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STABILITY AND PERIODICITY IN THE STRONG-FOCUSING ACCELERATOR

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Stability and Periodicity in the Strong-Focusing Accelerator (*).

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Summary. — A simple formalism is developed for the study of a strong-focusing accelerator of arbitrary geometry. It is shown to yield readily the necessary and sufficient conditions for stability and for periodicity (with any prescribed period) of the orbits. Criteria are also given which allow the determination of the range of initial conditions for which the orbits are stable.

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1. – We consider an accelerator consisting of circular sectors which, for a given mode of motion, act as focusing (fs) or defocusing (dfs) and of straight field-free sectors (ffs) disposed according to some periodic pattern which we need not specify. The behaviour of an accelerator of thys type (1) has been studied till now only for some special patterns and with methods which do not easily allow an understanding of its general properties. In particular, the effects of ffs's have been considered as perturbations, while, as we shall show, it is possible to treat them, quite simply, on the same footing as the fs's and dfs's.

Our treatment is valid as far as it is legitimate to adopt, for the description of the particle motion, the customary linearized equations for the vertical and radial displacements from the stable orbit, in the adiabatic approximation. Arguments in support of this procedure are given in the preceding work (2),

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(*) This work has been performed at the University of Rome.

(1) E. COURANT, S. LIVINGSTON and H. SNYDER: *Phys. Rev.*, **88**, 1190 (1952).

(2) E. R. CAIANIELLO: *Nuovo Cimento*, **10**, 581 (1953).

where the effects of non-linearity of the field are estimated to be unimportant in comparison with those arising from imperfect construction of the machine.

We study two separate questions:

1) the conditions under which *stability* (i.e. boundedness within the chamber walls) is guarantied for all the orbits of a given bundle;

2) the conditions which must be fulfilled in order that the orbits be *periodic* after traversing an arbitrary number of sectors. We shall find here, as a particular case, the well known periodicity after eight sections.

Only 1) is important if we restrict ourselves to the case of an «ideal» machine, as we do throughout this work. Knowledge of 2) becomes, however, important when considering the actual case, in which one wants to estimate the tolerances of the machine, and needs know whether small systematic perturbations are enhanced by resonances or not. This question is treated in detail in a very interesting work by SANDS and TOUSCHEK ([3]).

We consider only the vertical motion; everything we say applies also to the case of radial motion, due to the large value of the field gradient (n). We take $|n|$ constant in all fs's and dfs's: see, however, Part. 3·2).

2·1. – If z denotes the vertical displacement from the stable orbit, R the radius of this orbit, $|n|$ the magnitude of the field gradient, the standard linearized equations for z are:

in a fs:

$$z'' + \frac{|n|}{R^2}\, z = 0 , \tag{1}$$

in a dfs:

$$z'' - \frac{|n|}{R^2}\, z = 0 . \tag{2}$$

The derivative is taken with respect to the arc length $l = Rq$. (Comparison of (1) and (2) with formula (10) of ref. ([2]), from which they can be immediately derived, gives an idea of the approximation which is made by neglecting non-linearities of the field: $2\beta \approx |n|/R^2$).

Introduce now the two-component vector:

$$\boldsymbol{z} \equiv \left\{ z, \frac{R}{\sqrt{|n|}}\, z' \right\} .$$

(there is actually no particular reason for choosing the coefficient of z' in this fashion: nothing would be changed in the sequel by taking a different coef-

([3]) M. SANDS and B. TOUSCHEK: *Nuovo Cimento*, **10**, 604 (1953).

