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-SYNCHROTRON.

SCATTERING LOSSES IN WEAK FOCUSING ELECTRON-SYNCHROTRONS

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The problem of scattering losses in synchrotrons has been treated by several authors [1-6] with the aid of some approximations which consent a good estimate of the order of magnitude. The solution of the diffusion equation is very difficult because of the field of force (which gives the betatron oscillations) and the presence of absorbing walls : these are the main differences with the usual problem of multiple Coulomb scattering. A simplification results from the fact that only small angles are interested; moreover we may limit ourselves to the scattering in the plane of the vertical oscillations.

We wish to give another approach to the solution of the diffusion equation, which might be interesting for a clearer understanding of the role of the various parameters determining the losses; besides, it is perhaps more satisfactory from a mathematical point of view.

The damping of the betatron oscillations is not taken into account; we give only a short qualitative discussion of its role.

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We suppose the electrons injected at relativistic energies, that is $\beta = 1$. If t is the time from injection and c the velocity of light, then $x = ct$ is the path from injection. z is the normal distance from the median plane; $z' = dz/dx$ is the slope of the trajectory with respect to the principal orbit, which we may rectify and identify with the x axis. $2\pi\lambda$ is the wavelength of the vertical betatron oscillations. Then

$$A = (z^2 + \lambda^2 z'^2)^{1/2}$$

is the amplitude of the oscillations. If A_i is the injection value of A, the damping effect is described, for relativistic energies E, by

$$A/A_i = (E_i/E)^{1/2}. \tag{1}$$

E is a slow monotonic function of the time (of x); we assume

$$E = E_i + (x/c) \cdot (dE/dt) = E_i + (x/c) \cdot \dot{E} \tag{2}$$

with $\dot{E}$ constant.

The main features of the scattering are contained in two parameters :

k, the inverse of the mean free path, which does not depend on E

Θ, the screening angle (for projected scattering), which depends on E. Assuming Mollière's scattering cross-section [7],

$$\begin{cases} \Theta = W/E \\ W = 0.00454 \, Z^{1/3} \text{ Mev} \end{cases}$$

where Z is the atomic number of the scattering centers. Let us call $b/2$ the distance of the walls from the median plane; multiple scattering may build up large oscillation amplitudes if Θ does not diminish enough in a mean free path, even if $\lambda\Theta(E_i) \ll b/2$. When $\lambda\Theta$ is of the order of $b/2$ and Θ is a slow function of the time, single scattering losses dominate.

We want to compare now the effect of the damping and that of the scattering. As to the order of magnitude, we have

$$z' \rightarrow z' \pm \Theta$$

in a collision. The mean square amplitude variation is then

$$\overline{\Delta A^2} \approx \lambda^2 \Theta^2 = \lambda^2 W^2/E^2$$

for a collision. In a path dx there are $k\,dx$ collisions, so that

$$(dA^2)_{coll} \approx k \, (\lambda^2 W^2/E^2) \, dx$$

Because of the damping we have (from (1))

$$(dA^2)_{damping} = - (A^2/E) \, (\dot{E}/c) \, dx$$

Combining the two variations we obtain

$$\frac{dA^2}{dx} \simeq k \, \frac{\lambda^2 W^2}{E^2} - \frac{A^2}{E} \frac{\dot{E}}{c}$$

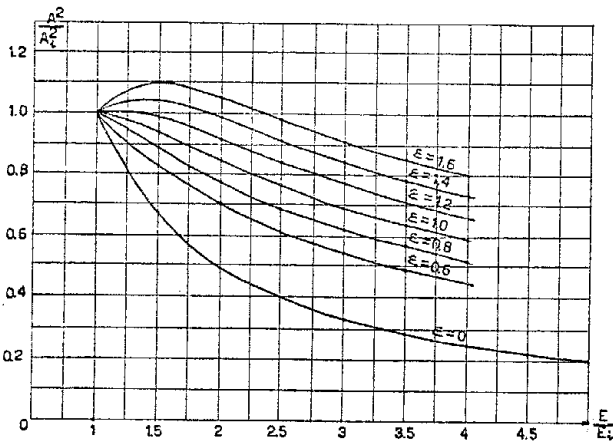that is

$$A^2/A_i^2 = \frac{E_i}{E} \, (1 + \varepsilon \, ln \, (E/E_i))$$

Fig. 1.

