M. Puglisi, I. F. Quercia: ELECTRONS LOADING ON A RADIO FREQUEN-
CY ECCELERATING CAVITY OF A 1 GeV SYNCHROTRON.

M. Puglisi and I.F. Quercia:
ELECTRONS LOADING ON A RADIO FREQUENCY ACCELERATING CAVITY
OF A 1-GeV SYNCHROTRON.-

Electrons in a Synchrotron are accelerated by azimuthal
electric fields, obtained by one or more radio frequency ac=
celerating gaps, along the trajectory of electrons.

Such accelerating gaps are, generally speaking, part of
radio frequency resonating structures, in which power is in=
jected by suitable electronic devices.[1] From the point of
view of the radio frequency supply, the system may be consi=
dered as one in which radio frequency power is transformed
in direct current power of the accelerated beam of electrons.

If the number of electrons in the beam is very low, the
power picked up from the resonator by the beam is only a
small fraction of the power lost in the resonator by resisti=
ve losses, then it may be neglected. When the number of ele=
ctrons in the beam increases,[2] the beam overloads the reso=
nator and this overloading must be taken into account in the
design of the Radio frequency accelerating system.

Let us suppose that electrons are already bunched, with
some azimuthal distribution. With respect to the radio fre=
quency electric field, the bunches of electrons are circula=
ting with different phases, according to their azimuthal di=
stribution. In general the bunches can be considered as a
periodic series of current pulses, which can be rapresented
by a Fourier expansion. The fundamental term of the series
is of course an a.c. current having the same frequency of the

[1] S. Ramo, J. Whinnery: FIELD AND WAVES IN MODERN RADIO -
(J. Wiley and Sons, Inc. - 1953) - Ch. X, pag. 442-444.

[2] D.R. Hamilton, J.K. Knipp, J.B.H. Kuper: KLYSTRON AND
MICROWAVE TRIODES - (McGraw Hill - M.I.T. - 1948).

radio frequency system, but shifted in phase with respect to the radio frequency voltage at the accelerating gap. This fact means that the fundamental component of the electrons current may be considered as the vectorial composition of two components: one resistive in phase, the other reactive and 90° out of phase.

The former increases the restive losses of the resonator, the latter drives the resonator out of resonance; both these effects lower the value of the electric field at the accele= rating gap of the resonator.

The problem arises of calculating the peak radio frequen= cy voltage at the gap of a cavity driven by a radio frequency amplifier, and loaded by a beam of electrons. In the follo= wing we shall write the equation for the more general case, and we shall give a numerical solution for the particular ca= se of a low voltage, frequency modulated, cavity of the 1 GeV Italian Electronsynchrotron.

§ 1 - We have to calculate the following vectorial functions inside the resonating cavity:

$\vec{H}$ = magnetic field

$\vec{E}$ = electric field

$\vec{J}$ = current density

It can be shown that the solenoidal part of each of these fields can be developed by a series of solenoidal orthogonal functions, $\vec{E}_n$; $\vec{H}_n$; and $\vec{J}_n$: [3]

$$\vec{E} = \sum_n \vec{E}_n \int_V \vec{E} \times \vec{E}_n \, dv$$

$$\vec{H} = \sum_n \vec{H}_n \int_V \vec{H} \times \vec{H}_n \, dv \qquad\qquad (1)$$

$$\vec{J} = \sum_n \vec{J}_n \int_V \vec{J} \times \vec{J}_n \, dv$$

Where the integrations are extended to the volume of the ca= vity.

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[3] J.C.Slater: MICROWAVE ELECTRONICS - (D.Van Nostrand Co., Inc. - 1950 (1951 reprinted) -) Ch.IV - We shall follow in this paper the general calculations and the notations used by J.C.Slater in the book referred.-

The cavity can be described by a volume defined by two kinds of surfaces:

I - Surfaces S' having perfect dieletric properties.

For such surfaces the tangential component of $\vec{H}_n$ vanishes, so that:

$$\vec{n} \wedge \vec{H}_n = 0 \qquad \text{on } S' \tag{2}$$

where $\vec{n}$ is the unity vector normal to S'.

II - Surfaces S having <u>non</u> perfect conducting properties.

For such surfaces the tangential component of $\vec{E}_n$ does not vanish then we have:

$$\vec{n} \wedge \vec{E}_n = (1 + j) \sqrt{\frac{\omega_n \mu}{2 \sigma}} \; \vec{H}_n \tag{3}$$

where μ is the permeability of the medium; σ is the surface conductivity of S; ω_n is the angular frequency of the fields. If we write the Maxwell equations for the fields and currents in the cavity, taking into account the series expansion (1) and the boundary conditions (2) and (3), we obtain the follo= wing equation:

$$\ddot{x}_n + (1 + j) \frac{\omega_0}{Q_W} \dot{x}_n + \omega_0 x_n = -j \frac{\omega_0}{\varepsilon} \int_V \vec{J} \times \vec{E}_n \; dv \tag{4};$$

where:
$$x_n \equiv \int_V \vec{E} \times \vec{E}_n \; dv;$$

$\omega_0 \equiv$ resonance angular frequency of the free cavity;

$Q_W \equiv$ quality factor of the cavity taking into account only the wall losses;

$\varepsilon \equiv$ dieletric constant of the medium.

The solutions of this equation x_n are obviously the coeffi= cients of the series expansion of the electric field $\vec{E}$ accor= ding to the first of (1).

It can be easily shown[4] that the coefficients of the series expansion of the field $\vec{H}$, are related to the x_n by the following relation:

$$y_n \equiv \int_V \vec{H} \times \vec{H}_n \; dv = j \sqrt{\frac{\varepsilon}{\mu}} \; x_n$$

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[4] See reference of note (3), Ch. IV.-

§ 2 - The general solution of equation (4) is of course the sum of two parts: a transient one due to the solution of the corresponding homogeneus equation; a steady one which takes into account the forced term due to the currents injected in the cavity. Of course we are interested in the latter solu= tion.