ficient, except the discussion of Part 3·1, which should have to be modified in an obvious manner). If z_0 denotes the value of z at entry of a section, one sees immediately from (1) and (2) that the value at exit of that section is given by:

$$(3)\qquad z = \Gamma_+(l_+)z_0\,, \qquad \text{with} \qquad \Gamma_+(l_+) = \begin{pmatrix} \cos\dfrac{\sqrt{|n|}}{R}\,l_+ & \sin\dfrac{\sqrt{|n|}}{R}\,l_+ \\ -\sin\dfrac{\sqrt{|n|}}{R}\,l_+ & \cos\dfrac{\sqrt{|n|}}{R}\,l_+ \end{pmatrix}$$

for a fs of length l_+ and by

$$(4)\qquad z = \Gamma_-(l_-)z_0\,, \qquad \text{with} \qquad \Gamma_-(l_-) = \begin{pmatrix} \cosh\dfrac{\sqrt{|n|}}{R}\,l_- & \sinh\dfrac{\sqrt{|n|}}{R}\,l_- \\ \sinh\dfrac{\sqrt{|n|}}{R}\,l_- & \cosh\dfrac{\sqrt{|n|}}{R}\,l_- \end{pmatrix}$$

for a dfs of length l_-. It is also readily verified that, at exit of a ffs of length l_0

$$z = \Gamma_0(l_0)z_0\,, \qquad \text{with} \qquad \Gamma_0(l_0) = \begin{pmatrix} 1 & \dfrac{\sqrt{|n|}}{R}\,l_0 \\ 0 & 1 \end{pmatrix}.$$

A convenient property of these matrices is that

$$(5)\qquad \det \Gamma_+ = \det \Gamma_- = \det \Gamma_0 = +1\,.$$

If the particle enters a sequence of sectors, say, dfs + ffs + fs + ffs′, it exits from it with

$$z = \Gamma_0(l_0')\Gamma_+(l_+)\Gamma_0(l_0)\Gamma_-(l_-)z_0\,.$$

The effect of crossing any number of sectors of various types is always described by a matrix which is the product (from right to left) of the matrices corresponding to the crossed sectors (from left to right).

The machine is a periodic structure, in which any given type of sector reappears again and again. We can consider it therefore as composed of consecutive identical « units », calling by this name the *smallest* set of contiguous sectors which can be regarded as the periodic element.

We leave, otherwise, completely arbitrary the number and type of sectors which, together, form the unit of the machine. We know from the preceding considerations that the unit is characterized by a 2×2 matrix Δ with real elements, such that $z = \Delta z_0$ is the exit value of z for a particle which has entered the unit with $z = z_0$. This is all we need to discuss most of the properties of the accelerator.

2·2. – Call $2x$ the spur of the matrix Δ. Since Δ is a product of matrices Γ_+, Γ_-, Γ_0, it follows from (5) that also $\det \Delta = +1$. The characteristic equation of Δ is therefore:

$$\lambda^2 - 2x\lambda + 1 = 0 \,. \tag{6}$$

Consider the roots λ_1, λ_2 of (6). Three cases are possible:

1) λ_1 and λ_2 are complex conjugate, therefore of the form

$$\lambda_1 = e^{i\varphi}\,, \qquad \lambda_2 = e^{-i\varphi}\,; \qquad \varphi \neq k\pi\,, \qquad x = \cos\varphi\,. \tag{7}$$

The conditions for this to happen is $|x| < 1$.

2) $\lambda_1 = \lambda_2$ real: $= 1$ if $x = 1$, $= -1$ if $x = -1$.

3) $\lambda_1 \neq \lambda_2$, both real. This happens if $|x| > 1$.

The following theorem holds (for the proof, see App. I): *The necessary and sufficient condition for the existence of stable orbits is* $|x| < 1$, i.e. that (6) have complex conjugate roots. The following corollary is then immediate: *A necessary condition for stability is that Δ be not symmetric* (or else, Δ being in this case Hermitean because of its reality, its eigenvalues would be real).

The stated theorem tells whether the machine allows stable orbits or not; it does not suffice, however, to delimit the bundle of orbits which are actually stable. (In other words, the initial conditions for which a particle, injected into the (ideal) machine, does remain at all times within the vacuum chamber). Information on this point, which is clearly of importance, can be had if one knows:

a) number, type and ordering of the sectors which together form the unit (smallest periodic element of the machine), in order to determine the maximum value that z can reach within the unit, for given initial conditions. This maximum depends upon the geometry of the unit. We outline in Part 3·1, the general procedure by treating in detaail simple example.

b) the maximum value that z can reach after crossing an arbitrarily large number of sectors. This equation is treated in the next section.

