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# A Theory of the Capture in a High Energy Injected Synchrotron.

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## 1. – Introduction.

In all synchrotrons for protons and in most of the modern electrosynchrotrons the particles are injected with a kinetic energy of the order of at least one MeV while the r.f. oscillating cavity is not yet excited, or is oscillating at a negligible amplitude. The particles accumulate in the donut, spiraling inwards because of the rising magnetic field, until the radio-frequency cavity starts oscillating, as suddenly as possible. At this instant, the particles that happen to be in a favourable phase are « captured » by the radio-frequency, and form one or more bunches that start being accelerated, while those that are caught in unfavourable phases are lost against the walls.

We will now expose an attempt to calculate the number of captured particles as a function of the radio-frequency peak voltage (and therefore of the initial equilibrium phase), of the width of the donut and other parameters of the machine. This calculation, although made under several simplifying assumptions, should give a criterium for choosing the initial equilibrium phases (for which we find an optimum value) and for evaluating the influence of the donut width on the output of the machine.

The simplifying assumptions are the following. We suppose that all the injected particles have the same energy and the same direction (parallel to the equilibrium orbit). The theory could be extended to the case of an injected beam having an energetic and angular spread by subdividing it in elementary beams.

We suppose also that the radio-frequency peak voltage across the cavity rises *istantaneously* from 0 to a given value $U$. By this we mean that the rise time should be short compared with the period $T_p$ of the phase oscillations *and* with $\sqrt{T_0\theta}$, where $T_0$ is the rotation period for the velocity $c$ and $\theta$ is the duration of the injection.

When the rise of the radio-frequency voltage cannot be considered istantaneous, the theory needs a numerical integration of the phase equation for each set of parameters, which has been worked out only for a few typical cases. It seems that the results are not very different from those of the istantaneous rise, even for a rise time of $T_x/h$ and of $4\sqrt{T_0\theta}$.

## 2. – Betatron Oscillations.

Let $t$ be the time and

$$B(r,t) = B_0(t)(R/r)^n\,, \tag{1}$$

be the vertical magnetic induction at radius $r$. Here $R$ is the radius of the so called « principal orbit » (running approximately in the middle of the donut section) and $B_0(t)$ is the magnetic induction on it: during the injection $B_0(t)$ can be considered a linear function.

The equilibrium orbit (e.o.) at time $t$ is defined by

$$r_i = 3.335\cdot 10^{-3}\,\frac{\beta E}{B(r_i,t)} = \tag{2}$$

($r_i$ in cm, $B$, $B_0$ in gauss)

$$= \left(3.335\cdot 10^{-3}\,\frac{\beta E}{B_0 R^n}\right)^{1/(1-n)} \tag{2'}$$

where $E$ is energy of the particles (included the rest energy) in eV, and $\beta$ is the velocity divided by $c$. The e.o. contracts at each tour approximately by

$$\sigma = \frac{R}{1-n}\,\frac{\dot{B}_0}{B_0}\,T\,, \tag{3}$$

($T$ being the duration of the revolution).

We take as origin of time the instant at which the e.o. passes through the injection point: so at time $t$ the e.o. will pass at a distance $\sigma t/T$ from that point.

We suppose that the radio-frequency cavity starts oscillating at the time $\theta$ at which $r_i = R$, and that during the interval from $t=0$ to $t=\theta$ the injector keeps injecting at a constant rate of $I$ particles per second. We consider this interval as the « injection interval »: probably some particles injected before $t=0$ or after $t=\theta$ can still be captured but we neglect them. So the duration of the injection is

$$\theta = \frac{aT}{2\sigma}\,, \tag{4}$$

where $a/2$ is the distance of the injection point from the principal orbit (and therefore $a$ is the useful donut width).

Of course, part of the particles injected between 0 and $\theta$ will be lost, before time $\theta$, owing to collision against the injector or the walls. We introduce therefore a quantity $\varrho(t) \leqslant 1$, representing the « *instantaneous injection efficiency* », such that $I\varrho(t)\,\mathrm{d}t$ is the number of particles injected between $t$ and $t+\mathrm{d}t$ still surviving at time $\theta$. The calculation of $\varrho$ does not belong to this paper. We can only say that, as a rule, it will increase from $\varepsilon$ to almost 1 during the first few revolutions and then remain near unity for most of the time $\theta$.