We shall suppose that the forced term is a time sinusoi= dal function with an angular frequency ω . In this case the steady solution of the equation (4) is of the type:

$$x_n = A\, e^{j\omega t} \tag{5}$$

By substituting (5) in (4) we have:

$$\left\{ j\left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right) - \frac{\omega}{\omega_0}(1+j)\frac{1}{Q_W} \right\} x_n =$$

$$= \frac{1}{\varepsilon \omega_0} \int_V \vec{J} \times \vec{E}_n \, dv \tag{6}$$

Let us now remember that:

$$x_n = \int_V \vec{E} \times \vec{E}_n \, dv$$

Dividing both sides of (6) by $\frac{\omega}{\omega_0} x_n$ we obtain the following equation:

$$j\left[\left(\frac{\omega_0^2 - \omega^2}{\omega_0^2}\right) - \frac{1}{Q_V}\right] - \frac{1}{Q_W} = \frac{1}{\varepsilon \omega}\frac{\int_V \vec{J} \times \vec{E}_n \, dv}{\int_V \vec{E} \times \vec{E}_n \, dv} \tag{7}$$

We shall base on (7) the calculations of the electronic loa= ding.

§ 3 - Let us consider a cylindrical cavity as sketched in fig. 1. In such a cavity we may assume that at any time the ele= ctric field $\vec{E}$ is mostly confined between the two planes of the 'condenser part' of the cavity.[5]
The magnetic field can be assumed as filling, with circular lines of force, the thoroidal part of the cavity.

The value of the radio frequency voltage between the con= denser plates at a distance d is therefore related to the

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[5] See reference of note (1), Ch.X, pag. 438-440.

value of the electric field by the simple formula:

$$V(t) = \left| \vec{E}(t) \right| d$$

This is also the voltage by which the electrons crossing the cavity in the axial direction at a time t are accelerated.
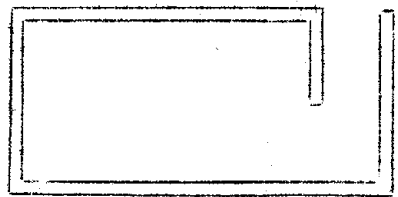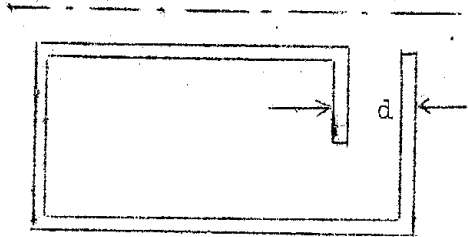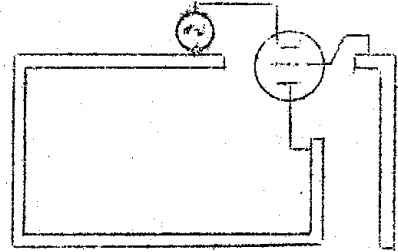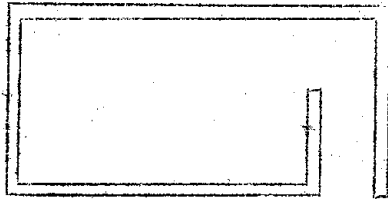

Fig.1 - Axial section of the cavity.-

Fig.2 - Coupling of the driving triode to the cavity.-

Moreover we suppose that the cavity is driven in the founda= mental mode (referred to as the "n" mode); so that:

$$\vec{E} = \vec{E}_n,$$ (8)

all other modes giving a negligible contribution to the se= ries expansion (1). In this case the integral at the deno= minator of the right side of (7) becomes:

$$\int_V \vec{E} \times \vec{E}_n \, dv = \left(\frac{V}{d} \right)^2 S \, d$$ (9)

S being the surface of the condenser part of the cavity.

We have now to take into account the current densities in= side the cavity. In this paper we shall suppose that the po= wer is supplied to the cavity by a direct coupling between the cavity and the anode circuit of a power triode runned as grounded grid amplifier shown schematically in fig.2.[6]

[6] L.B.Arguimbau: VACUUM TUBES CIRCUITS (J.Wiley and Sons, Inc. - 1948) - Ch.VI, pag 286-287.

E.K.Sandeman: RADIO ENGINEERING (Chapman and Hall, Ltd.- 1953) - Ch.X, pag 389-392.

This is just the case of the frequency modulated Radio Frequency System of the 1 GeV Italian Electronsynchrotron. In such a case two different electronic currents are circulating in the cavity:

a) the current $\vec{I}_v$ due to the electrons inside the triode;

b) the current $\vec{I}_e$ due to the electrons of the accelerated beam.

Both currents add to build up the total current density according to the formula:

$$\vec{J} = \vec{J}_e + \vec{J}_v = \vec{I}_e/a + \vec{I}_v/b$$

where a and b are the cross sections of the two currents. As a consequence the integral at the numerator of the right side of (7) can be written:

$$\int_V \vec{J} \times \vec{E}_n dv = \int_{V_e} \vec{J}_e \times \vec{E}_n dv + \int_{V_v} \vec{J}_v \times \vec{E}_n dv$$

the integrals being extended to the volumes in which the two currents are confined.

Taking into account position (8) we get:

$$\left. \begin{array}{c} \int_{V_e} \vec{J}_e \times \vec{E}_n dv = \dfrac{I_e}{a} \, a \left|\vec{E}\right| d = I_e \, V \\[3mm] \int_{V_v} \vec{J}_v \times \vec{E}_n dv = \dfrac{I_v}{b} \, b \left|\vec{E}\right| d = I_v \, V \end{array} \right\} \qquad (10)$$

where I_e and I_v are the scalar values of the beam and the triode currents.