2·3. – The only situation of interest is that described by (7). The discussion is facilitated by observing that Δ can be decomposed as follows:

$$\Delta = \cos\varphi + J \sin\varphi \tag{8}$$

(where J — which has, clearly, spur zero — can be immediately constructed as $(\Delta - \cos\varphi)/\sin\varphi$).

J has the remarkable property that:

$$J^2 = -1 \tag{9}$$

(this can be verified by using the fact that Δ must obey the same characteristic equation (6) as its eigenvalues; the same result is derived in App. I from a more pedestrian point of view). As a consequence of (9), we can evaluate any power of Δ as if Δ were a complex number:

$$\Delta^m = e^{mJ\varphi} = \cos m\varphi + J\sin m\varphi\,. \tag{10}$$

We can now answer question $b)$ of the preceding section. Write $m\varphi = \vartheta$ and consider, in the plane spanned by $\boldsymbol{z}$, the curve of equation:

$$\boldsymbol{z} = \cos\vartheta\,\boldsymbol{z}_0 + \sin\vartheta\,\boldsymbol{t}_0\,, \tag{11}$$

where $\boldsymbol{t}_0 = J\boldsymbol{z}_0$ (fig. 1). This is the equation of an ellipse, of which $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$ are two conjugate semidiameters. The explicit knowledge of J permits one to deduce, with simple geometrical considerations which we leave to the reader, direction and magnitude of the great semiaxis as a function of $\boldsymbol{z}_0$, from which the wanted information is readily gathered. We may just mention here that, once $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$ are given, a standard construction due to Chasles permits one to solve this problem graphically (see any textbook on projective geometry); also, if one is interested only in the magni-

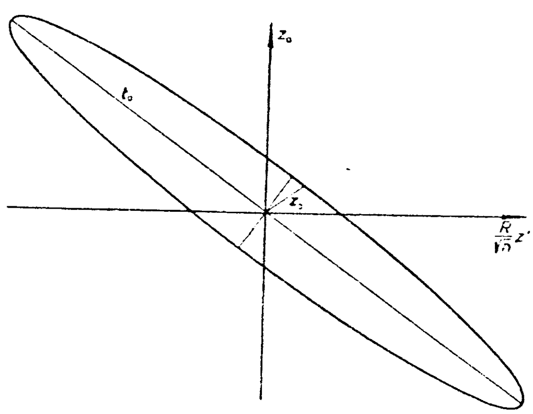

Fig. 1. – Numerical data corresponding to the figure: $|n| = 5\cdot10^3$; $R = 10^4$ cm; $\frac{\sqrt{|n|}}{R}\,L = \frac{\pi}{2}$, viz $L = 2.24\cdot10^2$ cm. $L_0 = 4\cdot10$ cm, viz $h = 2.8\cdot10^{-1}$; it follows $2x = -1.586$.

$$J = \frac{1}{0{,}613}\begin{pmatrix} 2.21 & 1{,}668 \\ -3{,}1535 & -2{,}21 \end{pmatrix}\cdot$$

$$\boldsymbol{z}_0 = \begin{pmatrix} 1 \\ R/\sqrt{|n|}\cdot10^{-2} \end{pmatrix};$$

it follows $|\boldsymbol{z}_0| = 1.75$ cm; $|\boldsymbol{t}_0| = 12{,}8$ cm; $\delta_0 = 109°$. In this numerical and graphical example, the great semiaxis differ in magnitude and direction very little from $\boldsymbol{t}_0$. It is not represented in the figure for the sake of clearness.

tude of the semiaxes, the following construction suffices (it is an immediate consequence of Apollonius' theorems on conics): let a and b be the magnitudes of the semiaxes, δ_0 the angle between $\boldsymbol{z}_0$ and $\boldsymbol{t}_0$: draw the parallelogram with sides of length $|\boldsymbol{z}_0|$ and $|\boldsymbol{t}_0|$, the angle between these two sides being taken $= (\pi/2) - \delta_0$: the two diagonals of this parallelogram have length $a+b$ and $a-b$ respectively.

