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Proposal for a Distributed Regenerative Action to Extract the Beam from a Weak-Focusing Synchrotron by Exciting the Resonance $\dfrac{2 \text{ Radial Oscillations}}{3 \text{ Revolutions}}$.

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1. – Introduction.

Till now the external electron beam has not been achieved in high energy circular machines. Indeed, the regenerative [1-6] solution is disadvantaged by the fact that the required nominal value of the regenerator strength is not obtainable in practice because of;

i) the narrowness of the linear field region;

ii) the form of the radial restoring force.

The radial equation of motion may be written — neglecting terms of order x/R with respect to unity —

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + x + \frac{R}{B_0}(B_z - B_0) = 0\,, \qquad x = r - R, \tag{1.1}$$

or

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + [1 - \overline{n}(x)]x = 0\,, \tag{1.2}$$

where

$$\overline{n}(x) = \frac{1}{x}\int_0^x n(x)\,\mathrm{d}x; \qquad n(x) = -\left[\frac{r}{B_z}\frac{\partial B_z}{\partial r}\right]_{z=0}. \tag{1.2'}$$

(1) A. V. CREWE and K. J. LE COUTEUR: *Rev. Sci. Instr.*, **26**, 725 (1955).
(2) A. V. CREWE and J. W. G. GREGORY: *Proc. Roy. Soc. London*, A **232**, 242 (1955).
(3) A. V. CREWE and U. E. KRUSE: *Rev. Sci. Instr.*, **27**, 5 (1956).
(4) J. L. TUCK and L. C. TENG: *Phys. Rev.*, **81**, 305 (1951).
(5) K. J. LE COUTEUR: *Proc. Phys. Soc.*, B **64**, 1073 (1951).
(6) S. COHEN and A. V. CREWE: *CERN Symposium on High Energy Accelerator* (1956), p. 140.

This means that it is very difficult to obtain an $(1/x)\int_0^x n(x)\,\mathrm{d}x$ required value having the following features: $\bar{n}_{\text{perturbed}} = \text{const} \gg 1$ only in a region of narrow angular width and only on one side of the equilibrium orbit. Equations (1.2), (1.2′) and their consequences are also valid for cyclotrons.

We shall now trace out the possibility of an azimuthally distributed regenerative action to extract the circulating electron beam from a racetrack having the operating point $[n_0, L/R]$ in the neighborhood of the resonance line (for betatron oscillations): 2 radial oscillations/3 revolutions. (This is, for example, the case of the 1 GeV Frascati Synchrotron: $\langle n_0 \rangle = 0.61$ designed field index value; four straight sections of lenght $L/R = 0.335$; resonance field index value $n_{\text{res}} = 0.635$, $R = 360$ cm).

For a right investigation of the betatron oscillations stability from this point of view, the racetrack must be replaced by a Circular Synchrotron having *the same periodicity* for radial oscillations.

For such a synchrotron the resonance $\langle n_0 \rangle$ value is $n_{\text{res}} = \frac{5}{9}$. The perturbing term most responsible for the radial oscillations growth is the non-linear term $\frac{1}{2}(\mathrm{d}n/\mathrm{d}x)^{(1,2)}x^2 \sin 2\theta$ derived from an $n(x, \theta)$ of the form

$$n(x, \theta) = \left(\frac{5}{9} - \delta\right) + \left(\frac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)} x \sin 2\theta \,, \tag{1.3}$$

where

$$\delta = n_{\text{res}} - n_0 = \text{distance in field index from resonance} .$$

The equation of motion relative to this $n(x, \theta)$ can be solved analytically using the Krylov-Bogoliubov technique (7).

One finds that the resonance buildup of radial oscillations is avoided for

$$\left| \text{radial amplitude} \cdot \left(\frac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)} \right| \ll 8\delta \,. \tag{1.4}$$

($8\delta = 0.2$ in the case of the Italian Synchrotron, and therefore we conclude that this machine lies within very satisfactory limits of stability).

2. – The azimuthally distributed regenerative perturbation.

We admit a beam cross section (at energies above 100 MeV) of about $d = 2$ cm in width by 1 cm in height. The 100 → 1000 MeV acceleration time being about $2 \cdot 10^{-2}$ s, the rise time of the proposed regenerative field lattice should be of the same order or shorter.