Now, a particle injected at time $t$ will be injected at a distance

$$x_M = \frac{\sigma t}{T}, \tag{5}$$

from its e.o.: so it will start betatron oscillations of this amplitude about the e.o. These oscillations will continue, practically undamped, following the e.o. while it contracts, so that at time $\theta$, when the radio-frequency is excited, the particle will oscillate about the principal orbit.

So, at time $\theta$, all surviving particles will be oscillating, with various amplitudes, about the principal orbit. Those, which have an amplitude between $x_M$ and $x_M+\mathrm{d}x_M$ will be those injected between $t$ and $t+\mathrm{d}t$, and their number will be

$$I\varrho(t)\,\mathrm{d}t = I\varrho T\frac{\mathrm{d}x_M}{\sigma}. \tag{6}$$

## 3. – Synchrotron Oscillations.

As soon as the radio-frequency is excited, synchrotron oscillations are superimposed on betatron oscillations. This means that the e.o., about which the betatron oscillations take place, is no longer slowly contracting, but contracting and expanding alternately about the principal orbit. Besides, the e.o. is no longer the same for all particles because each particle, at each revolution, will pick up a different (positive or negative) amount of energy according the the *phase* $\varphi$ in which it happens to cross the radio-frequency cavity.

Let us recall the law of synchrotron oscillations [1]. The phase $\varphi$ varies according to the equation

$$\ddot{\varphi} = M\left(1-\frac{U}{u}\sin\varphi\right), \tag{7}$$

[1] D. BOHM and L. FOLDY: *Phys. Rev.*, **70**, 249 (1946).

where $U$ is the peak value of the radio-frequency voltage across the cavity (in volt); $u$ is the energy (in eV) which must be supplied at each revolution to the particle in order that it keeps the same radius $R$ notwithstanding the rise of the magnetic field: it is given by

$$u = (1-n)\beta^2 E \frac{\sigma}{R}, \tag{8}$$

$M$ is defined by

$$M = 2\pi \frac{kK}{T^2}\frac{u}{E} = 2\pi\beta^2(1-n)\frac{kK}{T^2}\frac{\sigma}{R}, \tag{9}$$

where

$$K = 1 + \frac{1}{\beta^2}\left[\frac{1}{\Lambda(1-n)} - 1\right]. \tag{10}$$

$\Lambda$ is the total length of the principal orbit divided by the length of the curved parts; $k$ is an integer (usually called the « harmonic order ») giving the ratio between the radio-frequency and the frequency of revolution.

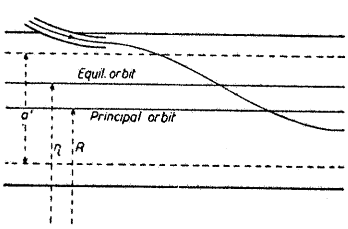

Fig. 1.

To the variations of the phase $\varphi$ are related the oscillations of the e.o. through the relation

$$X = \frac{-R}{(1-n)\beta^2 K\omega_e}\dot{\varphi}, \tag{11}$$

where: $X = R_i - R$ is the displacement of the e.o.;
$\omega_e = k(2\pi/T)$ is the pulsation of the radio-frequency field.

During the injection, that is when $U=0$, eq. (11) and (7) give, of course, the well known contraction of the e.o. at the rate $\sigma/T$.

After time $\theta$, $U$ has a constant positive value (greater than $u$). Equation (7) can be integrated once and gives

$$\dot{\varphi}^2 = 2MF(\varphi) + \text{const.}, \tag{12}$$

where

$$F(\varphi) = \frac{U}{u}\cos\varphi + \varphi. \tag{13}$$

If we define the « equilibrium phase » (or « synchronous » phase) $\varphi_s$ by

$$\sin\varphi_s = u/U, \qquad 0 \leqslant \varphi_s \leqslant \pi/2 \tag{14}$$

equation (13) can be written

$$F(\varphi)=\frac{\cos\varphi}{\sin\varphi_s}+\varphi\,. \tag{13'}$$

The function $F(\varphi)$ is plotted in Fig. 2 [1].

It has obviously a maximum at $\varphi=\varphi_s$.