where

$$\varepsilon = ck\lambda^2 W^2 / EE_i A_i^2$$

In fig. 1 a plot is given of the function A^2/A_i^2 versus E/E_i, for various ε values.

It is interesting to note that even neglecting the damping effect the amplitude does not increase indefinitely. It is easily found that, in any case,

$$A^2 \leqslant A_i^2 (1 + \varepsilon)$$

This means that the scattering alone is not catastrophic; because of that we can say that the situation without damping is only a little pessimistic.

Here we see that if

$$\varepsilon A_i^2 \ll b^2/4$$

the losses cannot be relevant. We shall find later the parameter

$$\frac{4\varepsilon A_i^2}{b^2} = 4 \frac{ck\lambda^2 W^2}{EE_i b^2} = \sigma \qquad (4)$$

together with

$$\alpha_i = 2\lambda W/bE_i \qquad (5)$$

to be the "characteristic parameters" of the diffusion problem. In fact, the losses depend only on these two non dimensional products formed with the various quantities involved, at least for simple enough forms of the scattering cross section. For a good machine operation, it must be

$$\alpha_i \ll 1 \ , \quad \sigma \ll 1 \qquad (6)$$

and these two rules may serve to simplify the use of many-entries tables accounting for pressure, chamber aperture, injection energy, E etc, variations.

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We call n (z, z', x) dzdz' the number of electrons which, after a path x, have coordinates z, z' in the element dz dz' of the plane Z, Z'. Each particle is a harmonic oscillator along the z axis, with equation of motion

$$z'' + z/\lambda^2 = 0$$

H ($\varkappa$, E) d$\varkappa$ is the projected scattering differential cross section at the energy E times the number of scattering centres per unit volume. With these notations, the diffusion equation may be written down with the standard technique :

$$\frac{\partial n}{\partial x} = -z' \frac{\partial n}{\partial z} + \frac{z}{\lambda^2} \frac{\partial n}{\partial z'} - kn + \int_{-\infty}^{+\infty} H (\varkappa, E) \, n (z, z' - \varkappa, x) \, d\varkappa \qquad (7)$$

This equation is intended to be valid for $|z| < b/2$. The distribution in z, z' is given at the time of injection :

$$n (z, z', 0) = n_0 (z, z') \qquad (8)$$

n_0 being a known function.

The losses are due to the fact that the electrons inpinging on the walls are absorbed by them; that is, there are no particles which enter through the walls into the vacuum chamber :

$$\begin{aligned} n (b/2, z', x) &= 0 \text{ if } z' < 0 \\ n (- b/2, z', x) &= 0 \text{ if } z' > 0 \end{aligned} \qquad (9)$$

We may suppose $n_0 (z, z') = n_0 (-z, -z')$ without loss of generality; because of the symmetry of the walls with respect to the median plane, for every x we can write

$$n (z, z', x) = n (- z, -z', x) \qquad (10)$$

This property allows sometimes useful simplifications.

