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AdS/CFT EQUIVALENCE TRANSFORMATION

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Abstract

We show that any conformal field theory in d-dimensional Minkowski space, in a phase with spontaneously broken conformal symmetry and with the dilaton among its fields, can be rewritten in terms of the static gauge $(d-1)$-brane on $\mathrm{AdS}_{(d+1)}$ by means of an invertible change of variables. This nonlinear holographic transformation maps the Minkowski space coordinates onto the brane worldvolume ones and the dilaton onto the transverse AdS brane coordinate. One of the consequences of the existence of this map is that any $(d-1+m)$-brane worldvolume action on $\mathrm{AdS}_{(d+1)} \times X^m$ (with X^m standing for the sphere S^m or more complicated curved manifold) admits an equivalent description in Minkowski space as a nonlinear and higher-derivative extension of some conventional conformal field theory action, with the conformal group being realized in a standard way. The holographic transformation explicitly relates the standard realization of the conformal group to its field-dependent nonlinear realization as the isometry group of the brane $\mathrm{AdS}_{(d+1)}$ background. Some possible implications of this transformation, in particular, for the study of the quantum effective action of $\mathcal{N}=4$ super Yang-Mills theory in the context of AdS/CFT correspondence, are briefly discussed.

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1 Introduction

The cornerstone of AdS/CFT correspondence [1, 2, 3, 4] is the hypothesis that the isometry group of an $\mathrm{AdS}_n \times S^m$ background in which some type IIB string theory and related supergravity live is identical to the standard conformal group (times the group of internal R symmetry) of the appropriate conformal field theory defined on the $(n-1)$-dimensional Minkowski space considered as a boundary of AdS_n. The full supersymmetric version of this correspondence deals with the bulk and boundary realizations of superconformal groups including conformal and R-symmetry groups as bosonic subgroups.

It was shown in [1], [5]-[7] that the invariance group of the worldvolume action of some probe brane in an $\mathrm{AdS}_n \times S^m$ background (e.g., a D3-brane in $\mathrm{AdS}_5 \times S^5$) can be realized as a field-dependent modification of the standard (super)conformal transformations of the worldvolume. In [8] it was demonstrated that such a realization of the AdS isometry corresponds to the choice of the special 'solvable subgroup' parametrization of the AdS background. In the spirit of the AdS/CFT correspondence (and some other hypotheses of similar nature), the AdS superbrane worldvolume actions are expected to appear as the result of summing up leading and subleading terms in the low-energy quantum effective actions of the corresponding Minkowski space (super)conformal field theories in the phase with spontaneously broken (super)conformal symmetry (e.g., the $\mathrm{AdS}_5 \times S^5$ D3-brane action [9] and its some modifications should be recovered in this way from the effective action of $\mathcal{N} = 4$ SYM theory in the Coulomb branch [10, 11, 1, 12]). In this connection it was suggested in [13, 14] that the modified (super)conformal transformations could be understood as a quantum deformation of the standard (super)conformal transformations of the classical field theory. The idea that the quantum effective action should be invariant just under the modified (super)conformal transformations was further advanced in [15].

In the present paper we take a different viewpoint on the interplay between the standard and modified (super)conformal transformations. We show that any conformal field theory in $d = p+1$-dimensional Minkowski space in the phase with spontaneously broken conformal symmetry, i.e. containing among its fields a Goldstone field (dilaton) associated with the broken scale generator, even at the classical level can be brought, by an invertible change of variables, into the form in which it respects invariance just under the above mentioned field-dependent conformal transformations. This change of variables essentially includes a field-dependent change of the Minkowski space-time coordinates y^μ ($\mu = 0, 1, \ldots, p$) and maps them on the worldsheet coordinates x^μ of the corresponding codimension-one brane in $\mathrm{AdS}_{(d+1)}$, while the dilaton is mapped on the brane transverse coordinate which completes x^μ to $\mathrm{AdS}_{(d+1)}$ in the solvable subgroup parametrization. Using this map between the conformal and AdS bases (it can naturally be called 'holographic map'), one can rewrite any conformal field theory containing the dilaton in terms of the variables of the corresponding AdS brane in a static gauge, and vice versa. The AdS images of the minimal conformally invariant Lagrangians (i.e. those containing terms with no more than two derivatives) prove to necessarily include non-minimal terms composed out of the extrinsic curvatures of the brane. On the other hand, the conformal field theory image of the minimal brane Nambu-Goto action is a non-polynomial and higher-derivative extension of the minimal Minkowski space conformal actions.

In this paper we restrict our study to the bosonic case only, having in mind to extend it to the full superconformal case in a forthcoming publication. We start with recalling basic facts about the standard nonlinear realization of conformal group $SO(2, p+1)$ in $p+1$-dimensional Minkowski space. Then we rewrite the algebra of $SO(2, p+1)$ in the solvable-subgroup basis of [8] as the $\mathrm{AdS}_{(p+2)}$ group algebra and show how to reproduce the static-gauge Nambu-Goto action of scalar p-brane in $\mathrm{AdS}_{(p+2)}$ background by applying to this group the nonlinear realizations techniques along the line of refs. [16, 17, 18, 19]. The $\mathrm{AdS}_{(p+2)}$ isometry group in the second nonlinear realization acts just as the field-modified conformal transformations of refs. [5]-[7]. Comparing two nonlinear realizations of $SO(2, p+1)$, the standard one and the one suitable to AdS branes, we establish the explicit relation between the coset parameters in both realizations. Finally, we give examples of various invariants in both bases, including the conformal basis form of the Nambu-Goto action, and discuss some possible implications of the relationship found.

2 Standard nonlinear realization of conformal group in d dimensions

The algebra of the conformal group $SO(2,d)$ of $d = p+1$-dimensional Minkowski space has the following form

$$\begin{aligned} &[M_{\mu\nu}, M^{\rho\sigma}] = 2\delta^{[\rho}_{[\mu} M_{\nu]}{}^{\sigma]} \ , \ [P_\mu, M_{\nu\rho}] = -\eta_{\mu[\nu} P_{\rho]} \ , \ [K_\mu, M_{\nu\rho}] = -\eta_{\mu[\nu} K_{\rho]} \ , \\ &[P_\mu, K_\nu] = 2\left(-\eta_{\mu\nu} D + 2M_{\mu\nu}\right) \ , \ [D, P_\mu] = P_\mu \ , \ [D, K_\mu] = -K_\mu \ , \end{aligned} \tag{2.1}$$

where

$$A_{[\mu\nu]} \equiv \frac{1}{2}\left(A_{\mu\nu} - A_{\nu\mu}\right) \tag{2.2}$$

and $\eta_{\mu\nu} = \mathrm{diag}(+ - \ldots -)$. In what follows this standard basis of conformal algebra will be called 'conformal' to distinguish it from the 'AdS basis' to be specified below.