By making use of the perturbation

$$\begin{cases} \bar{n}(x, \theta) = \left(\dfrac{5}{9} - \delta\right) + \varepsilon \sin 2\theta & \text{when} \quad x > \dfrac{d}{2} \,, \\[2ex] \bar{n}(x, \theta) = \left(\dfrac{5}{9} - \delta\right) - \varepsilon \sin 2\theta & \text{when} \quad x < -\dfrac{d}{2} \,, \end{cases} \tag{2.1}$$

(7) N. M. Krylov and N. N. Bogoliubov: *Introduction to Non-Linear Mechanics* (a free translation by S. Lefschetz from two Russian monographs), Princeton University Press.

$$(2.1') \qquad \bar{n}(x, \theta) = \left(\frac{5}{9} - \delta\right) + \frac{1}{2}\left(\frac{dn}{dx}\right)^{(1,2)} x \sin 2\theta \qquad \text{when} \qquad -\frac{d}{2} < x < \frac{d}{2},$$

($\varepsilon > 0$; $\frac{4}{9} \gg \varepsilon \gg |\delta|$) the desired buildup process can be achieved for radial oscillations without increasing at the same time the axial oscillations. The (2.1) $n(x, \theta)$ required may be get by inserting the $\pi/2$ wide regenerator field shown in Fig. 1 (« $\pm\eta$ » region)

As long as (1.4) is satisfied in the $\pm d/2$ radial extent, the amplitude of radial and axial oscillations remains bounded. Radial deflection begins when $(dn/dx)^{(1,2)}$ is made to increase until (1.4) is violated. The increasing radial oscillation carries the particles into the field lattice (2.1) ($|x| > d/2$). When $|x|_{max}$ differs from $d/2$ by a relatively large value ($|x|_{max}/(d/2) \approx 4$) the equation of motion takes the simple form

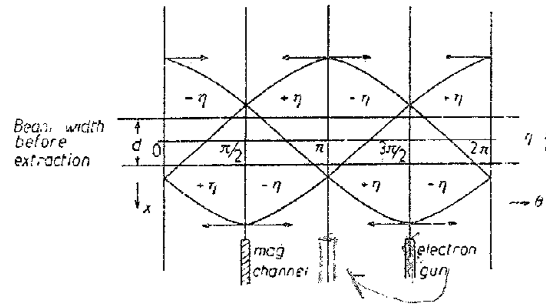

Fig. 1.

$$(2.2) \qquad \frac{d^2x}{d\theta^2} + \left(\frac{4}{9} + \delta\right) x = \varepsilon |x| \sin 2\theta ,$$

and only negligible errors are thus introduced.

Solving this equation after the quoted procedure (7), the first approximation solution is

$$(2.3) \qquad x(\theta) = a(\theta) \sin\left[\tfrac{2}{3}\theta + \Phi(\theta)\right],$$

where

$$(2.4) \qquad \begin{cases} \dfrac{da}{d\theta} = \dfrac{\varepsilon}{2\pi}\, a \sin 3\Phi , & (a(\theta) \geqslant 0), \\[2ex] \dfrac{d\Phi}{d\theta} = \dfrac{\varepsilon}{2\pi} \cos 3\Phi + \dfrac{3}{4}\,\delta . & (0 \leqslant \Phi(\theta) \leqslant 2\pi). \end{cases}$$

The condition to be met to allow an efficient deflection is $|\sigma| < 1$, where $\sigma = (3\pi/2)(\delta/\varepsilon)$. In this case the following results: *Whatever* the initial phase value $\Phi(0)$ is, the oscillation tends to a constant phase value determined by

$$(2.5) \qquad \begin{cases} \cos 3\Phi_\infty = -\sigma , \\[1ex] \sin 3\Phi_\infty = +\sqrt{1-\sigma^2}, \end{cases}$$

that is

$$\begin{cases} \Phi_\infty = 0 & \text{for} \quad \sigma = -1, \\[2ex] \Phi_\infty = \dfrac{\pi}{6} & \text{for} \quad \sigma = 0, \\[2ex] \Phi_\infty = \dfrac{\pi}{3} & \text{for} \quad \sigma = +1 . \end{cases}$$