Injecting (9) and (10) in the (7) we obtain the following equation:

$$j \left[\left(\frac{\omega_o^2 - \omega^2}{\omega_o^2} \right) - \frac{1}{Q_W} \right] - \frac{1}{Q_W} = \frac{1}{\omega C} \left[\frac{I_e}{V} - \frac{I_v}{V} \right] \qquad (11)$$

where: $C = S/d$ is the capacity between the plates of the condenser part of the cavity, taking into account also the anode-grid capacity of the triode.

§ 4 - The two quantities in brackets at right side of (11) can be considered as the admittances of the system:

$Y_e \gtrless I_e/V$ being the admittance of the beam load.

$Y_v = I_v/V$ being the admittance of the triode.

Both the beam current and the anode current of the triode, supposed to be driven as Class C amplifier,[7] are periodic short pulses of current appearing at some phase with respect to the radio frequency voltage $V(t)$, as shown in fig.3.- Supposing the $V(t)$ to be a sinusoidal function of the time with angular frequency ω, both the periodic cur= rents wave forms can be expanded in Fourier series, giving rise to a fundamental component of angular fre= quency ω, and to other higher harmo= nics terms.[8] Taking into account the fundamental component only, we can represent the currents as vec= tors rotating with the angular frequency ω. In other words,

Fig.3 - Voltage and currents waveforms in the cavity.-

the two admittances can be represented by complex numbers as follows:

$$\left. \begin{aligned} Y_e &= I_e/V = g_e + j\, b_e \\ Y_v &= I_v/V = g_v + j\, b_v \end{aligned} \right\} \tag{12}$$

Substituting (12) in (11), and separating the real and ima= ginary parts, we obtain the following two equations:

$$\frac{\omega\, C}{Q_W} = - (g_e + g_v) \tag{13}$$

$$\omega\, C \left[\left(\frac{\omega_o}{\omega} - \frac{\omega}{\omega_o} \right) - \frac{1}{Q_W} \right] = b_e + b_v \tag{14}$$

§ 5 - As a first approach to the solution of the problem let we assume that the cavity is driven at a fixed angular fre= quency ω', which is also the angular frequency of revolution

[7] See the first reference of note (6), Ch.VI, pag.267-277.

[8] J.P.Heyboer, P.Zijlstra: TRASMITTING VALVES (Philips' technical Library - 1951) - Ch.III, pag. 26-60.

of electrons of the beam.[9] Moreover we shall assume that ω' is just the frequency of resonance of the cavity, taking in= to account the losses on the walls. In this case (14) redu= ces to:

$$b_e + b_v = 0 \qquad\qquad (14a)$$

because the term:

$$\omega' c \left(\left[\frac{\omega_o}{\omega'} - \frac{\omega'}{\omega_o} \right] - \frac{1}{Q_W} \right)$$

in (14) vanishes.

In the following we shall assume as independent variable the peak value V of the radio frequency voltage at the gap of the cavity. The question we have to solve is:

'' If V is the value of the voltage at the gap of the ca= vity without any beam load, what will be the value V' of the voltage when synchronous bunches of electrons are accelera= ted by the cavity?''-

Let us consider the situation of the currents and volta= ges in the cavity, represented in fig.4 as electrical vectors rotating at the angular frequency ω'.
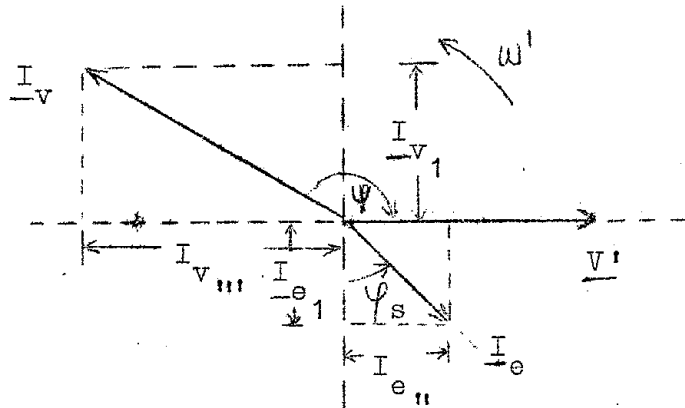

Fig.4 - Vectorial relationship between currents and voltages in the cavity.-

According to (14a) the two components of the currents at 90° out of phase with the voltage have an equal amplitude and opposite sign:

$$I_{v_1} = - I_{e_1}$$

or:

$$|\underline{I}_v| \sin \Psi = - |\underline{I}_e| \cos \varphi_s \qquad\qquad (15)$$

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[9] Or an upper harmonic of the revolution frequency. In our case that is the 4th harmonic.-

According to (13), between the two components of the currents in phase with V' the following relation must be satisfied:

$$\frac{\omega' C}{Q_w} \left| \underline{V}' \right| = - \left[I_{v_{//}} + I_{e_{//}} \right]$$

or:

$$\frac{\omega' C}{Q_w} \left| \underline{V}' \right| = - \left\{ \left| \underline{I}_v \right| \cos \psi + \left| \underline{I}_e \right| \sin \psi_s \right\} \tag{16}$$

Let us now introduce the following approximation: 'The wave form of the current of the triode is not modified by the reactive load[10] introduced by the beam'.