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We change for convenience the notations :

$$\xi = 2z/b \qquad -1 \leqslant \xi \leqslant 1$$

$$\eta = 2\lambda z'/b \qquad -\infty \leqslant \eta \leqslant \infty$$

$$\alpha = 2\lambda x/b$$

$$s = x/\lambda$$

$$\mu = k\lambda$$

$$H (\varkappa, E) = (2k\lambda/b) \cdot P(s, \alpha)$$

$$n (z, z', x) \rightarrow n (\xi, \eta, s) \qquad (11)$$

The diffusion equation becomes

$$\frac{\partial n}{\partial s} = - \eta \frac{\partial n}{\partial \xi} + \xi \frac{\partial n}{\partial \eta} - \mu n + \mu \int_{-\infty}^{+\infty} P(s,\alpha)\, n\,(\xi, \eta - \alpha, s)\, d\alpha \qquad (12)$$

The integral term suggests a Fourier transform in η :

$$\psi\,(\xi, \omega, s) = \int_{-\infty}^{+\infty} d\eta\, e^{-i\omega\eta}\, n\,(\xi, \eta, s)$$

Like $n\,(\xi, \eta, s)$, this ψ is defined for $|\xi| \leqslant 1$. The equation for ψ does not contain integral terms :

$$\frac{\partial \psi}{\partial s} = - i \frac{\partial^2 \psi}{\partial \xi \partial \omega} + i\omega \xi \psi - Q(\omega, s)\, \psi \qquad (13)$$

where $\qquad 1 - \dfrac{1}{\mu}\, Q(\omega, s) = \displaystyle\int_{-\infty}^{+\infty} P(s, \alpha)\, e^{-i\omega\alpha}\, d\alpha$

Then, defining

$$J\,(\sigma, \omega, s) = \int_{-1}^{1} e^{-i\sigma\xi}\, \psi\,(\xi, \omega, s)\, d\xi$$

we obtain

$$\frac{\partial J}{\partial s} = \sigma \frac{\partial J}{\partial \omega} - \omega \frac{\partial J}{\partial \sigma} - Q(\omega, s)\, J - B\,(\sigma, \omega, s) \qquad (14)$$

where

$$B(\sigma, \omega, s) = i e^{-i\sigma}\left(\frac{\partial \psi}{\partial \omega}\right)_{\xi = 1} - i e^{i\sigma}\left(\frac{\partial \psi}{\partial \omega}\right)_{\xi = -1} \qquad (14')$$

If B were known, the problem would be exactly solved (as we shall see soon). This is not the case, but we can obtain in this way a new starting point for the formal solution of equation (14), treating B "as if" it were known.

Changing to polar coordinates

$$\sigma = r \sin \theta$$
$$\omega = r \cos \theta$$

and putting

$$G\,(r, \theta, s) = J\,(r \sin \theta, r \cos \theta, s)$$
$$G_0\,(r, \theta) = J\,(r \sin \theta, r \cos \theta, 0)$$
$$T\,(r, \theta, s) = B\,(r \sin \theta, r \cos \theta, s)$$

the equation (14) becomes

$$\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial \theta}\right) G\,(r, \theta, s) = - Q\,(r \cos \theta, s)\, \dot{G}\,(r, \theta, s) - T\,(r, \theta, s) \qquad (15)$$

The general solution of this new equation (for known T) is

$$G\,(r, \theta, s) = \exp\left\{-\int_0^s Q\,[r \cos(\theta - s + s'), s']\, ds'\right\}$$
$$\cdot \left\{G_0\,(r, \theta-s) - \int_0^s ds'\, T\,(r, \theta + s' - s, s')\right.$$
$$\left. \exp \int_0^{s'} Q\,[r \cos(\theta - s + s''), s'']\, ds''\right\} \qquad (16)$$

The various terms in this expression are made clear by the following considerations : suppose the walls removed to infinity and that initially there are no particles at infinity. Then, there are never particles at infinity and, in this conditions, T (or B), which is calculated on the boundaries, will vanish. The surviving term in (16) represents the solution without walls :

$$G_\infty\,(r, \theta, s) = G_0(r, \theta-s) \exp\left\{-\int_0^s Q\,[r \cos(\theta - s + s'), s']\, ds'\right\} \qquad (17)$$

while the term containing T is responsible for the losses. We see that the diffusion equation without walls is exactly solved for whatever scattering law; but the presence of the walls restores again the difficulty of handling an integral equation.