One sees from (10) that, *for arbitrary m and a stable orbit, all points $\Delta^m \boldsymbol{z}_0$ lie on the ellipse* (11). The following theorem is also evident:

The necessary and sufficient condition for all bounded orbits to have a period of m units is $\Delta^m = +1$, i.e.: from (7) and (10)):

$$\varphi = \frac{2k}{m}\pi\,, \qquad \frac{2k}{m} \neq \text{integer}\,, \tag{12}$$

which gives distinct roots only for $0 < 2k < m$. If m is not a prime one finds, of course, also the periodicity conditions relative to its prime factors: thus, for $m = 6$, one finds $\varphi_1 = \pi/3$ and $\varphi_2 = 2\pi/3$, the second being the periodicity condition for $m = 3$. Periodicity is forbidden for $n = 1$ (see App. I), and then also for $n = 2$.

Assume now that one wants to build a machine such that all orbits have a period of m (and not less than m):

1) a value of φ must be obtained from (12) which secures this periodicity (and not less). This sets one condition on the geometry of the unit, by fixing $x = (\Delta_{11} + \Delta_{22})/2 = \cos\varphi$;

2) a second condition is given by assigning the total number of fs, dfs and ffs contained in the machine.

These two condition may or may not suffice to determine uniquely the geometry. A very simple pattern is that corresponding to a unit dfs + ffs + fs + + ffs, with the dfs and fs having both length L, and the ffs's having all length L_0. The two stated conditions suffice in this case to determine uniquely the geometry, i.e. the values of L and L_0. Δ is given in this case, from (3)-(5), by (writing: $\cos(\sqrt{|n|/R})L = a$, $\cosh(\sqrt{|n|/R})L = a'$, $(\sqrt{|n|/R})L_0 = h$, $b = \sqrt{1-a^2}$, $b' = \sqrt{a'^2 - 1}$)

$$\begin{aligned}
\Delta_{11} &= -bb'h^2 + (2ab' - ba')h + (aa' + bb')\\
\Delta_{12} &= -ba'h^2 + (2aa' - bb')h + (ab' + ba')\\
\Delta_{21} &= -bb'h + (ab' - ba')\\
\Delta_{22} &= -ba'h + (aa' - bb')
\end{aligned}$$

and

$$2x = -bb'h^2 + 2(ab' - ba')h + 2aa'\,.$$

We record in App. II, for possible future reference, some formulae which give an alternative expression of Δ^m.

3·1. – We study here in detail a very simple example which shows clearly the role of the fs's and dfs's and indicates the procedure to follow in general to answer question *a*) of 2·2.

Consider the unit as composed only of one dfs of length L and one fs of length $(\pi/2)(R/\sqrt{|n|})$, so that

$$\Delta = \Gamma_+\left(\frac{\pi R}{2\sqrt{|n|}}\right)\Gamma_-(L) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \cosh\dfrac{\sqrt{|n|}\,L}{R} & \sinh\dfrac{\sqrt{|n|}\,L}{R} \\ \sinh\dfrac{\sqrt{|n|}\,L}{R} & \cosh\dfrac{\sqrt{|n|}\,L}{R} \end{pmatrix}.$$

One finds immediately that $\Delta^2 = -1$, *regardless of the value of* L. Periodicity occurs therefore after four units, i.e. eight sectors. We want to determine the initial conditions for which a particle remains bounded in the vacuum chamber.