The standard nonlinear realization of the conformal group (see, e.g. [20]) corresponds to choosing the Lorentz group $SO(1,p) \propto M_{\mu\nu}$ as the stability (linearization) subgroup and so it is defined as left shifts of the following coset element

$$g = e^{y^\mu P_\mu} e^{\Phi D} e^{\Omega^\mu K_\mu} \ . \tag{2.3}$$

The left shifts with parameters a^μ, b^μ and c related to the generators P_μ, K_μ and D induce the familiar conformal transformations of the coset coordinates

$$\delta y^\mu = a^\mu + c\, y^\mu + 2\,(yb) y^\mu - y^2\, b^\mu \ , \ \delta\Phi = c + 2\, yb \ , \ \delta\Omega^\mu = e^\Phi\, b^\mu + 2(\Omega b) y^\mu - 2(y\Omega)\, b^\mu \ . \tag{2.4}$$

We define the left-covariant Cartan 1-forms as follows

$$\begin{aligned} g^{-1}dg \ &= \ e^{-\Phi} dy^\mu P_\mu + \left(d\Phi - 2e^{-\Phi}\Omega_\mu dy^\mu\right) D - 4e^{-\Phi}\Omega^\mu dy^\nu M_{\mu\nu} \\ &\quad + \left[d\Omega^\mu - \Omega^\mu d\Phi + e^{-\Phi}\left(2\Omega_\nu dy^\nu \Omega^\mu - \Omega^2 dy^\mu\right)\right] K_\mu \ . \end{aligned} \tag{2.5}$$

The vector Goldstone field $\Omega^\mu(x)$ is redundant as it can be covariantly expressed through the only essential one, dilaton $\Phi(x)$, by imposing the covariant Inverse Higgs constraint [21]

$$\omega_D = 0 \Rightarrow \Omega_\mu = \frac{1}{2}\, e^\Phi \partial^y_\mu \Phi \ . \tag{2.6}$$

The remaining 1-forms associated with the coset generators then read

$$\omega_P^\mu = e^{-\Phi} dy^\mu \ , \ \omega_K^\mu = d\Omega^\mu - e^{-\Phi}\Omega^2 dy^\mu \ . \tag{2.7}$$

The covariant derivative of Ω^μ is defined by the relation

$$\begin{aligned} &\omega_K^\mu = \omega_P^\nu \mathcal{D}_\nu \Omega^\mu \quad \Rightarrow \\ &\mathcal{D}_\nu \Omega^\mu = e^\Phi \partial_\nu \Omega^\mu - \Omega^2 \delta_\nu^\mu = \frac{1}{2}\, e^{2\Phi} \left[\partial_\nu \partial^\mu \Phi + \partial_\nu \Phi \partial^\mu \Phi - \frac{1}{2} \left(\partial\Phi\partial\Phi\right) \delta_\nu^\mu \right]. \end{aligned} \tag{2.8}$$

The covariant derivative of some non-Goldstone ('matter') field $\Psi^a(y)$, where a is an index of the Lorentz group representation, is defined by

$$\begin{aligned} &d\Psi^a - 4e^{-\Phi}\Omega^\mu dy^\nu (M_{\mu\nu})^a_b \Psi^b = \omega_P^\mu \mathcal{D}_\mu \Psi^a \quad \Rightarrow \\ &\mathcal{D}_\mu \Psi^a = e^\Phi\, \partial_\mu \Psi^a + 4\Omega^\nu (M_{\mu\nu})^a_b \Psi^b. \end{aligned} \tag{2.9}$$

When y^μ is transformed according to (2.4), the field Ψ^a, as well as the covariant derivatives (2.8) and (2.9), undergo an induced Lorentz rotation with respect to their Lorentz indices, e.g.,

$$\delta\Psi^a(y) = \Psi^{a\prime}(y') - \Psi^a(y) = \beta^{\mu\nu}(M_{\mu\nu})^a_b \Psi^b(y) \ , \ \ \beta^{\mu\nu} = -4\, y^{[\mu} b^{\nu]} \ . \tag{2.10}$$

The conformally invariant measure of integration over $\{y^\mu\}$ is defined as the exterior product of d 1-forms ω_P^μ

$$S_1 = \int \mu(y) = \int d^{(p+1)}y\, e^{-(p+1)\Phi} \ . \tag{2.11}$$

It can be treated as the conformally invariant potential of dilaton.

The covariant kinetic term of Φ can be constructed as

$$S_\Phi^{kin} = \int d^{(p+1)}y\, e^{-(p+1)\Phi}\, \mathcal{D}_\mu \Omega^\mu = \frac{1}{4}(p-1) \int d^{(p+1)}y\, e^{(1-p)\Phi}\, \partial\Phi\partial\Phi \tag{2.12}$$

(while passing to the final form of (2.12), we integrated by parts). For the special case $d = 2\, (p = 1)$ the Lagrangian in (2.12) is reduced to a full derivative. In this case one can still define the non-tensor kinetic term which is invariant under (2.4) up to a shift by full derivative

$$S_\Phi^{kin(2)} = \frac{1}{2} \int d^2y\, \partial\Phi\partial\Phi \ . \tag{2.13}$$

Conformally invariant Lagrangians of matter fields Ψ^a are obtained by replacing ordinary derivatives by the covariant ones (2.9) and promoting $d^{(p+1)}y$ to the conformally invariant measure (2.11). E.g., the standard Maxwell field strength can be covariantized as

$$\tilde{F}_{\mu\nu} = \mathcal{D}_\mu \tilde{A}_\nu - \mathcal{D}_\nu \tilde{A}_\mu = e^{2\Phi} F_{\mu\nu} \ , \tag{2.14}$$

where $\tilde{A}_\mu$ is transformed according to the generic law (2.10) and its covariant derivative is defined by (2.9). It is related in the following way to the ordinary Maxwell vector potential A_μ having the same conformal transformation law as the partial derivative ∂_μ and the standard gauge transformation law

$$\tilde{A}_\mu = e^{-\Phi} A_\mu \ , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{2.15}$$

The conformally invariant action of A_μ then reads

$$S_M^{(c)} = -\frac{1}{4}\int d^{(p+1)}y\, e^{(3-p)\Phi} F^{\mu\nu}F_{\mu\nu}\ . \tag{2.16}$$

At $d = 4$ ($p = 3$) it coincides with the standard Maxwell action which is conformal in its own right only in this dimension.

This formalism of nonlinear realizations of conformal symmetry is universal in the following sense. In any theory in which conformal symmetry is spontaneously broken, it is always possible to make a field redefinition which splits the full set of scalar fields of the theory into the dilaton Φ with the transformation law (2.4) and the subset of fields which are scalars of weight zero under conformal transformations. For instance, let us consider the free action of N massless scalar fields ϕ^I, $I = 1, \ldots N$ ($p \neq 1$):

$$S = \int d^{(p+1)}y\ \partial\phi^I\partial\phi^I\ . \tag{2.17}$$

It is invariant under (2.4) (up to a shift of the Lagrangian by a full derivative) if ϕ^I are transformed with the appropriate weight

$$\delta\phi^I = \frac{1}{2}\,(1-p)(c+2yb)\phi^I\ , \quad \delta|\phi| = \frac{1}{2}\,(1-p)(c+yb)|\phi|\ . \tag{2.18}$$

If some field develops a non-zero vacuum value, $<\ \phi^{I_0}\ >= v \neq 0$ (e.g. due to the presence of some conformally invariant potential term which should be added to (2.17)), the conformal symmetry is spontaneously broken and one can perform the equivalence field redefinition

$$\phi^I = \frac{|\phi|}{v}\,\hat{\phi}^I\ , \quad \hat{\phi}^I\hat{\phi}^I = v^2\ , \quad \delta\hat{\phi}^I = 0 \tag{2.19}$$

$$|\phi| = v + \tilde{\phi} + \ldots = v\, e^{\frac{1}{2}(1-p)\Phi}\ , \quad \phi^{I_0} \equiv \tilde{\phi} + v\ . \tag{2.20}$$

Then the action (2.17), up to an overall coefficient and surface terms, can be rewritten as

$$S = \int d^d y e^{(1-p)\Phi}\left[\frac{1}{4}(1-p)^2\partial\Phi\partial\Phi + \partial\hat{\phi}^I\partial\hat{\phi}^I\right]. \tag{2.21}$$

The first term coincides with the universal dilaton action (2.12) while the second term is the action of a nonlinear sigma model of the internal symmetry group realized on the indices I.