Till now we did not introduce any approximation for the diffusion equation. We may simplify its new form (16) by using the fact that $k\lambda \ll 1$ and the energy increases very slowly.

Consider for instance the function

$$Q\,[r \cos(\theta - s + s'), s']$$

and, keeping fixed the second argument, develop it in a Fourier series of $\tilde{\theta} = \theta - s + s'$:

$$Q\,(r \cos \tilde{\theta}, s') = Q_0\,(r, s') + Q_1\,(r, s')\, e^{i\tilde{\theta}} \qquad (18)$$

If $k\lambda \ll 1$ and $\dot{E}$ is small enough, $Q_n(r, s)$ is a slow function of s. Thus, only $Q_0\,(r, s)$ contributes appreciably to

$$\int_0^s Q\,[r \cos(\theta - s + s'), s']\, ds'$$

The same holds true for T (r, θ, s), although it is less evident. Putting

$$T_0(r, s) = \frac{1}{2\pi} \int_0^{2\pi} T(r, \theta, s)\, d\theta \tag{19}$$

$$L(r, s) = \exp\left[-\int_0^s Q_0(r, s')\, ds' \right] \tag{19'}$$

We have

$$G(r, \theta, s) \approx L(r, s) \left\{ G_0(r, \theta-s) - \int_0^s \frac{T_0(r, s')}{L(r, s')}\, ds' \right\}$$

$$= G_\infty(r, \theta, s) - L(r, s) \int_0^s \frac{T_0(r, s')}{L(r, s')}\, ds' \tag{20}$$

The $n(\xi, \eta, s)$ is related to $G(r, \theta, s)$ by a double antitransformation of the Fourrier-Bessel type :

$$4\pi^2 n(\xi, \eta, s) = \sum_{-\infty}^{+\infty} e^{in\varphi} \int_0^\infty rdr J_n(rA) \int_0^{2\pi} e^{in\theta} G(r, \theta, s)\, d\theta$$

where

$$\xi = A \cos \varphi$$

$$\eta = A \sin \varphi$$

and the J_n are Bessel functions.

If the initial distribution n_0 is only amplitude-dependent, G is a function of r only and then n does not depend on φ for every s. The assumption

$$n_0(\xi, \eta) = n_0(A) \tag{21}$$

is rather simplifying and certainly realistic, so we adopt it. It follows that

$$2\pi n(A, s) = \int_0^\infty rdr J_0(rA) G(r, s) \tag{22}$$

where the arguments φ and θ are withdrawn. Coming back to (14'), and taking the average with respect to θ as in (19) we obtain easily :

$$T_0(r,s) = \int_{-\infty}^{+\infty} \eta J_0(r\sqrt{1+\eta^2})\, [n(1, \eta, s)(-1, \eta, s)]\, d\eta$$

But, for the boundary conditions (9) (or better with one of them and the symmetry property (10)) this becomes

$$T_0(r, s) = 2 \int_0^\infty \eta J_0(r\sqrt{1+\eta^2})\, n(1, \eta, s)\, d\eta$$

We saw that n is practically only amplitude dependent; thus, since at the boundary

$$1 + \eta^2 = A^2$$

the function $n(1, \eta, s)$ becomes $n(A, s)$ and

$$T_0(r, s) = 2 \int_1^\infty A J_0(rA) n(A, s)\, dA \tag{23}$$

Substituting into (20) and then into (22) we have

$$n(A, s) = n_\infty(A, s) - \frac{1}{\pi} \int_0^s ds' \int_0^\infty rdr J_0(rA) \frac{L(r, s')}{L(r, s)} \cdot$$

$$\times \int_1^\infty A' J_0(r A') n(A', s')\, dA' \tag{24}$$

where

$$n_\infty(A, s) = 1/2\pi \int_0^\infty rdr J_0(rA) G_\infty(r,s). \tag{25}$$

This is a new equation for the amplitude distribution function which takes automatically into account the presence of the absorbing walls. We may obtain a very simple "continuity equation" from (24) in this way :

put

$$L(r,s) v(r,s) = \int_0^s ds' L\left(L(r,s') \int_1^\infty A J_0(rA) n(A,s')\, dA \right)$$

From the fact that L (o, s) = 1 (see appendix), we obtain

$$v(o,s) = \int_0^s ds' \int_1^\infty A n(A,s')\, dA = \int_0^s ds' \frac{N_e(s')}{2\pi} \tag{26}$$

where $N_e(s)$ is the total number of particles having $A > 1$ at s.