Under this assumption the amplitude of the vector $\underline{I}_v$ is just the same as in the case of no reactive load:

$$\left| \underline{I}_v \right| = I_v^o \quad ;$$

where I_v^o is the current in the triode with a radio frequency voltage V'. Putting this value in (16) instead of $\left| \underline{I}_v \right|$ and dividing both sides by V', we get:

$$\frac{\omega' C}{Q_w} = - g_v^o \cos \psi - g_e \tag{17}$$

where g_v^o is the conductance of the triode without electron beam loading and g_e is the conductance of the electron beam; of course both of them are functions of the voltage V'.

Still we have to determinate the value of the electrical phase ψ of the triode current, which is connected to the phases of the beam current ψ_s by the (15).

Therefore we have:

$$\cos \psi = - \left[1 - \sin^2 \psi \right]^{1/2} = - \left[1 - \left(\frac{\left| \underline{I}_e \right| \cos \psi_s}{\left| \underline{I}_v \right|} \right)^2 \right]^{1/2}$$

Supposing that $\sin \psi \ll 1$, we shall use the following approximate value:

$$- \cos \psi \simeq 1 - \frac{1}{2} \left(\frac{\left| \underline{I}_e \right| \cos \psi_s}{I_v^o} \right)^2 \tag{18}$$

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[10] This assumption is true only for pentodes, and quite good for the high μ power television tubes. See the reference of note (8), Ch.IV.-

§ 6 - For the numerical computation we shall use the follo=
wing procedure:

- First we fix the working conditions of the triode employed:

Vacuum tube	Siemens type RS 1001
Anode supply voltage	E_{bb} = 8.000 V
Fixed grid bias	E_g = - 200 V
Peak radio frequency vol. bet= ween grid and cathode	v_g = 350 V
Amplification coefficient	μ = 65
Plate resistence	v_p = 1.600 ohm

From a graphical analysis [11] of the characteristic curves
of this tube we have calculated the values of $g_v^o(V)$, refer=
red in Table I°.

TABLE I° - Values of the tube conductance v/s R.F. swing

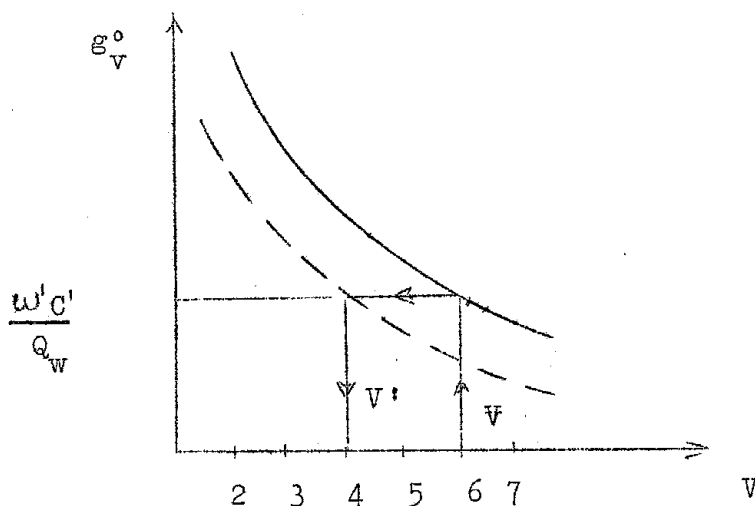| V(volt)	2000	3000	4000	5000	6000	7000
$g_v^o \times 10^3$ (mho)	4,35	2,50	1,60	1,15	0,815	0,445

Fig.5 - g_v: conductance of the system v/s voltage
V at the gap.-

If we plot data as in the solid curve of fig.5, with the
measured value of the R.F. voltage V at the gap of the cavi=
ty without beam loading, we are able to find the value of:

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[11] See the second reference of note (6), Ch.X, pag.372-378.

$$\frac{\omega' C}{Q_w} = g_v^o \ ,$$

which is a mechanical characteristic of the cavity employed, and therefore is not affected by any loading of the system.

We can now plot on the same diagram the values of the function of V:

$$- g_v^o \cos \psi - g_e \ ,$$

they are shown by the dotted curve in fig.5. The abscissa of the point where this curve crosses the value $\omega' C/Q_w$, is just the voltage V' at which the cavity loaded by the electron beam works.

To draw the latter curve according to (17) and (18) we need the values of:

ψ_s = phase angle between the electrons current of the beam and the radio frequency voltage.

$|I_e|$ = the absolute value of the beam current, given by the number of electrons in the beam, and by the phase distribution of electrons in the bunches.

Let us assume that the bunches of electrons are distributed around a phase ψ_s which corresponds to the phase of the 'synchronous particle' accelerated by the synchrotron. The phase angle ψ_s is defined by:

$$\sin \psi_s = \frac{u}{V}$$

where u is the minimum value of the R.F. voltage needed to accelerate electrons.[12] In the following we shall assume

$$u = \text{cost.} = 2.000 \text{ Volt.}$$

Moreover we have calculated the absolute value of the electrons current $|I_e|$, for two different number of accelerated electrons; to say:

$$N = 10^{11} \ ; \qquad\qquad N = 10^{12} \text{ electrons.}$$

For the distribution of the bunches of electrons around

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[12] E.Persico: TEORIA DELLA CATTURA RAPIDA IN UN SINCROTRONE INIETTATO AD ALTA ENERGIA (relaz.n°: T4 della Sezione Acceleratore dell'I.N.F.N. - 1953).-

the synchronous phase φ_s we have assumed three different

shapes:

a) All the electrons travel at phase φ_s. The bunches have the shape of a delta function.

b) The electrons are uniformly distributed between the phase φ_s - 45° and φ_s + 45°. The bunches have a rectangular shape around φ_s.

c) The electrons are uniformly distributed between the fixed phases 110° and 200°.

In fig..6 the assumed distributions are indicated.