An example of the system admitting such a field redefinition is supplied, e.g., by the scalar fields sector of $\mathcal{N} = 4, d = 4$ SYM action in the Coulomb branch. Consider, e.g. the simplest case of $SU(2)$ gauge group. When some scalar field valued in the Cartan subalgebra $u(1)$ acquires a non-zero expectation value (which is a solution of classical equations of motion for the full action including the conformally invariant quartic potential of the scalar fields), the gauge group gets broken to $U(1)$ and there remain 6 scalar massless fields in the theory which form a vector of the R-symmetry group $SO(6) \sim SU(4)$. The norm of this vector is just the dilaton associated with the spontaneous breaking of conformal symmetry $SO(2,4)$. The remaining 5 independent fields appear as the solution of the algebraic constraint in (2.19) and parametrize the internal sphere

$S^5 \sim SO(6)/SO(5)$. Thus the set of 6 massless bosonic fields of $SU(2)$ $N=4$ SYM theory in the Coulomb branch naturally splits into the $SO(6)$ invariant dilaton sector and the sector of a nonlinear sigma model on S^5.

In the special case of $d=2$ ($p=1$) the field ϕ^I is a scalar of the conformal weight zero, so no redefinition like (2.19), (2.20) is needed. The kinetic and potential terms of dilaton (2.13), (2.11) can be independently added, if necessary. An example of such $d=2$ system, which, like $\mathcal{N}=4$ SYM is conformal (and superconformal) both on classical and quantum levels, is provided by $\mathcal{N}=(4,4)$ supersymmetric $SU(2)$ WZW sigma model [22]. Its bosonic sector includes four scalar fields, one of which is a dilaton and three remaining ones possess zero conformal weight and parametrize the coset $S^3 \sim SU(2)\times SU(2)/SU(2)$. The conformally invariant bosonic action is a sum of free action of the dilaton and standard $SU(2)$ WZW action [23].

3 The AdS nonlinear realization

In the AdS basis we introduce the following generators

$$\hat{K}_\mu = mK_\mu - \frac{1}{2m}P_\mu\ ,\ \hat{D} = mD\ , \tag{3.1}$$

where m will be identified with the inverse radius of AdS space.

The same conformal algebra (2.1) in the AdS basis (3.1) reads

$$\begin{aligned} &\left[\hat{K}_\mu, \hat{K}_\nu\right] = -4M_{\mu\nu}\ ,\ \left[P_\mu, \hat{K}_\nu\right] = 2\left(-\eta_{\mu\nu}\hat{D} + 2mM_{\mu\nu}\right), \\ &\left[\hat{D}, P_\mu\right] = mP_\mu\ ,\ \left[\hat{D}, \hat{K}_\mu\right] = -\left(P_\mu + m\hat{K}_\mu\right) \end{aligned} \tag{3.2}$$

(commutators with the Lorentz generators $M_{\mu\nu}$ are of the same form as in (2.1)).

The basic difference of (3.2) from (2.1) is that the generators $(\hat{K}^\mu, M_{\mu\nu})$ generate the semi-simple subgroup $SO(1,d)$ of $SO(2,d)$, while the subgroup $(K^\mu, M_{\mu\nu})$ has the structure of a semi-direct product. As a result, in the coset element (2.3) rewritten in the new basis

$$g = e^{x^\mu P_\mu} e^{q\hat{D}} e^{\Lambda^\mu \hat{K}_\mu}\ , \tag{3.3}$$

the coordinates x^μ and $q(x)$ are parameters of the coset manifold $SO(2,d)/SO(1,d)$ which is none other than $\mathrm{AdS}_{(d+1)}$. This parametrization of $\mathrm{AdS}_{(d+1)}$ was called in [8] 'the solvable subgroup parametrization', since the generators P_μ and $\hat{D}$ with which the $\mathrm{AdS}_{(d+1)}$ coordinates are associated as the coset parameters constitute the maximal solvable subgroup of $SO(2,d)$. One more convenience of the basis (3.2) with the manifestly included dimensionful parameter m is that one can perform the contraction $m=0$ in (3.2), which takes it just into the $(d+1)$-dimensional Poincaré group $ISO(1,d)$, with the set $(P_\mu, \hat{D})$ becoming the generators of $(d+1)$-translations. In this limit x^μ and $\frac{1}{\sqrt{2}}q$ are recognized as the coordinates of $(d+1)$-dimensional Minkowski space, the standard $R=\infty$ limiting case of $\mathrm{AdS}_{(d+1)}$. This confirms the interpretation of the parameter m as the inverse $\mathrm{AdS}_{(d+1)}$ radius.

In the new basis the Cartan forms (2.5) read

$$g^{-1}dg \ = \ \left[e^{-mq}\left(dx^\mu + \frac{\lambda^\mu\lambda_\nu dx^\nu}{1-\frac{\lambda^2}{2}}\right) - \frac{\lambda^\mu dq}{1-\frac{\lambda^2}{2}}\right]P_\mu$$

$$
\begin{aligned}
&+\frac{1+\frac{\lambda^2}{2}}{1-\frac{\lambda^2}{2}}\left[dq-2\frac{e^{-mq}\lambda_\mu dx^\mu}{1+\frac{\lambda^2}{2}}\right]\hat{D} \\
&+\frac{1}{1-\frac{\lambda^2}{2}}\left[d\lambda^\mu - m\lambda^\mu dq - me^{-mq}\left(\lambda^2 dx^\mu - 2\lambda^\mu\lambda_\nu dx^\nu\right)\right]\hat{K}_\mu \\
&+\omega_M^{\mu\nu}M_{\mu\nu}\ ,
\end{aligned} \tag{3.4}
$$

where

$$
\lambda^\mu = \frac{\tanh\sqrt{\frac{\Lambda^2}{2}}}{\sqrt{\frac{\Lambda^2}{2}}}\Lambda^\mu\ . \tag{3.5}
$$

and the new basis form of $\omega_M^{\mu\nu} = -4\,e^{-\Phi}\Omega^{[\mu}y^{\nu]}$ can be found using the explicit relation between the parameters of the coset elements (2.3) and (3.3) which will be given in the next Section.