Call d the half-aperture of the dfs. Boundedness requires that $|z| < d$ in the dfs. Assume that

$$(13) \qquad |z_0| < k, \qquad \left|\frac{Rz_0'}{\sqrt{|n|}}\right| < k.$$

Since

$$|z| \leq |z_0|\cosh\frac{\sqrt{|n|}\,L}{R} + \left|\frac{Rz_0'}{\sqrt{|n|}}\right|\sinh\frac{\sqrt{|n|}\,L}{R} < k\left(\cosh\frac{\sqrt{|n|}\,L}{R} + \sinh\frac{\sqrt{|n|}\,L}{R}\right),$$

one finds the condition

$$(14) \qquad k \leq \frac{d}{\cosh\dfrac{\sqrt{|n|}\,L}{R} + \sinh\dfrac{\sqrt{|n|}\,L}{R}}.$$

In the next fs. from (3):

$$(15) \qquad |z| < d(|\cos\theta| + |\sin\theta|) \leq \sqrt{2}\,d.$$

If the fs has a half aperture $D > \sqrt{2}\,d$, all the orbits (13), (14) remain bounded (clearly, once boundedness is insured in Δ, it is insured throughout, since, here, $\Delta^2 = -1$). The role of the dfs is to fix k by means of (14); thus on finds, with $d \leqslant D\sqrt{2}$:

$$\frac{\sqrt{|n|}\,L}{R} = \frac{\pi}{2} \qquad k = \frac{d}{4.8}\,,$$

$$\text{»} \quad = \frac{3\pi}{4} \qquad k = \frac{d}{10.5}\,,$$

$$\text{»} \quad = \pi \qquad k = \frac{d}{23.1}\,.$$

3·2. – We add a few final remarks to the preceding considerations.

1) It is possible to describe by means of matrices of type Γ also the sectors in which an electric field is applied to accelerate the particles. The net action of these sectors would immediately show up, by doing so, as a damping effect. We forego this discussion, since stability should be secured independently of the existence of accelerating fields.

2) It is not essential to the preceding treatment that the magnetic field gradient $|n|$ have the same value for all fs's and dfs's. The generalization would be quite immediate, consisting essentially in slight changes in notation. Also, it is not essential that $\det \Gamma = 1$, $\det \Delta = 1$.

3) The example considered in the previous section shows that the dfs's might be given, in principle, an aperture $2d$ smaller than that $2D$ of the fs's the ratio D/d depending upon the geometry of the machine.

4) From (3), (4) and (5) one readily gathers that the matrices Δ can be given a simple geometrical interpretation in the plane of fig. 1. $\Gamma_+(l_+)$ rotates the vector $\boldsymbol{z}_0$ by an angle $\sqrt{|n|}\,l_+/R$ in the positive direction, the maximum value of z in the fs being given by the maximum z reached in the rotation. $\Gamma_-(l_-)$ causes the end point of $\boldsymbol{z}$ to move along a hyperbola (a simple construction could be likewise found); $\Gamma_0(l_0)$ adds to $\boldsymbol{z}_0 \equiv \{z_0,\ Rz_0'/\sqrt{|n|}\}$ the vector $\{l_0 z_0',\ 0\}$. A detailed study of this fact, in conjunction with the considerations made with regard to fig. 1, would afford a quite simple means of determining graphically the trajectory of a particle entering the vacuum chamber with given initial conditions.

In conclusion, the authors wish to thank Prof. E. AMALDI for his constant encouragement in this work, and him, Drs. M. SANDS and B. TOUSCHEK, for many interesting discussions.

APPENDIX I.

Proof of Theorem of 2·2.

We start by considering case 3): $\lambda_1 \neq \lambda_2 = 1/\lambda_1$, both real. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ denote the (real) eigenvectors of Δ. Any vector $\boldsymbol{z}$ can be expressed as a linear combination of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$:

$$\boldsymbol{z} = \alpha_1 \boldsymbol{u}_1 + \alpha_2 \boldsymbol{u}_2 .$$

It follows:

$$\Delta^m \boldsymbol{z} = \alpha_1 \lambda_1^m \boldsymbol{u}_1 + \alpha_2 \frac{1}{\lambda_1^m} \boldsymbol{u}_2$$

which proves the theorem in this case.