But for (24)

$$n(A, s') = n_\infty(A, s) - \frac{1}{\pi} \int_0^\infty r J_0(rA) \, v(r, s) \, dr$$

and, by Fourier-Bessel antitransformation

$$v(r, s) = \pi \int_1^\infty A J_0(rA) [n_\infty(A, s) - n(A, s)] \, dA \quad (27)$$

Now, in absence of walls the total number of electrons is conserved, so that

$$\int_0^\infty A n_\infty(A, s) \, dA = \int_0^\infty A n_0(A) \, dA = \frac{N_0}{2\pi}$$

while

$$2\pi \int_0^\infty A n(A, s) \, dA = N(s)$$

is the number of particles surviving after a path s when the walls are present.

From (27) we have

$$v(o, s) = \frac{1}{2} \{ N_0 - N(s) \}$$

and combining this with (26) eventually we obtain :

$$dN/ds = - N_e/\pi \qquad N(0) = N_0 \qquad (28)$$

This equation confirms that the number of electrons lost at a given time in a half period of betatron oscillations is equal to the number of particles which, at that time, have amplitude greater than the aperture of the vacuum chamber.

We may now proceed to try some approximations for solving equation (24). Let us write :

$$n(A, s) = n_\infty(A, s) - \delta n_i(A, s) \text{ for } A < 1$$

$$n(A, s) = \delta n_e(A, s) \text{ for } A > 1$$

If the losses are not catastrophic one should find :

(a) δn_i small as compared with n_∞ nearly for every $A < 1$ amplitude, except a small range, below $A = 1$, where δn_i becomes comparable with n_∞

(b) δn_e small as compared with n_∞ for every $A > 1$ amplitude and for every time except a short interval at the very beginning of the acceleration.

With the just introduced notations eq. (24) splits into

$$\delta n_i(A,s) = \frac{1}{\pi} \int_0^s ds' \int_0^\infty r dr \, J_0(rA) \frac{L(r,s')}{L(r,s)}$$

$$\times \int_1^\infty A' J_0(rA') \, \delta n_e(A', s') \, dA'$$

for $A < 1$

$$\delta n_e(A, s) = n_\infty(A, s) - \frac{1}{\pi} \int_0^s ds' \int_0^\infty r dr \, J_0(rA) \frac{L(r,s')}{L(r,s)}$$

$$\times \int_1^\infty A' J_0(rA') \, \delta n_e(A', s') \, dA'$$

for $A > 1$

$$(24')$$

We can start with the approximation

(a') $\delta n_i = 0$; this implies not only smallness of the losses but also exaggerated sharpness of the distribution derivative approaching the walls. Note that $\delta n_i = 0$ does not imply $\delta n_e = 0$.

(b') $\delta n_e = 0$ in the left hand side of the second equation (24').

As consequence we clearly get

$$N(s) \simeq N_0 - N_{e\infty}(s) \qquad (29)$$

where

$$N_{e\infty}(s) = 2\pi \int_1^\infty A n_\infty(A_1 s) \, dA$$

Next, using (28) we have

$$N_e(s) \simeq \pi \, dN_{e\infty}/ds$$

That is : the particles would accumulate (in absence of walls) outside the limit $A = 1$ at a rate $\pi \cdot dN_{e\infty}/ds$ in a half period of betatron oscillations; but the walls swallow them at approximatily the same rate, the approximation depending on the accuracy of eq. (29).