In figs. 7,8,9 the curves for the two values of N and for three cases a), b), c) are plotted. It is easy to see that our assumption about the shape of the bunches does not af= fect too much the results.
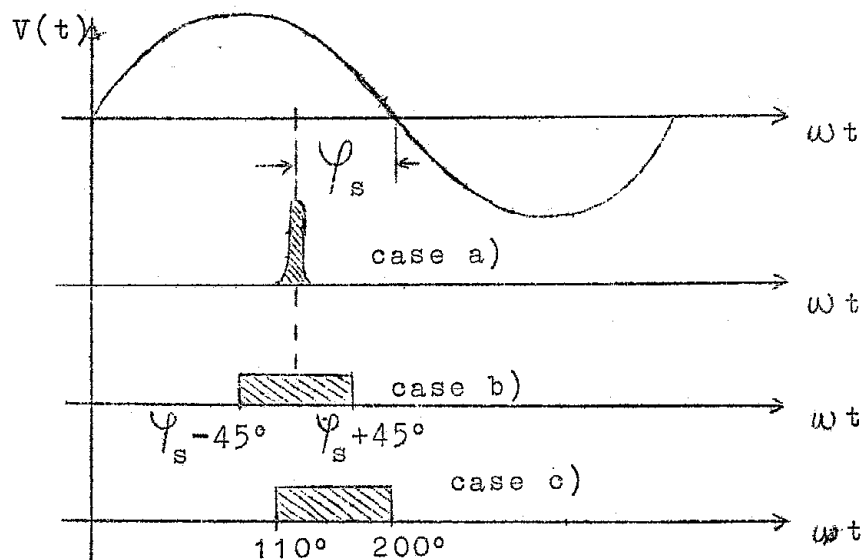

Fig. 6 - The assumed shapes of the beam current and corresponding phases.-

For instance, with the unloaded cavity working at V = 6 kV, the value V' of the voltage, reached when the load is of 10^{12} electrons, is in any case: V' $\geq$ 4,5 kV. Moreover it is shown by the curves that in our case the system is quite in= sensitive to a load of 10^{11} electrons.

§ 7 - As a second approach to the solution of our problem let us consider the fact that during the first part of the accelerating cycle the cavity is driven at a frequency which
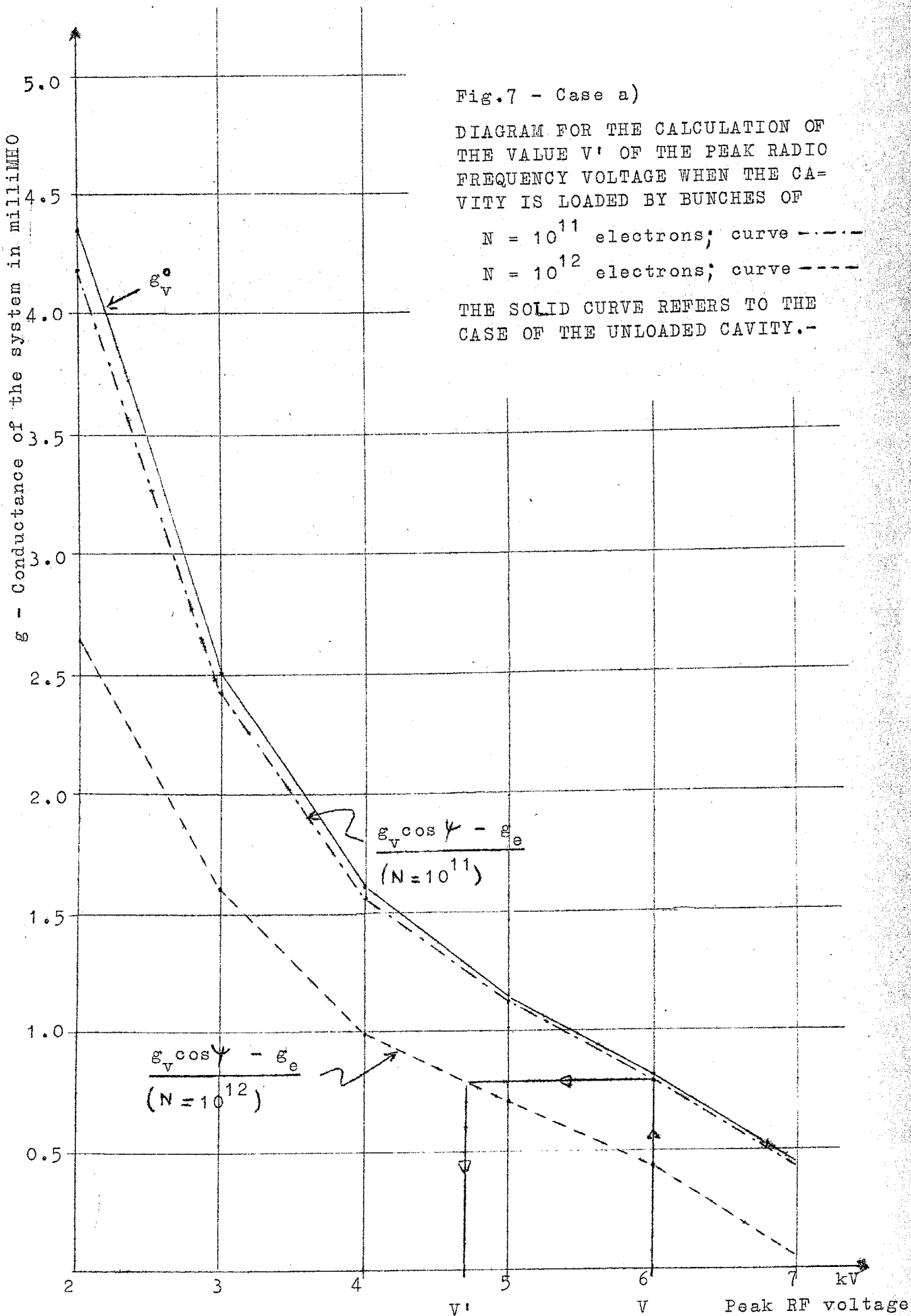
- 12 -