The inverse Higgs constraint (2.6) is rewritten in the AdS basis as

$$
\begin{aligned}
&\omega_{\hat{D}} = 0 \quad \Rightarrow \quad \frac{\lambda_\mu}{1+\frac{\lambda^2}{2}} = \frac{1}{2}e^{mq}\,\partial_\mu q\ , \\
&\lambda_\mu = e^{mq}\,\frac{\partial_\mu q}{1+\sqrt{1-\frac{1}{2}e^{2mq}(\partial q\partial q)}}\ .
\end{aligned} \tag{3.6}
$$

On the surface of this covariant constraint the remaining coset space Cartan forms are given by the expressions:

$$
\begin{aligned}
\omega_P^\mu &= e^{-mq}\left(\delta_\nu^\mu - \frac{\lambda^\mu\lambda_\nu}{1+\frac{\lambda^2}{2}}\right)dx^\nu \equiv E_\nu^\mu dx^\nu = e^{-mq}\hat{E}_\nu^\mu dx^\nu\ , \\
\omega_{\hat{K}}^\mu &= \frac{1}{1-\frac{\lambda^2}{2}}\left(d\lambda^\mu - m\lambda^2\omega_P^\mu\right).
\end{aligned} \tag{3.7}
$$

The covariant derivative, with the Lorentz connection part omitted, is defined by

$$
dx^\mu\partial_\mu = \omega_P^\mu\mathcal{D}_\mu \Rightarrow \mathcal{D}_\mu = e^{mq}\left(\delta_\mu^\nu + \frac{\lambda_\mu\lambda^\nu}{1-\frac{\lambda^2}{2}}\right)\partial_\nu \equiv (E^{-1})_\mu^\nu\partial_\nu = e^{mq}(\hat{E}^{-1})_\mu^\nu\partial_\nu\ . \tag{3.8}
$$

The covariant derivative of the $SO(1,d+1)/SO(1,d)$ Goldstone field λ^μ is defined by the formula analogous to (2.8)

$$
\begin{aligned}
&\omega_{\hat{K}}^\nu = \omega_P^\mu\mathcal{D}_\mu\lambda^\nu\ , \\
&\mathcal{D}_\mu\lambda^\nu = \frac{1}{1-\frac{\lambda^2}{2}}\left[e^{mq}\left(\delta_\mu^\rho + \frac{\lambda_\mu\lambda^\rho}{1-\frac{\lambda^2}{2}}\right)\partial_\rho\lambda^\nu - m\lambda^2\delta_\mu^\nu\right].
\end{aligned} \tag{3.9}
$$

It is straightforward to find the transformation laws of $x^\mu, q(x)$ and $\lambda^\mu(x)$ under the left shifts of (3.3)

$$
\delta x^\mu = a^\mu + c\,x^\mu + 2\,(xb)x^\mu - x^2\,b^\mu + \frac{1}{2m^2}\,e^{2mq}b^\mu\ ,\ \delta q = \frac{1}{m}(c+2\,xb)\ , \tag{3.10}
$$

$$
\delta\lambda^\mu = \frac{1}{m}\left(1+\frac{\lambda^2}{2}\right)e^{mq}\,\hat{E}_\nu^\mu\,b^\nu + 2(\lambda b)x^\mu - 2(x\lambda)\,b^\mu\ , \tag{3.11}
$$

where all group parameters are the same as in (2.4). It is easy to check that (3.10) are perfectly consistent with the inverse Higgs expression (3.6) for $\lambda^{\mu}(x)$.

The transformations of x^{μ} and $q(x)$ are just the field-dependent conformal transformations which were discussed in [1], [5]-[7] in connection with the AdS branes and were shown in [8] to naturally arise as the AdS isometries in the above solvable-subgroup parametrization of AdS groups. To see how this interpretation is recovered in the present approach, let us first write the $\mathrm{AdS}_{(d+1)}$ metric

$$ds^2 = \omega_P^{\mu}\omega_{P\mu} = e^{-2mq}dx^{\mu}\eta_{\mu\nu}dx^{\nu} - dq^2 \ . \tag{3.12}$$

The change of variables (we assume $p \neq 1$)

$$e^{-2mq} = \left(\frac{U}{R}\right)^{\frac{4}{p-1}} \frac{2}{(p-1)^2} \ , \quad R = \frac{1}{m} \ , \tag{3.13}$$

brings (3.12) (up to a factor) and transformation rules (3.10) into the form

$$ds^2 = \left(\frac{U}{R}\right)^{\frac{4}{p-1}} dx^{\mu}\eta_{\mu\nu}dx^{\nu} - \left(\frac{R}{U}\right)^2 dU^2 \ , \tag{3.14}$$

$$\begin{aligned}
\delta x^{\mu} &= a^{\mu} + c\,x^{\mu} + 2\,(xb)x^{\mu} - x^2\,b^{\mu} + \frac{1}{4}(p-1)^2\,\frac{R^{2\frac{p+1}{p-1}}}{U^{\frac{4}{p-1}}}\,b^{\mu} \ , \\
\delta U &= -\frac{1}{2}(p-1)(c+2xb)U \ ,
\end{aligned} \tag{3.15}$$

which coincide with those given e.g. in [5] (up to a rescaling of x^{μ} and a different choice of the signature of Minkowski metric).

The simplest invariant of the nonlinear realization considered is again the covariant volume of x-space obtained as the integral of wedge product of $(p+1)$ 1-forms ω_P^{μ}. The difference from (2.11) is that this invariant is basically the static-gauge Nambu-Goto (NG) action for p-brane in $\mathrm{AdS}_{(p+2)}$

$$\begin{aligned}
S_{NG} &= -\int d^{(p+1)}x \left[\det E - e^{-(p+1)mq}\right] = \int d^{(p+1)}x\, e^{-(p+1)mq}\left(1 - \frac{1-\frac{\lambda^2}{2}}{1+\frac{\lambda^2}{2}}\right) \\
&= -\int d^{(p+1)}x\, e^{-(p+1)mq}\left[\sqrt{1-\frac{1}{2}e^{2mq}(\partial q\partial q)} - 1\right],
\end{aligned} \tag{3.16}$$

where we used the relations

$$\det E = e^{-(p+1)mq}\det\hat{E} \ , \quad \det\hat{E} = \frac{1-\frac{\lambda^2}{2}}{1+\frac{\lambda^2}{2}} = \sqrt{1-\frac{1}{2}e^{2mq}(\partial q\partial q)} \tag{3.17}$$

and subtracted 1 to obey the standard requirement of absence of the vacuum energy (corresponding to $q = const$) [1]. Note that the subtracted term

$$S_2 = \int d^{(p+1)}x\, e^{-(p+1)mq} \tag{3.18}$$

is invariant under (3.10) (up to a shift of the integrand by a full derivative) on its own. In most interesting cases it is a part of some WZ (or CS) term in a static gauge. The action (3.16) is universal, in the sense that it describes the radial (pure AdS) part of any $\mathrm{AdS}_n \times S^m$ $(n+m-2)$-brane action corresponding to 'freezing' (setting equal to constants) all other fields on the brane (e.g., the gauge fields and angular S^5 fields in the case of $\mathrm{AdS}_5 \times S^5$ D3-brane) and also to neglecting some further possible WZ-type terms on the brane worldvolume. Actually, this universality extends to the branes on $\mathrm{AdS}_n \times X^m$ where X^m can stand for some m-dimensional curved manifold different from the sphere, e.g. one of the manifolds considered in [24] while analysing the AdS/CFT correspondence for a general $\mathcal{N}=4$ SYM theory in the Coulomb branch.