Equation (12) permits plotting $\dot\varphi$ (and therefore $X$) as a function of $\varphi$ (Fig. 3). If we consider the « *synchronous particle* » running on the principal orbit at an angular velocity $\omega_e/k$ with the constant phase $\varphi_s$, and call $\theta_s(t)$ its azimut, we have

$$\theta-\theta_s=(\varphi-\varphi_s)/k\,,$$

(since $\dot\varphi=k\dot\theta$) and so the abscissa in Fig. 3 represents, in a convenient scale, the azimuth of a generical particle referred to that of the synchronous particle. On the other hand the ordinate, through equation (11), represents its radial displacement (apart from betatron oscillations). Therefore the curves of Fig. 3 represent the mouvement of any particle, apart from betatron oscillations, with respect to the synchronous particle.

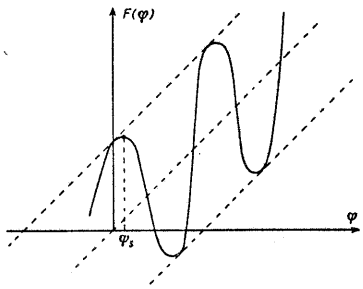


Fig. 2.

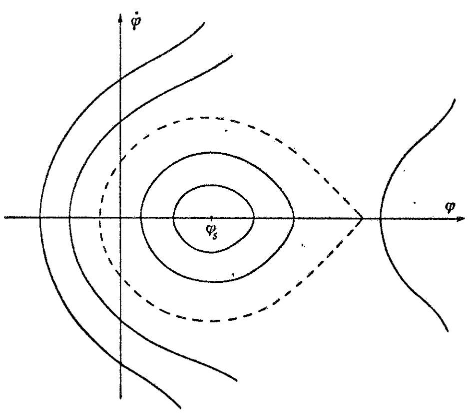


Fig. 3.

These curves fall into two classes, according to the value of the constant in eq. (12). Some of them are open curves, and represent particles that, sooner or later, collide with the walls and are lost. The remaining, and more interesting, curves, are closed, and represent particles oscillating around the synchronous particle.

Let us consider one of these closed curves. If we call $\varphi_m$ and $\varphi_M$ ($\varphi_m<\varphi_s<<\varphi_M$) the extreme values of $\varphi$, we can assume one of these quantities, f.i. $\varphi_M$, as a parameter to label the curve. The other extreme, $\varphi_m$, is related to $\varphi_M$ by a universal function

$$\varphi_m=g(\varphi_M,\varphi_s)\,,$$

defined by $F(\varphi_m)=F(\varphi_M)$, which has been numerically calculated for different values of $\varphi_s$. The widest of all closed curves (dotted line in Fig. 3) corresponds to $\varphi_M = \pi - \varphi_s$.

Equation (12) can now be written

$$\dot{\varphi}^2 = 2M[F(\varphi) - F(\varphi_M)] . \tag{15}$$

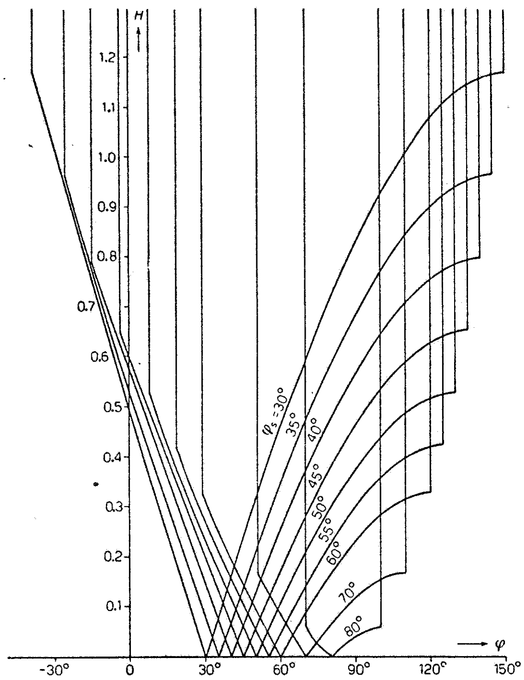


Fig. 4.