Fig.7 - Case a)

DIAGRAM FOR THE CALCULATION OF THE VALUE V' OF THE PEAK RADIO FREQUENCY VOLTAGE WHEN THE CAVITY IS LOADED BY BUNCHES OF

$N = 10^{11}$ electrons; curve —·—·—

$N = 10^{12}$ electrons; curve ————

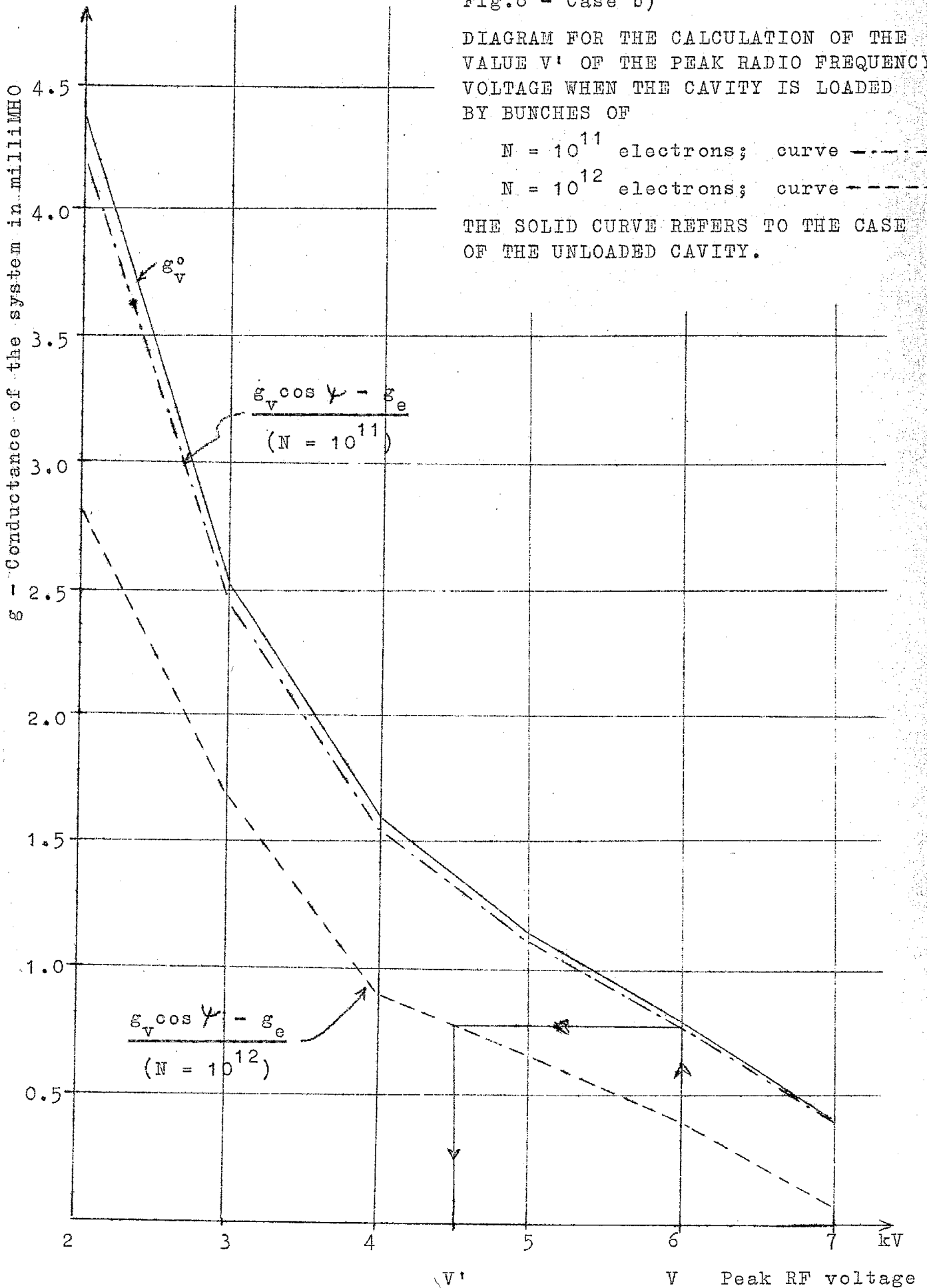THE SOLID CURVE REFERS TO THE CASE OF THE UNLOADED CAVITY.—

Fig.8 - Case b)

DIAGRAM FOR THE CALCULATION OF THE VALUE V' OF THE PEAK RADIO FREQUENCY VOLTAGE WHEN THE CAVITY IS LOADED BY BUNCHES OF

$N = 10^{11}$ electrons; curve —·—·—·—

$N = 10^{12}$ electrons; curve ——————

THE SOLID CURVE REFERS TO THE CASE OF THE UNLOADED CAVITY.

g - Conductance of the system in milliMHO

g_v^o

$\dfrac{g_v \cos \psi - g_e}{(N = 10^{11})}$

$\dfrac{g_v \cos \psi - g_e}{(N = 10^{12})}$

V'   V   Peak RF voltage

kV

Fig.9 - Case c)