The minimal covariant actions of various 'matter' fields are obtained via replacing the ordinary derivatives by the covariant ones and inserting $\det E$ into the integration measure. E.g., the covariant kinetic term of some scalar field $\phi(x)$ is given by

$$S_\phi = \int d^{(p+1)}x \, \det\hat{E} \, e^{(p-1)mq} \, \hat{G}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \ , \tag{3.19}$$

where

$$\hat{G}^{\mu\nu} = \eta^{\omega\rho} (\hat{E}^{-1})^\mu_\omega (\hat{E}^{-1})^\nu_\rho = \eta^{\mu\nu} + e^{2mq} \frac{2}{1 - \frac{1}{2} e^{2mq} (\partial q \partial q)} \partial^\mu q \partial^\nu q \tag{3.20}$$

is the inverse of the induced metric

$$\hat{G}_{\mu\nu} = \eta_{\omega\rho} E^\omega_\mu E^\rho_\nu = \eta_{\mu\nu} - \frac{1}{2} e^{2mq} \partial_\mu q \partial_\nu q \tag{3.21}$$

(with the factors $e^{\pm 2mq}$ detached).

As the last topic of this Section, let us clarify the geometric meaning of the covariant derivative (3.9) which plays an important role in our construction. We will show that it is the tangent-space projection of the first extrinsic curvature of the brane. For simplicity, we shall consider the limiting case $m=0$ in (3.9) and (3.6) which corresponds to the p brane in the flat $(p+2)$-dimensional Minkowski background. The generalization to the AdS case is straightforward.

One defines the extrinsic curvature by the relation (see, e.g.[25]-[27])

$$\nabla_\mu \partial_\nu X^A n_A = K_{\mu\nu} \ , \tag{3.22}$$

where X^A are target brane coordinates, $X^A = (x^\mu, -\frac{1}{\sqrt{2}} q)$ in the considered static gauge, $\eta_{AB} = (\eta_{\mu\nu}, -1)$, $n_A = (n_\mu, n)$ is a normal to the brane worldsheet

$$\partial_\mu X^A \, n_A = 0 \ , \quad n^A n_A = n^\mu n_\mu - n^2 = -1 \tag{3.23}$$

and

$$\nabla_\mu \partial_\nu X^A = (\partial_\mu \partial_\nu - \Gamma^\rho_{\mu\nu} \partial_\rho) X^A \ . \tag{3.24}$$

The induced metric $G_{\mu\nu}$ in the static gauge and its inverse $G^{\mu\nu}$ are given by (3.21), (3.20) with $m=0$. We find

$$\Gamma^\rho_{\mu\nu} = G^{\rho\omega} \Gamma_{\mu\nu\,\omega} \ , \quad \Gamma_{\mu\nu\,\omega} = \frac{1}{2} \left(\partial_\mu G_{\nu\omega} + \partial_\nu G_{\mu\omega} - \partial_\omega G_{\mu\nu} \right) = -\frac{1}{2} \partial_\mu \partial_\nu q \partial_\omega q \ , \tag{3.25}$$

and

$$\nabla_\mu \partial_\nu q = \frac{1}{1 - \frac{1}{2}(\partial q \partial q)} \partial_\mu \partial_\nu q \ , \quad \nabla_\mu \partial_\nu x^\rho = \frac{1}{2} \frac{1}{1 - \frac{1}{2}(\partial q \partial q)} \partial_\mu \partial_\nu q \partial^\rho q \ . \tag{3.26}$$

Further, the orthogonality condition (3.23) in the static gauge is reduced to [1]

$$n_\mu + \frac{1}{\sqrt{2}}\partial_\mu q\,\sqrt{1+n^\nu n_\nu} = 0 \quad \Rightarrow \quad n_\mu = -\frac{1}{\sqrt{2}}\,\frac{\partial_\mu q}{\sqrt{1-\frac{1}{2}(\partial q\partial q)}}\ . \tag{3.27}$$

After substituting all this into the definition (3.22), we obtain

$$K_{\mu\nu} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1-\frac{1}{2}(\partial q\partial q)}}\,\partial_\mu\partial_\nu q \tag{3.28}$$

and

$$\mathcal{D}_\mu\lambda_\nu = \frac{1}{\sqrt{2}}(E^{-1})^\rho_\mu(E^{-1})^\omega_\nu K_{\rho\omega}\ . \tag{3.29}$$

4 An equivalence relation between CFT and AdS bases

In both nonlinear realizations described above we deal with the same coset manifold $SO(2,d)/SO(1,d-1)$, in which the coset parameters are divided into the space-time coordinates and Goldstone fields in two different ways. In the first realization the coordinates y^μ parametrize the d-dimensional Minkowski space considered as a coset of $SO(2,d)$ identified with the corresponding conformal group.[2] All other parameters are Goldstone fields, the essential one being dilaton $\Phi(y)$ associated with the spontaneous breaking of scale invariance. In the second realization the space-time coordinates x^μ on their own do not constitute a coset manifold of $SO(2,d)$ and therefore do not form a closed set under the left action of this group. However, together with the Goldstone field $q(x)$ they parametrize the coset $SO(2,d)/SO(1,d) \sim \mathrm{AdS}_{(d+1)}$ and this extended set is closed under the action of $SO(2,d)$. These coset parameters admit a clear interpretation as the worldvolume (x^μ) and transverse (q) coordinates of $(d-1)$-brane evolving in $\mathrm{AdS}_{(d+1)}$.

Apart from this essential difference in the interpretation, the fact that both these realizations (with vector Goldstone fields Ω_μ and λ_μ included) are in fact defined on the same full coset of $SO(2,d)$, viz. $SO(2,d)/SO(1,d-1)$, suggests the existence of relation between these two different coset parametrizations. This relation can be straightforwardly extracted from comparison of (2.3) and (3.3)

$$y^\mu = x^\mu - \frac{e^{mq}}{2m}\lambda^\mu\ ,\ \Phi = mq + \ln\left(1-\frac{\lambda^2}{2}\right)\ ,\ \Omega^\mu = m\lambda^\mu\ . \tag{4.1}$$

We see that it is invertible at any finite non-zero $m = 1/R$. It is straightforward to check that the Minkowski space conformal transformations (2.4) are mapped by (4.1) on the field-dependent ones (3.10) and vice versa. Since this change of variables maps the geometric objects living in the $\mathrm{AdS}_{(d+1)}$ bulk on those defined on its Minkowski boundary, it seems natural to name it 'holographic transformation'. It is important to emphasize

[1] Actually, this condition is another form of the inverse Higgs constraint (3.6) at $m=0$, with n_μ being related via a field redefinition to the Goldstone field λ_μ.