To justify (a'), (b') we may attach eq. (24') with a development in power series of μ, that is of the ratio between the betatron wavelength and the mean free path. When scattering is missing we have :

$$n(A, s) = n_\infty(A, s) = n_0(A) ;$$

for $\mu > 0$ we can develop n in a series

$$n (A, s) = n_0 (A) + \mu n_1 (A, s) + \ldots$$

and similarly

$$n_\infty (A, s) = n_0 (A) + \mu n_{1\,\infty} (A, s) + \ldots$$

$$L (r, s) = 1 + \mu l_1 (r, s) + \ldots = 1 - \int_0^s Q_0 (r, s') \, ds' + \ldots$$

Substituting into eq. (24) we get the results

$$n_1 (A, s) = n_{1\infty} (A, s) \qquad \text{for } A < 1$$

$$n_1 (A, s) = n_{1\infty} (A, s) - \frac{1}{\pi} \int_0^s ds' \, n_1 (A, s') \qquad \text{for } A > 1$$

The first order term is thus discontinuous at $A = 1$, so confirming that assumption (a') is correct at least to the order μ. The $A > 1$ equation serves only to pass to the μ^2 approximation; it gives

$$n_1(A,s) = \int_0^s \frac{\partial n_{1\infty}}{\partial s'} \exp\left(-\frac{s-s'}{\pi}\right) ds' \simeq \pi \frac{\partial n_{1\infty}}{\partial s}$$

Going further on we obtain

$$n_2(A,s) = n_{2\infty}(A,s) - \frac{1}{\mu} \int_0^s ds' \int_0^r rdr J_0 (rA) Q_0 (r,s')$$

$$\times \int_1^\infty A'J_0(rA')n_{1\infty} (A',s') \, dA' \qquad \text{for } A < 1$$

We need not the $A > 1$ second order solution if we stop the accuracy at the second order for the losses.

Thus, to the order μ^2

$$N(s) \simeq [N_0 - N_{e\infty} (s)]_{\text{II appr.}}$$

$$- 2\pi\mu \int_0^s ds' \int_0^\infty dr \, J_1(r) \, Q_0(r,s') \int_1^\infty A'J_0(rA') \, n_{1\infty}(A',s')dA$$

This formula is valid only for s small enough and cannot give informations about the losses on the whole acceleration cycle. The point is now that the series development in powers of μ gives a justification for keeping (29) as a good estimate only when the energy variation in a mean

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* See however for more accurate cross sections [2,8].

free path is large enough.

On the other hand we may estimate what is the probability that a particle, once scattered to a $A > 1$ amplitude, comes back to an allowed $A < 1$ amplitude by a second scattering when there are no walls: clearly the $A < 1$ open-space distribution differs from the wall-bounded one because of such reentering particles. We can largely overestimate indeed this probability supposing that

1. the energy is constant;

2. s is large so that two collisions certainly occur

3. the eventuality of a third collision bringing again the particle to $A > 1$ is not accounted for.

Detailed calculations are very tedious and we need only a rough information; it can be easily shown that in the screened Rutherford scattering approximation the ratio between reentered and lost particles is $\ll 20\%$ and cannot affect (29) in a serious manner.

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We consider the case of an initial uniform amplitude distribution function (for $A < 1$). If N_0 is the total number of injected electrons we have

$$G_0 (r) = 2 N_0 J_1 (r)/r \qquad (31)$$

It follows that

$$n_\infty(A, s) = \frac{N_0}{\pi} \int_0^\infty J_0(rA) J_1 (r) L (r, s) \, dr$$

and

$$N_0 - N_{e\infty} (s) = 2 N_0 \int_0^\infty \frac{J_1^2(r)}{r} L (r,s) \, dr$$

We take for the differential cross section the simple form *

$$H (\varkappa, E) d\varkappa = \tfrac{1}{2} k \frac{d(\varkappa/\Theta)}{\left[1 + \left(\frac{\varkappa}{\Theta}\right)^2\right]^{3/2}}, \quad \Theta = \frac{W}{E} \qquad (32)$$