Now we are interested in the amplitude $X_M$ of the synchrotron oscillation, and therefore (eq. (11)) in the maximum ordinate $\dot{\varphi}_M$ of the curve. It is seen immediately that it corresponds to $\varphi = \varphi_s$ and is given by

$$\dot{\varphi}_M = \sqrt{2M}\,\sqrt{F(\varphi_s) - F(\varphi_M)} .$$

We have plotted, in Fig. 4, the universal function

$$H(\varphi) = \sqrt{F(\varphi_s) - F(\varphi)} ,$$

for several values of $\varphi_s$. By means of these curves, we can immediately determine the amplitude $X_M$ of the synchrotron oscillation as a function of one of the extremes ($\varphi_m$ or $\varphi_M$) of the phase oscillation. In effect, from (11) and (15) we get

$$X_M = CH(\varphi_m) = CH(\varphi_M) , \tag{16}$$

where

$$C = \sqrt{\frac{\sigma R}{\pi k K(1-n)\beta^2}} . \tag{17}$$

## 4. – Theory of Capture.

Since the cavity starts oscillating at the instant $\theta$ at which $X = \dot{\varphi} = 0$, the initial conditions for the synchrotron oscillations are: $\dot{\varphi}_0 = 0$, and $\varphi_0$ distributed with uniform probability (2) between $-\pi$ and $\pi$. This initial

(2) The phase of the particle at time $\theta$ is related to the instant of its injection $t$ by: $\varphi_0 = \text{const.} - \omega_0 t$. So the particles injected in an interval $\Delta t = T/k = t\pi/\omega_i$ have phases spread uniformely between $-\pi$ and $\pi$, and since $T/k \ll \theta$ we can practically assume $\Delta t$ as an elementary interval $dt$.

value $\varphi_0$ will be the $\varphi_m$ or the $\varphi_M$ of the phase oscillations, according whether it is $<\varphi_s$ or $>\varphi_s$.

Those particles, whose $\varphi_0$ happens to be in the region of closed curves, i.e. between $\pi-\varphi_s$ and $g(\pi-\varphi_s)$, initiate their oscillations around the synchronous particle, while the others are thrown against the walls. But only a part of the former group of particles can really perform their oscillations without hitting the walls, namely those for which the *total* amplitude of oscillation (synchrotron amplitude $X_M$ plus betatron amplitude $x_M$) is less than the *useful* half width of the donut. We consider as « useful » width $a'$ the radial width $a$ diminished by a certain amount to take into account the gas scattering and the distortion of the orbit due to field irregularities and frequency errors. Therefore, once chosen the value of $a'$, the capture condition can be written

$$X_n + x_n \leqslant \frac{a'}{2}. \tag{18}$$

Now we want to calculate how many particles satisfy this condition.

Let us consider first only the particles injected between $t$ and $t+dt$. They are $I\varrho\, dt$ and perform betatron oscillations given by (5): so the capture condition for them is

$$X_M \leqslant \frac{a'}{2} - \frac{\sigma t}{T},$$

or, because of (16)

$$H(\varphi_0) = \frac{1}{C}\left(\frac{a'}{2} - \frac{\sigma t}{T}\right).$$

If, in Fig. 4, we draw an horizontal line of ordinate

$$H = \frac{1}{C}\left(\frac{a'}{2} - \frac{\sigma t}{T}\right) \tag{19}$$

and call $\Phi(H)$ the length intercepted on it by the curve of given $\varphi_s$, divided by the abscissa length corresponding to 360°, this $\Phi$ represents the captured fraction of the particles injected in $dt$. So the total number of captured particles will be

$$N = \int_0^{\theta} \Phi\left(\frac{a'}{2C} - \frac{\sigma t}{CT}\right) I\varrho\, dt. \tag{20}$$

Let us take $H$ (eq. (19)) as integration variable and remark that for $H < 0$

we must assume $\Phi(H)=0$, because these particles have $x_M > a'/2$. Then equation (20) becomes

$$(21) \qquad N=\frac{CT}{\sigma}\int_0^\alpha \Phi(H) I_\varrho \,\mathrm{d}H\,,$$

where

$$(22) \qquad \alpha=\frac{a'}{2C}\,.$$

We shall discuss later the factor $\varrho$, for the moment let us call $\bar{\varrho}$ a convenient average of $\varrho$ and suppose that $I$ is constant.

Then

$$(23) \qquad N=I\bar{\varrho}\,\frac{CT}{\sigma}\int_0^\alpha \Phi(H)\,\mathrm{d}H\,.$$

The integral represents the area between the curve of Fig. 4 and the horizontal line of ordinate $\alpha$. It has been calculated numerically as a function of $\alpha$ for different $\varphi_s$ and is plotted in Fig. 5. Remark that these curves represent universal functions, not depending on the parameters of the machine.