[2] To be more rigorous, it is the compactified Minkowski space which can be treated as a coset manifold of conformal group.

that this holographic transformation essentially involves the Goldstone field λ^μ (or Ω_μ) which basically becomes the derivative of $q(x)$ (or $\Phi(y)$) after imposing the covariant constraint (3.6) (or its conformal basis counterpart (2.6)). However, for the existence of map (4.1) it does not matter whether (3.6) or (2.6) are imposed or not, the only necessary condition is the presence of vector parameters $\Omega^\mu(y)$ and $\lambda^\mu(x)$ in both cosets. In other words, (4.1) could not be guessed solely in the framework of the pure $\mathrm{AdS}_{(d+1)}$ geometry, i.e. by dealing with the AdS coordinates x^μ and q alone; it can be defined only when considering extended coset manifolds $\{y^\mu, \Phi, \Omega_\mu\}$ and $\{x^\mu, q, \lambda_\mu\}$. Another characteristic feature of the map (4.1) is that it is well defined only for non-zero and finite values of AdS radius $R = 1/m$.

Using the holographic transformation, any conformal field theory in Minkowski space with a dilaton among its basic fields can be projected onto the variables of AdS brane and vice versa. To find the precise form of various $SO(2,d)$ invariants in two bases, the conformal and AdS ones, let us first define the transition matrix

$$\frac{\partial y^\nu}{\partial x^\mu} \equiv \mathcal{A}_\mu^\nu = \delta_\mu^\nu - \frac{\lambda_\mu \lambda^\nu}{1+\frac{\lambda^2}{2}} - \frac{e^{mq}}{2m}\partial_\mu \lambda^\nu = \left(1 - \frac{\lambda^2}{2}\right) \hat{E}_\mu^\rho T_\rho^\nu \ , \tag{4.2}$$

where

$$T_\rho^\nu = \delta_\rho^\nu - \frac{1}{2m}\mathcal{D}_\rho \lambda^\nu \ , \tag{4.3}$$

the matrix $\hat{E}_\nu^\mu$ is defined by (3.7) and $\mathcal{D}_\rho\lambda^\nu$ is the covariant derivative of λ^ν defined in (3.9) (it is an extrinsic curvature of the brane). We then have the following general formula for the Jacobian of the change of space-time coordinates in (4.1)

$$J \equiv \det \mathcal{A} = \left(1 - \frac{\lambda^2}{2}\right)^{p+1} \det \hat{E} \det T \ . \tag{4.4}$$

Making the change of variables (4.1) in the invariant dilaton Lagrangians (2.11) and (2.12), we obtain, respectively,

$$\begin{aligned} S_1 &= \int d^{(p+1)}y\, e^{-(p+1)\Phi} = \int d^{(p+1)}x\, e^{-(p+1)mq} \det \hat{E} \det T \\ &= \int d^{(p+1)}x\, e^{-(p+1)mq} \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q \partial q)} \det T \ , \end{aligned} \tag{4.5}$$

$$\begin{aligned} S_\Phi^{kin} &= \int d^{(p+1)}y\, e^{-(p+1)\Phi}\ \mathcal{D}_\mu \Omega^\mu = \frac{1}{2}\int d^{(p+1)}y\, e^{(1-p)\Phi} \left[\Box\Phi + \frac{1}{2}(1-p)(\partial\Phi\partial\Phi)\right] \\ &= m\int d^{(p+1)}x\, e^{-m(p+1)q} \det\hat{E}\ \left[\det T\, (T^{-1}\mathcal{D}\lambda)_\mu^\mu\right] \\ &= m\int d^{(p+1)}x\, e^{-m(p+1)q} \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q\partial q)}\ \left[\det T\, (T^{-1}\mathcal{D}\lambda)_\mu^\mu\right] . \end{aligned} \tag{4.6}$$

We observe a surprising fact that the AdS image of the potential term of dilaton contains the NG part of the AdS p-brane action (3.16) modified by the higher-derivative covariants collected in $\det(I - \frac{1}{2m}\mathcal{D}\lambda) = 1 - \frac{1}{2m}\mathcal{D}_\mu\lambda^\mu + \ldots$. As we saw, $\mathcal{D}_\mu\lambda^\nu$ is basically the extrinsic curvature of the p-brane. So already the simplest conformal invariant in Minkowski space proves to produce, on the AdS side, a rather complicated action which is the standard p-brane action in AdS$_{(p+2)}$ plus corrections composed out of the extrinsic curvature tensor. The leading (with two derivatives) term in the r.h.s. of (4.5) comes both from the NG square root and the terms $\sim \partial_\mu \lambda^\mu$, λ^2 in $\mathcal{D}_\mu \lambda^\mu$ (see (3.9) and (3.6))

$$S_1 = \int d^4x \, e^{-(p+1)mq} \left[1 - \frac{1}{8}(p+1)e^{2mq}(\partial q \partial q) + \ldots \right]. \tag{4.7}$$

Note that in the flat case $m = 0$ the extrinsic curvature terms are capable to produce only higher-order (in fields and derivatives) corrections to the minimal NG p-brane action (as follows from the expression (3.9) at $m = 0$). On the other hand, the AdS image of the kinetic term of dilaton, eq. (4.6), starts with the correct kinetic term of q:

$$S^{kin}_{\Phi} = \frac{m^2}{4} \int d^4x \left[e^{-(p-1)mq}(p-1)\,(\partial q \partial q) + \ldots \right]. \tag{4.8}$$

Note, however, that it comes solely from the extrinsic curvature term, not from the NG square root. The latter is always multiplied by degrees of the extrinsic curvature in (4.6).

A way to elude this paradox of generating kinetic terms from the pure potential ones via the change of variables could be to start from the reasonable field theory action on the CFT side, having from the beginning both kinetic and potential dilaton terms, i.e. from the action

$$S = S^{kin}_{\Phi} + \gamma S_1 \, , \tag{4.9}$$

where γ is a coupling constant. To the second order in $\partial_\mu q$ it is

$$S = \int d^4x \left(\gamma \, e^{-(p+1)mq} + \frac{1}{4}[m^2(p-1) - \frac{1}{2}\gamma(p+1)](\partial q \partial q) + \ldots \right), \tag{4.10}$$

and we observe that the holographic transformation (4.1) merely renormalizes the coefficient before the kinetic term. Nevertheless, the paradox still persists because one can fully eliminate the kinetic term of q by choosing $\gamma = 2m^2 \, \frac{p-1}{p+1}$. Then on the CFT side we still have quite reasonable field theory, while on the AdS side we get an action admitting no standard weak-field expansion. These observations suggest that the map (4.1) is not the standard equivalence transformation preserving the canonical structure of the given theory. This peculiarity of (4.1) is manifested, first, in that the essential part of (4.1) is a non-linear field-dependent transformation of the space-time coordinate starting with a derivative of q and, second, in that the relation between Φ and q contains a shift by kinetic term of q, $\Phi = mq - \frac{1}{8}(\partial q \partial q) + \ldots$. Note that for the conformal actions containing no potential terms of dilaton the relations (4.1) can be still treated as setting a genuine equivalence map, since they always take the kinetic term of Φ into that of q (up to rescaling by m) plus some terms of higher order in q and its derivatives. The same remains true when bringing the minimal AdS brane action (3.16) with vanishing vacuum energy into the conformal basis (see next Section).