This means that the maxima of this asymptotic solution can take place only within the azimuthal extent

$$\begin{array}{ccccccc} \frac{\pi}{4} & \frac{\pi}{2} & \frac{3\pi}{4} & \text{and} & \frac{5\pi}{4} & \frac{3\pi}{2} & \frac{7\pi}{4} \\ \text{for} \quad \sigma=+1 & \sigma=0 & \sigma=-1 & & \sigma=+1 & \sigma=0 & \sigma=-1\,. \end{array}$$

Having discussed the stationary regime for the phase, we shall consider now the first equation of (2.4) for the amplitude. One obtains ($\varepsilon > 0$)

$$(2.6)\qquad \begin{cases} a_\infty(\theta) = a(0)\exp\left[\dfrac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right] & \text{for} \quad \Phi(0) \text{ near to } \Phi_\infty\,, \\ a(\theta) \;\;= a(0)\exp\left[-\dfrac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right] & \text{for} \quad \Phi(0) \text{ opposite to } \Phi_\infty\,, \end{cases}$$

i.e. there is an *unique* stationary behaviour of oscillations, represented by

$$(2.7)\qquad x(\theta) = a(0)\exp\left[\frac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right]\sin\left[\frac{2}{3}\theta + \Phi_\infty\right].$$

It is clear that this is a triple turn extraction mode, and therefore the realizable gain is expressed by

$$(2.8)\qquad g = \exp\left[3\varepsilon\sqrt{1-\sigma^2}\right].$$

By way of example, say $\langle n_0 \rangle = 0.61$, $L/R = 0.335$ i.e. $\delta = +0.025$. In this case in Fig. 2 g and Δ ($=\pi/2$ — azimutal point at wich maximum occurs) are plotted versus $\eta = (\pi/4)\varepsilon$. For $n_0 = 0.66$, $L/R = 0.335$ the results relative to g are the same, and Δ must be replaced by $-\Delta$.

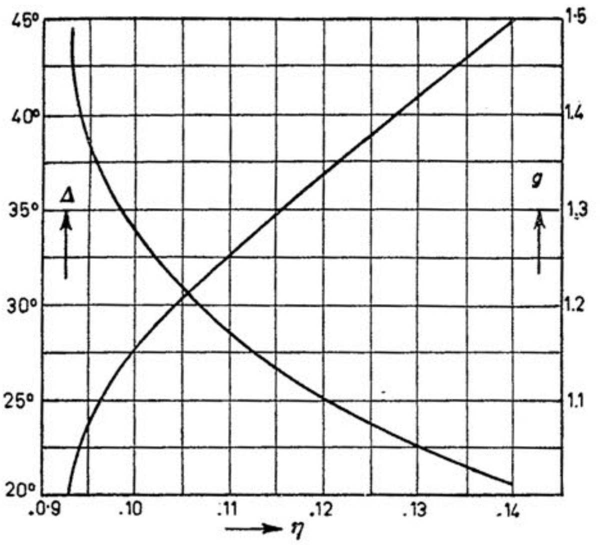

Fig. 2.

From the point of view of axial oscillations one can see that the particle is captured into a motion through a field structure of the form (for the Italian Synchrotron)

$$(2.9)\qquad n \cong \frac{3}{4} + \frac{\varepsilon}{2}\cos\left[\frac{4}{3}\theta - \frac{\pi}{6}\right].$$

The resulting equation of motion may be transformed into a Mathieu equation having divergent solutions [8]

[8] N. W. Mc Lachlan: *Theory and Application of Mathieu Functions* (Oxford Clarendon Press).

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only when $\varepsilon > 1.4$. This, however, is not our case and vertical stability of motion is therefore mantained.

The most disturbing terms for this ordinated buildup of radial oscillation (the linear damping term βx and the $1.3n(x, \theta)$ variation) cannot invalidate the essential aspects of our extraction method when

$$\begin{cases} \left| \left(\dfrac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)} \cdot a(0) \right| \ll 2.2\eta \\ \beta \ll 0.4\eta \,. \end{cases}$$

These conditions seem reasonably weak.

Finally, it seems that the proposed perturbation cannot excite other kinds of non linear resonances.