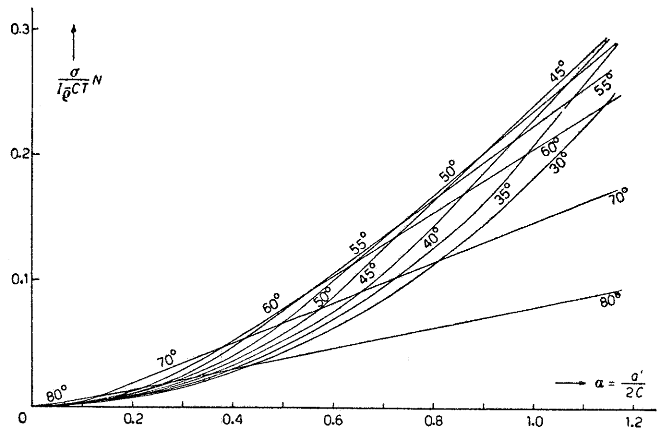


Fig. 5.

Now, this diagram shows that the family of curves has an envelope on the upper side. Therefore, for each value of $\alpha$ there exists a value of $\varphi_s$ for which the number of captured particles is a maximum: it is the value belonging to that curve, which is tangent to the envelope at the point of abscissa $\alpha$. So, if we call $J(\alpha)$ the ordinates of the envelope, the number of captured particles corresponding to the optimum $\varphi_s$ is

$$N_{\max} = I\bar{\varrho}\,\frac{CT}{\sigma}\,J\left(\frac{a'}{2C}\right) \tag{24}$$

or, using (22)

$$N_{\max} = I\bar{\varrho}\,\frac{a'T}{2\sigma}\,\frac{J(\alpha)}{\alpha}\,. \tag{24'}$$

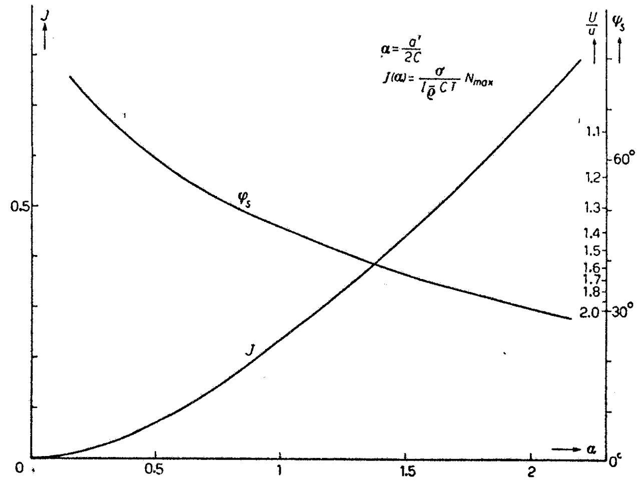


Fig. 6.

The function $J(\alpha)$, numerically calculated, is given in Table I. The same table gives also the optimum $\varphi_s$ and $U/u = \operatorname{cosec} \varphi_s$ as functions of $\alpha$. This table enables one to find immediately the most convenient radiofrequency voltage and the corresponding number of captured particles. For $\alpha < 2$ one can also use the diagram (Fig. 6) where the functions $J$, $\varphi_s$, $U/u$ are plotted against $\alpha$.

TABLE I.

| $\alpha$ | $J(\alpha)$ | $\varphi_s(\alpha)$ | $U/u$ |
|---|---|---|---|
| 0.0 | 0.000 | | |
| 0.5 | 0.072 | 59.5° | 1.16 |
| 1.0 | 0.234 | 46.0 | 1.39 |
| 1.5 | 0.447 | 36.8 | 1.67 |
| 2.0 | 0.687 | 30.6 | 1.96 |
| 2.5 | 0.952 | 26.3 | 2.26 |
| 3.0 | 1.24 | 23.1 | 2.55 |
| 3.5 | 1.55 | 20.4 | 2.87 |
| 4.0 | 1.87 | 18.4 | 3.17 |
| 4.5 | 2.20 | 16.6 | 3.50 |
| 5.0 | 2.53 | 15.1 | 3.84 |