from which it follows that

$$Q(\omega,s) = \mu\,[1 - \omega\,\hat{\alpha}\,K_1(\omega\hat{\alpha})] \quad \text{(see appendix)}$$

where

$$\hat{\alpha} = 2\lambda\Theta/b \qquad (33)$$

and K_1 is a Bessel function of the second kind. If the residual gas is air, $W \simeq 0.00876$ Mev and, at ordinary temperature, for air

$$k = 1.22 \times 10^{-2} \, p \, cm^{-1}$$

where p is the pressure in mm. Hg.

Then

$$Q_0(r,s) = Q_0(\hat{r\alpha}) = \mu r\hat{\alpha} I_1(\hat{r\alpha}/2) K_0(\hat{r\alpha}/2) \qquad (34)$$

where I_1 is a modified Bessel function of the first kind. From (33) and remembering that E is linear in s we have

$$ds = -\gamma \; d\hat{\alpha}/\hat{\alpha}^2$$

where

$$\gamma = 2Wc/b\dot{E}$$

Putting $\hat{\alpha}(E_i) = \alpha_i$ and $\hat{\alpha}(E) = \varepsilon\alpha_i$, we have

$$ds = -\gamma/\alpha_i \cdot d\varepsilon/\varepsilon^2 \quad 0 \leqslant \varepsilon \leqslant 1$$

Defining

$$\mathcal{E}(y) = \int_0^y \frac{I_1(x)K_0(x)}{x} \, dx$$

we obtain

$$\int_0^s Q_0(r_1 s') \, ds' = \gamma\mu r \left\{ \mathcal{E}\left(\frac{r\alpha_i}{2}\right) - \mathcal{E}\left(\frac{r\varepsilon\alpha_i}{2}\right) \right\} \qquad (35)$$

With a good approximation, the function $y\mathcal{E}(y)$ is

$$y \, \mathcal{E}(y) \approx 2y^2/(1 + 2y) \quad \text{(see appendix)}$$

Then we have

$$L(r,s) \approx \exp \left\{ -\sigma \frac{r^2(1-\varepsilon)}{(1 + r\alpha_i)(1 + r\varepsilon\alpha_i)} \right\}$$

$$L(r,\infty) \approx \exp \left(-\sigma \frac{r^2}{1 + r\alpha_i} \right)$$
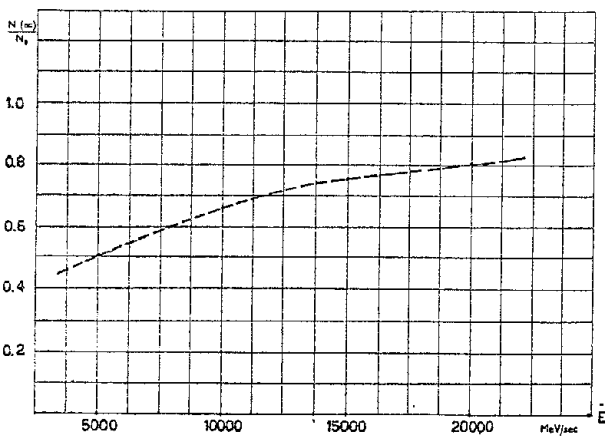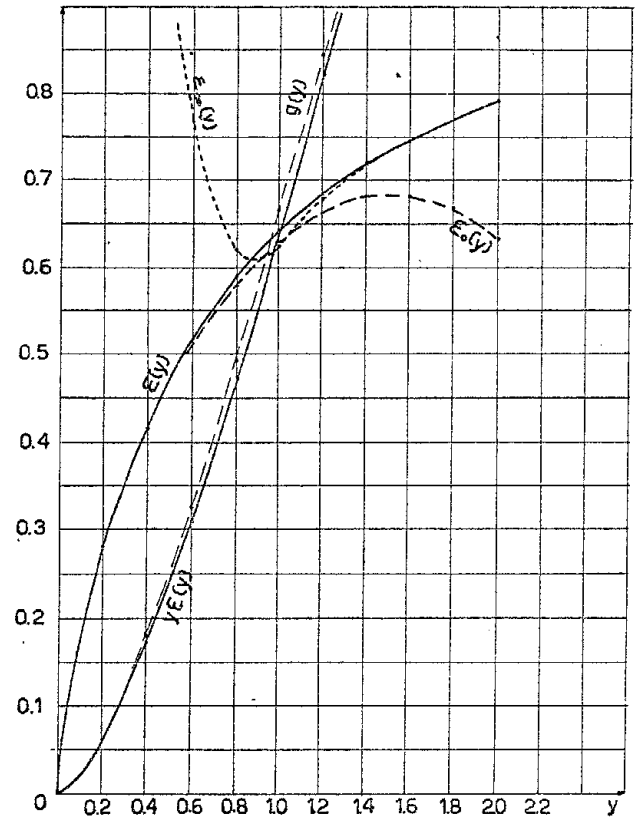