In the special $d = 2\,(p = 1)$ case the conformally invariant kinetic term of Φ is given by the non-tensor Lagrangian (2.13). Its AdS image is also of non-tensor form, in contradistinction to the manifestly invariant term (4.6) for $d \neq 2$

$$\begin{aligned} S^{kin(2)}_{\Phi} &= \int d^2y \, (\partial\Phi\partial\Phi) = 4m^2 \int d^2x \frac{e^{-2mq}\,\lambda^2}{\left(1 - \frac{\lambda^2}{2}\right)^2} \det \mathcal{A} \\ &= 4m^2 \int d^2x \, e^{-2mq} \, \lambda^2 \det \hat{E} \det T \, . \end{aligned} \tag{4.11}$$

It is not easy to check the invariance of (4.11) under the transformations (3.10). For proving that (4.11) is indeed invariant, up to a shift of the Lagrangian by a full derivative, one needs to use the explicit form of $\det \mathcal{A}$ for this case

$$\begin{aligned}\det \mathcal{A} &= \frac{1}{2}\left[(\mathrm{Tr}\,\mathcal{A})^2 - \mathrm{Tr}\,\mathcal{A}^2)\right] = \frac{1-\frac{\lambda^2}{2}}{1+\frac{\lambda^2}{2}}\left[1 - \frac{e^{mq}\partial_\mu\lambda^\mu}{2m} - \frac{e^{mq}\lambda^\mu\lambda^\nu\partial_\mu\lambda_\nu}{2m\left(1-\frac{\lambda^2}{2}\right)}\right] \\ &+ \frac{e^{2mq}}{8m^2}\left[(\partial_\mu\lambda^\mu)^2 - \partial_\mu\lambda^\nu\partial_\nu\lambda^\mu\right]. \end{aligned} \tag{4.12}$$

The AdS images of the conformally invariant kinetic terms of 'matter' fields can be obtained by making the variable change (4.1) in the corresponding actions. For instance, for a scalar field $\Psi(y)$ we find

$$\begin{aligned} &S_\psi = \int d^{(p+1)}y\, e^{(p-1)\Phi}\,\partial^\mu\Psi\partial_\mu\Psi = \int d^{(p+1)}x \det E\, \mathcal{L}(q,\Psi)\ , \\ &\mathcal{L}(q,\Psi) = \det T\ \eta^{\mu\nu}(T^{-1})_\mu^\omega(T^{-1})_\nu^\tau\ \mathcal{D}_\omega\Psi\mathcal{D}_\tau\Psi = G^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + O(\mathcal{D}\lambda)\ , \\ &\mathcal{D}_\mu\Psi = (E^{-1})_\mu^\nu\partial_\nu\Psi\ , \quad G^{\mu\nu} = \eta^{\rho\tau}(E^{-1})_\rho^\mu(E^{-1})_\tau^\nu\ . \end{aligned} \tag{4.13}$$

We see that this expression differs from the minimal covariantization (3.19) by couplings to the brane extrinsic curvatures.

The change (4.1) brings the conformal Maxwell action (2.16) into the form

$$S_M = -\frac{1}{4}\int d^{(p+1)}x \det E\, \mathcal{H}^{\mu\nu}\mathcal{H}_{\mu\nu}\ , \tag{4.14}$$

where

$$\begin{aligned} &\mathcal{H}_{\mu\nu} = (T^{-1})_\mu^\rho(T^{-1})_\nu^\omega\mathcal{F}_{\rho\omega}\ ,\ \mathcal{F}_{\mu\nu} = (E^{-1})_\mu^\rho(E^{-1})_\nu^\omega\hat{F}_{\rho\omega}\ , \\ &\hat{F}_{\rho\omega} = \partial_\rho^x\hat{A}_\omega - \partial_\omega^x\hat{A}_\rho\ ,\ \hat{A}_\mu = \mathcal{A}_\mu^\nu A_\nu\ . \end{aligned} \tag{4.15}$$

Once again, a difference from the minimal invariant Lagrangian $\sim \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = G^{\mu\nu}G^{\omega\lambda}$ $\hat{F}_{\mu\omega}\hat{F}_{\nu\lambda}$ is the presence of extra couplings with the extrinsic curvature.

It is instructive to give how $\hat{A}_\nu$ and $\hat{F}_{\mu\nu}$ are transformed under (3.10). Their transformation laws follow from the property that A_μ is transformed under the conformal group as the derivative ∂_μ^y, while the matrix $\mathcal{A}_\nu^\mu = \partial y^\mu/\partial x^\nu$ as

$$\delta\mathcal{A}_\nu^\mu = 2(yb - xb)\mathcal{A}_\nu^\mu + 2\mathcal{A}_\nu^\rho(b_\rho y^\mu - y_\rho b^\mu) - 2\left(b_\nu x^\rho - x_\nu b^\rho + \frac{1}{4m^2}\partial_\nu e^{2mq}\, b^\rho\right)\mathcal{A}_\rho^\mu\ . \tag{4.16}$$

Then

$$\delta\hat{A}_\mu = -(c + 2xb)\hat{A}_\mu - 2\left(b_\mu x^\rho - x_\mu b^\rho + \frac{1}{4m^2}\partial_\mu e^{2mq}\, b^\rho\right)\hat{A}_\rho\ , \tag{4.17}$$

or

$$\delta^*\hat{A}_\mu = \hat{A}'_\mu(x) - \hat{A}_\mu(x) = \delta_c^*\hat{A}_\mu - \frac{1}{2m^2}e^{2mq}\, b^\rho\hat{F}_{\rho\mu} - \frac{1}{2m^2}\partial_\mu\left(e^{2mq}b^\rho\hat{A}_\rho\right), \tag{4.18}$$

where δ_c^* denotes the conventional conformal (including no q-dependent terms) part of the complete variation. The transformation of $\hat{F}_{\mu\nu}$ is of standard form

$$\delta\hat{F}_{\mu\nu} = -(\partial_\mu\delta x^\rho)\,\hat{F}_{\rho\nu} - (\partial_\nu\delta x^\rho)\,\hat{F}_{\mu\rho}\ .$$

5 AdS brane actions in the conformal basis

In the previous section we have found how the simplest conformally invariant Lagrangians in Minkowski space look after passing to the AdS basis. It is of interest also to see what the AdS brane action (3.16) looks like in the conformal basis, with the conventionally realized spontaneously broken conformal symmetry. The helpful relations are

$$\mathcal{D}_\mu\Omega^\nu = m(T^{-1})_\mu^\omega \mathcal{D}_\omega \lambda^\nu \ , \quad (T^{-1})_\mu^\nu = \delta_\mu^\nu + \frac{1}{2m^2}\mathcal{D}_\mu\Omega^\nu \ , \tag{5.1}$$

where $\mathcal{D}_\mu\Omega^\nu$ was defined in (2.8).

