-

---

References.

(1) R.F.Christy: Synchrotron Beam Loss due to Quantum Fluctuations in the Radiation - California Institute of Technology (1957) - unpublished

(2) M.Sands, N. Cim., 15, 599 (1960)

(3) M.Sands, Phys.Rev., 97, 470 (1955)
A.A.Kolomensky, A.N.Lebedev, N.Cim., 11,458 (1959)

(4) F.Bloch and A.Nordsieck, Phys.Rev., 52, 54 (1937)

(5) J.Schwinger, Phys.Rev., 75, 1912 (1949)
L.Landau and E.Lifshitz, Theory of Fields (Moscow,1948)
D.Ivanenko,A.Sokolov, Classical Theory of Fields