DIAGRAM FOR THE CALCULATION OF THE VALUE V' OF THE PEAK RADIO FREQUENCY VOLTAGE WHEN THE CAVITY IS LOADED BY BUNCHES OF

$N = 10^{11}$ electrons; curve — · — · —

$N = 10^{12}$ electrons; curve — — — —

THE SOLID CURVE REFERS TO THE CASE OF THE UNLOADED CAVITY.
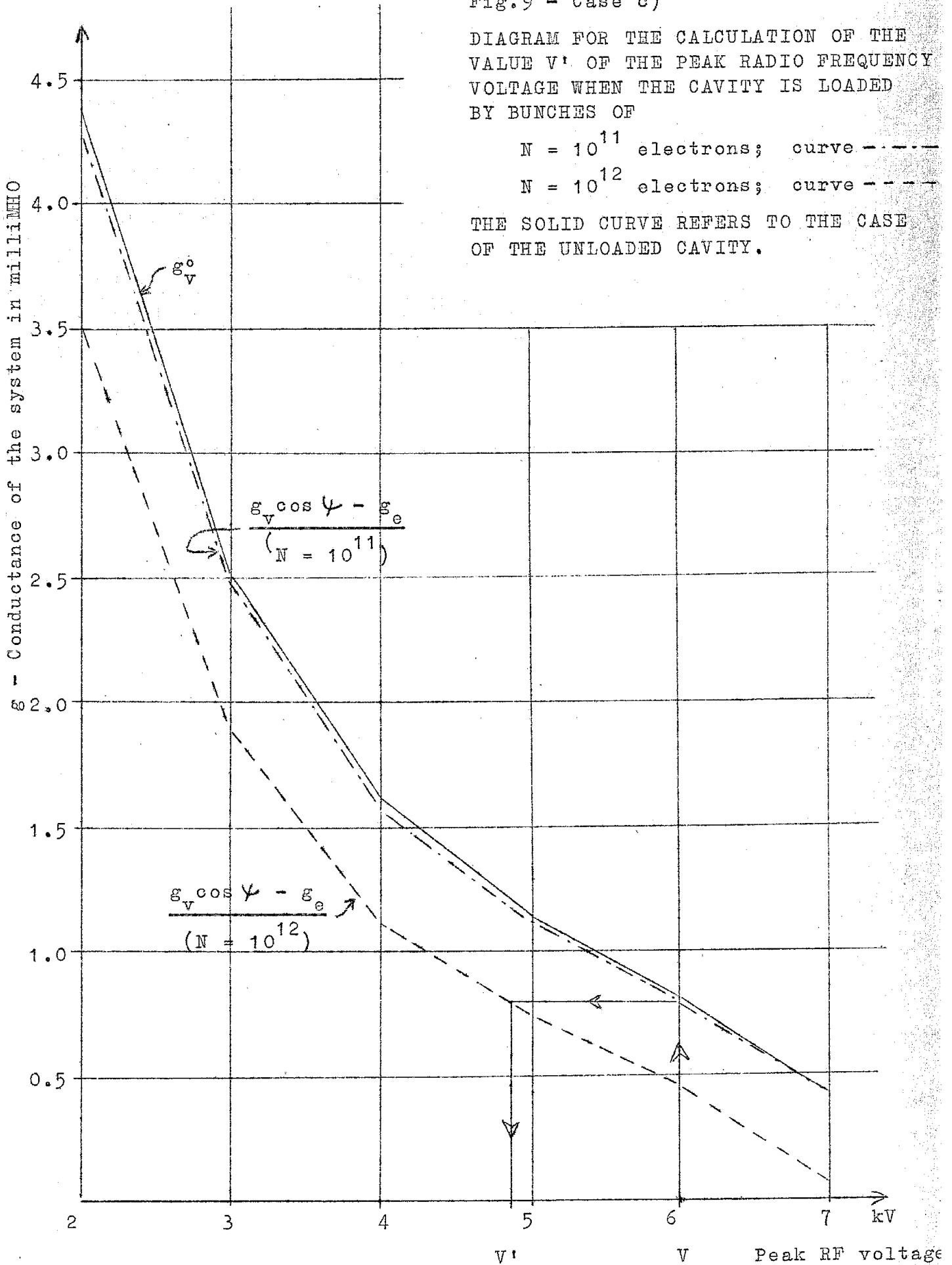
is about 2% lover than the resonating one.[13] If we consi=
der equation (14) of § 4 we see that in this case the left
side does not vanish.

Working at an angular frequency ω, which is $\Delta\omega$ lower
than the resonating frequency ω', eq.(14) becomes:

$$2 C \Delta\omega = b_e + b_v \qquad\qquad (14b)$$

or by multipling both sides by V':

$$2 C \Delta\omega V' = I_{e_1} + I_{v_1}$$

In other words the tube reactive current I_{v_1} must balance
both the beam reactive current I_{e_1}, and the detuned load
reactive current $2 C \Delta\omega V'$.

In our case the situation is rapresented in fig.10 where
the currents are indicated as vectors rotating at the fre=
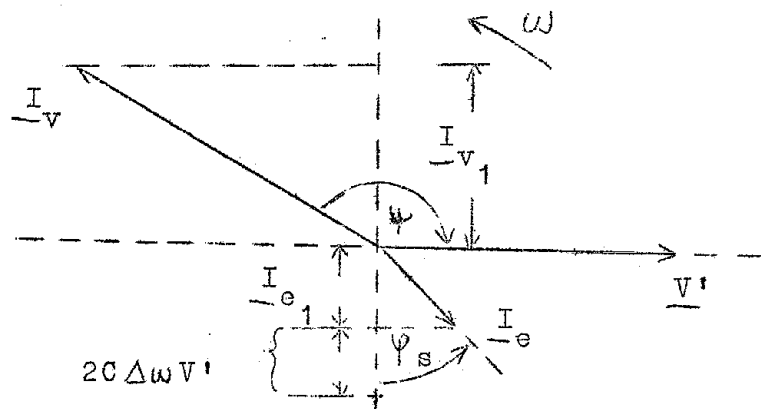quency ω.