Consider now case 2): $\lambda_1 = \lambda_2 = \pm 1$, for $x = \pm 1$. Two cases are possible: *a*) Δ is not symmetric: then it can be reduced, by transformation with a suitable matrix, at most to the form $\begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}$, from which one readily sees that the theorem is again true; *b*) Δ is symmetric and can be reduced therefore to the form ± 1. It is not possible to discard this case a priori by considering exclusively the behaviour of the ideal machine. The argument that leads to its rejection is that the condition $x = \pm 1$ is very critic, because small deviations from this value due to the unavoidable imperfections of the actual machine might lead to instability (case (3)). See also ref. (3).

To study case 1): $\lambda_1 = \bar{\lambda}_2 = e^{i\varphi}$, $x = \cos\varphi \neq \pm 1$, we express again $\boldsymbol{z}$ as a l.c. of the eigenvectors $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$ (with complex coefficients): then

$$\boldsymbol{z} = \alpha \boldsymbol{u} + \bar{\alpha} \bar{\boldsymbol{u}},$$

$$\Delta^m \boldsymbol{z} = \alpha \exp[im\varphi]\boldsymbol{u} + \bar{\alpha}\exp[-im\varphi]\bar{\boldsymbol{u}} = \cos m\varphi\, \boldsymbol{z} + \sin m\varphi\, \boldsymbol{t},$$

with:

$$\boldsymbol{t} = i(\alpha \boldsymbol{u} - \bar{\alpha}\bar{\boldsymbol{u}}) = \frac{\Delta - \cos\varphi}{\sin\varphi}\boldsymbol{z} \quad \text{(from the above, for } m = 1\text{)};$$

setting $J = (\Delta - \cos\varphi)/\sin\varphi$, comparison of

$$\Delta^m = \cos m\varphi + J \sin m\varphi$$

with the analogous expression for complex numbers, shows that $J^2 = -1$. (This method of proof could be applied also to case 3), but we need not insist on this point).

APPENDIX II.

Alternative study of $\Delta^m = 1$ for $|x| < 1$.

Write
$$\gamma = 2x = 2\cos\varphi .$$
From:
$$\Delta^2 - \gamma\Delta + 1 = 0 ,$$
one finds by recursion:
$$\Delta_m = \begin{pmatrix} P_{m-1}(\gamma)\Delta_{11} - P_{m-2}(\gamma) & P_{m-1}(\gamma)\Delta_{12} \\ P_{m-1}(\gamma)\Delta_{21} & P_{m-1}(\gamma)\Delta_{22} - P_{m-2}(\gamma) \end{pmatrix},$$
where
$$P_0(\gamma) = 1 ,$$
$$P_1(\gamma) = \gamma ,$$
$$P_m(\gamma) = \gamma P_{m-1}(\gamma) - P_{m-2}(\gamma) = \begin{vmatrix} \gamma & 1 & 0 & 0 & \dots & 0 \\ 1 & \gamma & 1 & 0 & \dots & 0 \\ 0 & 1 & \gamma & 1 & \dots & 0 \\ . & . & . & . & . & . \\ . & . & . & . & 1 & \gamma \end{vmatrix} ,$$
the determinant being of order m. The condition for periodicity after m units $\Delta^m = +1$ is
$$P_{m-1}(\gamma) = 0 , \qquad P_{m-2}(\gamma) + 1 = 0 .$$

The roots common to these two equations can be shown to be all and only the roots of:
$$(m = 2k) \qquad P_{k-1}(\gamma) = 0$$
$$(m = 2k + 1) \qquad P_{k-1}(\gamma) + P_k(\gamma) = 0 .$$

They coincide with (12), as is seen from the relation:
$$P_k(2\cos\varphi) = \frac{\sin (k+1)\varphi}{\sin\varphi} .$$

RIASSUNTO

Viene indicato un semplice metodo per lo studio delle proprietà di un acceleratore a « focalizzazione forte » di struttura arbitraria. Si mostra come da esso derivino prontamente le condizioni necessarie e sufficienti per la stabilità e per la periodicità (con periodo assegnato a piacere) delle orbite. Si danno infine criteri che consentono di determinare il campo dei valori iniziali in corrispondenza dei quali le orbite risultano stabili.