Fig. 3.

where

$$\sigma = \gamma\mu\alpha_i = \frac{kc\,E_i}{E} \alpha_i^2$$

is the quantity (6) already seen at the beginning. The first approximation to the surviving fraction of electrons was given by

$$\frac{N(\infty)}{N_0} \approx 1 - \frac{N_{e\infty}(\infty)}{N_0}$$

and this is plotted in fig. (2) as a function of $\dot{E}$ for b = = 7 cm., p = 10^{-5} mm. Hg, E_i = 2.5 Mev, λ = 500 cm.

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Apendix

Remembering the definition (11) of P(s,α) and (18) of Q_0 we have

Fig. 2.

$$1 - \frac{1}{\mu} Q_0(r,s) = \int_{-\infty}^{+\infty} P(s,\alpha) J_0(r\alpha) \, d\alpha$$

For the special form of P adopted

$$P(s,\alpha) = \frac{1}{2} \frac{\hat{\alpha}^2}{(\alpha^2 + \hat{\alpha}^2)^{3/2}}$$

We obtain

$$Q_0(r,s) = \mu[1 - M(r\hat{\alpha})]$$

where

$$M(u) = \int_0^\infty \frac{J_0(u\alpha)}{(1 + \alpha^2)^{3/2}} \, d\alpha$$

It is easy to show that

$$M(u) = \frac{u}{2} \{I_0(u/2) K_1(u/2) - I_1(u/2) K_0(u/2)\}$$

and, because

$$I_0(x) K_1(x) + I_1(x) K_0(x) = 1/x$$

finally we obtain (34).

Then

$$\int_0^s Q_0(r,\bar{s}) d\bar{s} = \gamma \int_0^{\alpha_i} \frac{Q_0(r\alpha)}{\alpha^2} \, d\alpha$$

$$-\gamma \int_0^{\hat{\alpha}} \frac{Q_0(r\alpha)}{\alpha^2} \, d\alpha = \gamma\mu r \left\{ \mathcal{E}\left(\frac{r\alpha_i}{2}\right) - \mathcal{E}\left(\frac{r\hat{\alpha}}{2}\right) \right\} \quad (35)$$

It is easily shown that

for $y \to \infty$

$$\mathcal{E}(y) \to 1 - \frac{1}{2y} - \frac{1}{8y^3} + \ldots = \mathcal{E}_\infty(y) + \ldots$$

for $y \to 0$

$$\mathcal{E}(y) \to y/2 \, (1 + y^2/_8)(1 - \ln C y) + \ldots = \mathcal{E}_0(y) + \ldots$$

with $\ln C = -0.1159$

The region $y \approx 1$ has been covered by numerical integration. The function

$$2y^2/(1 + 2y) = g(y)$$

is a good approximation (within 5%) over the whole range of y values, for y $\mathcal{E}(y)$ (see fig. 3),.

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