Fig.10 - Vectorial relationship between currents and
voltages in the cavity.-

It is easy to show that also in this case it is possible to
write an equation like the (17):

$$\frac{\omega C}{Q_w} = - g_v^o \cos\psi - g_e \; \qquad\qquad (17a)$$

but the meaning of $\cos\psi$ is now:

[13] See the reference of note (8), Ch.III, pag.63-72.-

$$- \cos \psi \cong 1 - \frac{1}{2} \left[\frac{\dfrac{I_{e_1}}{V'} + 2\,C\,d\,\omega}{\dfrac{I^o_v}{V'}} \right]^2 \qquad (18a)$$

By the (17a) and (18a) we calculated numerically the effect of detuning, in the following case:

- Detuning of the cavity $\Delta\omega$ = 2% with ω' = 43 x 10^6 c/s.
- Capacity of the cavity and of the triode C = 60 pF.
- Shape of electron bunches as in the case a) of § 6.
- All other conditions as in § 6.

We obtained the curves of fig.11. In that case it is shown that if the cavity works at 6 kV driven at the resonating frequency and without any beam load, with a beam loading of 10^{12} electrons the Radio Frequency voltage is lowered to about 3,7 kV, and with a loading of 10^{11} electrons it is lowered to about 5,2 kV, by the combined effect of loading and detuning.-

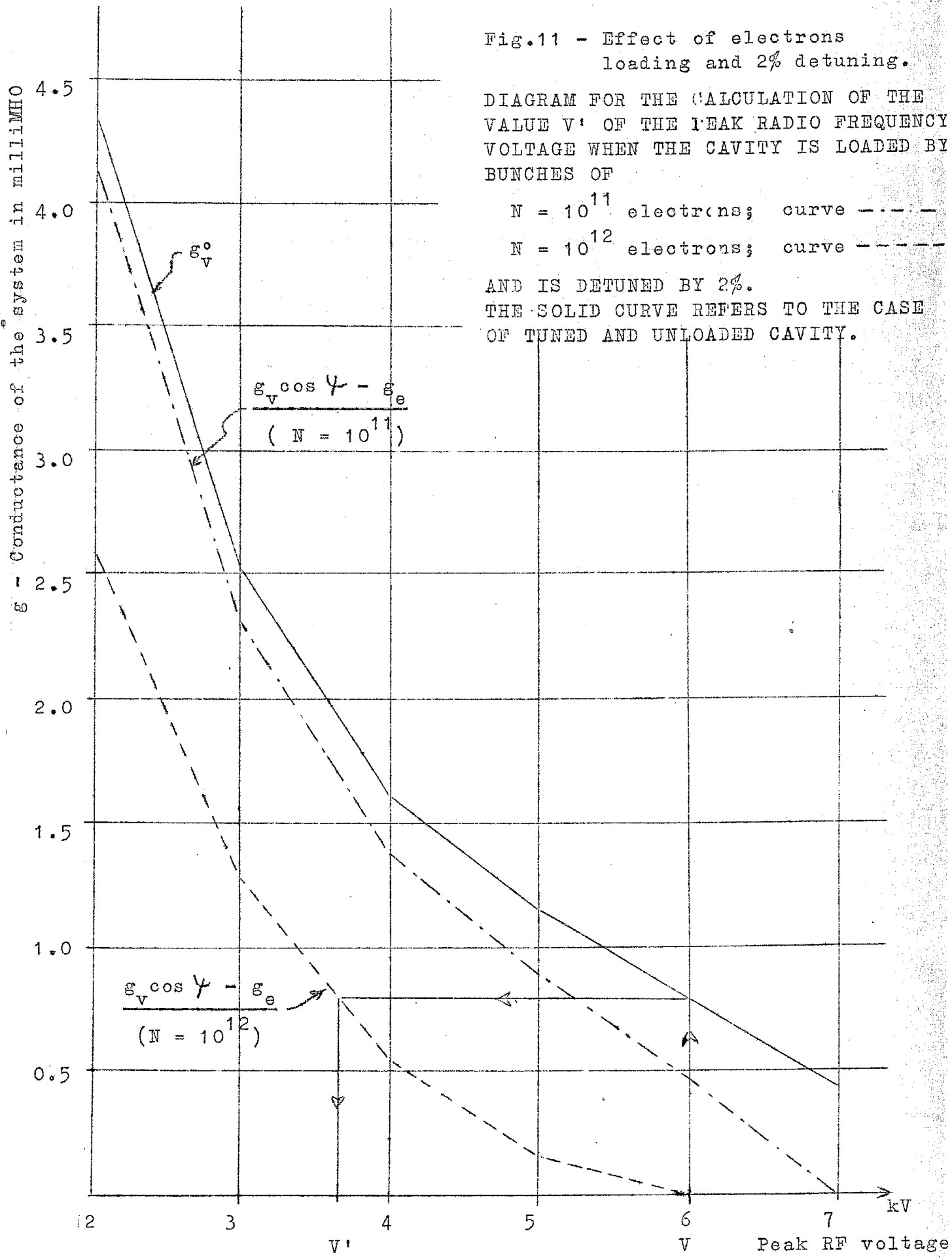Fig.11 - Effect of electrons
loading and 2% detuning.

DIAGRAM FOR THE CALCULATION OF THE
VALUE V' OF THE PEAK RADIO FREQUENCY
VOLTAGE WHEN THE CAVITY IS LOADED BY
BUNCHES OF

$N = 10^{11}$ electrons; curve ---·---

$N = 10^{12}$ electrons; curve ---·---

AND IS DETUNED BY 2%.
THE SOLID CURVE REFERS TO THE CASE
OF TUNED AND UNLOADED CAVITY.