## 5. – Discussion of the Results.

To discuss the influence of various parameters on the number of captured particles it is convenient to approximate the function $J(\alpha)$ by an empirical formula. A good one is

$$J(\alpha) = 0.242\alpha^{\frac{3}{2}} . \tag{25}$$

Then eq. (24′) becomes

$$N_{\max} = 0.11 \cdot I\bar{\varrho}T \left(\frac{kK(1-n)\beta^2}{R}\right)^{\frac{1}{4}} \frac{a'^{\frac{3}{2}}}{\sigma^{\frac{5}{4}}} \tag{26}$$

In the case of an electrosynchrotron we can assume $\beta = \sim 1$; then we get

$$N_{\max} = 0.11 \cdot I\bar{\varrho}T \left(\frac{k}{\Lambda R}\right)^{\frac{1}{4}} \frac{a'^{\frac{3}{2}}}{\sigma^{\frac{5}{4}}} . \tag{26′}$$

This formula can be written (expressing all lengths in cm)

$$N_{\max} = 0.23 \cdot 10^{-10} I\bar{\varrho}k^{\frac{1}{4}}(\Lambda R)^{\frac{3}{4}} \frac{a'^{\frac{3}{2}}}{\sigma^{\frac{5}{4}}} .$$

The most interesting result contained in these formulae is the proportionality of $N_{\max}$ to $k^{1/4}a'^{3/2}\sigma^{-5/4}$. It is, however, based on the assumption that the « injection efficiency » $\varrho(t)$ is practically constant over most of the injection time.

If the r.f. voltage $U$ is not given its optimum value, the number of captured particles decreases in a measure that can be easily calculated from the curves of Fig. 5.

## 6. – Influence of the Injection Losses.

Let us now discuss the influence of the variation of the injection efficiency $\varrho(t)$ during the injection time.

If the spiral pitch $\varrho$ is very small, a good deal of particles will be lost against the deflector or the walls in the first two or three revolutions, while in the subsequent revolutions these losses will be negligible. That means that the integral in (21) represents now a sort of « weight » of the shaded area of Fig. 4 with a density which is $< 1$ in a strip near the upper border and almost $= 1$ in the remaining part of the area.

This suggests dividing the injection time $\theta$ in a first, short interval $\theta_1$, (corresponding to the upper strip) in which $\varrho < 1$, while in the remaining interval $\theta - \theta_1$ it can be assumed $\varrho = 1$. The two parts may give comparable contributions to the integral, because the low density corresponds to the wider part of the area, that is, because the early injected particles have lesser betatron amplitude and so are captured in a wider phase interval.

Putting

$$\alpha_1 = \frac{1}{C}\left(\frac{a'}{2} - \sigma\frac{\theta_1}{T}\right), \tag{27}$$

equation (21) becomes

$$N = IC\frac{T}{\sigma}\left[\int_0^{\alpha_1}\Phi(H)\,\mathrm{d}H + \int_{\alpha_1}^{\alpha}\Phi(H)\varrho\,\mathrm{d}H\right].$$

The first integral can be calculated be means of the curves of Fig. 5. The second integral, in most cases, will extend over an interval of $\alpha$ in which the curve of Fig. 4 is vertical, that is $\Phi$ is constant. In such cases it will be

$$N = IC\frac{T}{\sigma}\int_0^{\alpha_1}\Phi(H)\,\mathrm{d}H + I\theta_1\bar{\varrho}_1\Phi(\alpha),$$

where $\bar{\varrho}_1$ is now the average of $\varrho(t)$ during the interval $\theta_1$.

The determination of the optimum $\varphi_s$ and of the corresponding $N_{\max}$ must be done in this case by trial and error.

This theory has been extended, by I. Solomon, to a strong focusing synchrotron. The final formulae are the same, provided $K$ and $C$ are replaced by

$$K = 1 + \frac{1}{\beta^2}\left[\frac{1}{\Lambda Q^2} - 1\right],$$

$$C = \frac{1}{\sqrt{Q^2 - 1}}\sqrt{\frac{R\sigma}{\pi k}},$$

where $Q$ is defined by

$$\frac{\mathrm{d}r}{r} = \frac{1}{Q^2}\frac{\mathrm{d}p}{p},$$

$p$ being the momentum of the particle.