We start with the 'potential' term of q, eq. (3.18). Making in (3.18) the change of variables inverse to (4.1), we find

$$S_2 = \int d^{(p+1)}y\, e^{-(p+1)\Phi}\, \frac{1+\frac{1}{8m^2}\, e^{2\Phi}\,(\partial\Phi\partial\Phi)}{1-\frac{1}{8m^2}\, e^{2\Phi}\,(\partial\Phi\partial\Phi)} \det\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right) \ . \tag{5.2}$$

For the pure NG-part of the action (3.16) we obtain rather simple expression

$$S = \int d^{(p+1)}y\, e^{-(p+1)\Phi} \det\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right) . \tag{5.3}$$

Then the full brane action (3.16) takes the form

$$S_{NG} = \frac{1}{4m^2}\int d^{(p+1)}y\, e^{(1-p)\Phi}\, \frac{(\partial\Phi\partial\Phi)}{1-\frac{1}{8m^2}e^{2\Phi}(\partial\Phi\partial\Phi)} \det\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right) . \tag{5.4}$$

Thus we have found an equivalent representation of the static-gauge action (3.16) of p-brane in $\mathrm{AdS}_{(p+2)}$ as a non-linear extension of the conformally-invariant dilaton action in $(p+1)$ dimensional Minkowski space. Note that the conformal image of the brane action is nonlinear and non-polynomial, however it is a rational function of Φ and its derivatives. We also note that, despite the simplicity of the standard conformal transformations (2.4), it is rather tricky to directly check that (5.4) or (5.2) are indeed invariant under them. The difficulty originates from the property that the Lagrangian densities in (5.4), (5.2), like their AdS images (3.16), (3.18), are not tensors, they are shifted by a full derivative under (2.4) (as distinct from the Lagrangian in (5.3) which is manifestly invariant). Though the conformal variation of S_{NG} (5.4) can easily be found

$$\delta_c S_{NG} = \frac{1}{m^2}\int d^{(p+1)}y\, e^{(1-p)\Phi}\, \frac{b^\mu\,\partial_\mu\Phi}{\left[1-\frac{1}{8m^2}e^{2\Phi}(\partial\Phi\partial\Phi)\right]^2} \det\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right) , \tag{5.5}$$

it is far from obvious that the integrand in (5.5) is a full derivative. To see this, one should demonstrate that the variational derivative of (5.5) is identically vanishing,

$$\frac{\delta}{\delta\Phi(y)}(\delta_c S_{NG}) = 0 \ .$$

The proof makes use of the explicit expressions (2.8) and (2.6) and is somewhat tiresome, though straightforward. Notice the crucial importance of terms with two derivatives on Φ

coming from the determinant in (5.5). As a simpler exercise, one can directly check that (5.5) is reduced to a full derivative in the first order in $1/m^2$ (since transformations (2.4) do not include m^2, each term in the expansion of (5.4) in powers of $1/m^2$ should be invariant separately). It would hardly be possible to guess such a non-tensor conformal invariant, staying solely in the framework of the standard nonlinear realization of conformal group.

Our last example will be the conformal field theory image of the full bosonic part of D3-brane on $\mathrm{AdS}_5 \times S^5$. Neglecting the 'magnetic' part of the Chern-Simons term, the action in the static gauge can be written as (see, e.g. [28])

$$S_5 = -C \int d^4x \, \frac{|X|^4}{R^4} \left[\sqrt{-\det\left(\eta_{\mu\nu} - \frac{R^4}{|X|^4} \, \partial_\mu X^i \partial_\nu X^i + \frac{R^2}{|X|^2} \, \hat{F}_{\mu\nu} \right)} - 1 \right], \tag{5.6}$$

where $i = 1, \ldots 6$, $|X| = \sqrt{X^i X^i}$, C is some positive renormalization constant the precise form of which is of no interest in the present context and the signs are adjusted in accordance with our choice of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(+---)$.

Firstly we rewrite (5.6) in our notation, using the field redefinition

$$\frac{R}{|X|} = \frac{1}{\sqrt{2}} \, e^{mq} \,, \quad m = \frac{1}{R} \,, \tag{5.7}$$

which is the particular $p = 3$ case of the redefinition (3.13). We obtain

$$S_5 = -4C \int d^4x \, e^{-4mq} \left[(\det \hat{E}) \, \sqrt{-\det\left(\eta_{\mu\nu} + \frac{1}{2} \mathcal{F}_{\mu\nu} - \frac{1}{2} \mathcal{D}_\mu \tilde{X}^i \mathcal{D}_\nu \tilde{X}^i \right)} - 1 \right], \tag{5.8}$$

where $\mathcal{D}_\mu$ and $\mathcal{F}_{\mu\nu}$ were defined in (3.8), (4.15) and $\tilde{X}^i$ parametrize the sphere S^5,

$$\tilde{X}^i \tilde{X}^i = R^2 \,.$$

For constant $\tilde{X}^i$ and $\hat{A}_\mu$ the action (5.8) is reduced to the pure $\mathrm{AdS}_{(d+1)}$ action (3.16) with $d = 4$.

Now, making in (5.8) the change of variables inverse to (4.1), we obtain the conformal basis form of the $\mathrm{AdS}_5 \times S^5$ action

$$\begin{aligned} S_5 \;\; &= \;\; 4C \int d^4y \, e^{-4\Phi} \det\left(I + \frac{1}{2m^2} \mathcal{D}\Omega \right) \left\{ \frac{1 + \frac{1}{8m^2} \, e^{2\Phi} \, (\partial\Phi\partial\Phi)}{1 - \frac{1}{8m^2} \, e^{2\Phi} (\partial\Phi\partial\Phi)} \right. \\ &\left. - \sqrt{-\det\left[\eta_{\mu\nu} + \frac{1}{2} \, e^{2\Phi} T^\rho_\mu \, T^\omega_\nu \left(F_{\rho\omega} - \partial_\rho \tilde{Y}^i \partial_\omega \tilde{Y}^i \right) \right]} \right\}, \end{aligned} \tag{5.9}$$

where

$$\tilde{Y}^i(y) \equiv \tilde{X}^i(x(y)) = \frac{R}{|Y|} Y^i \,, \; \frac{R}{|Y|} = \frac{1}{\sqrt{2}} \, e^{\Phi} \, \frac{1}{1 - \frac{1}{8m^2} e^{2\Phi} (\partial\Phi\partial\Phi)} \,. \tag{5.10}$$

Thus we have succeeded in equivalently rewriting the effective bosonic action of D3-brane in the $\mathrm{AdS}_5 \times S^5$ background (5.6) or (5.8) as a conformally invariant nonlinear action of the coupled system of the following set of fields in 4-dimensional Minkowski space $\{y^\mu\}$: dilaton $\Phi(y)$, five independent scalar fields $\tilde{Y}^i(y)$, $\tilde{Y}^i \tilde{Y}^i = R^2$, parametrizing the sphere S^5, and an abelian gauge field $A_\mu(y)$. For $\tilde{Y}^i$ and A_μ we still have a version of the Dirac-Born-Infeld action promoted to a conformally-invariant one due to couplings to the dilaton $\Phi(y)$. It also includes extra conformal couplings to the curvature $\mathcal{D}_\mu \Omega^\nu$ (through the common factor det $\left(I + \frac{1}{2m^2}\mathcal{D}\Omega\right)$ and the matrices T^ρ_μ in the determinant under the square root). The dilaton $\Phi(y)$ itself, with all other fields neglected, is described by the nonlinear higher-derivative action (5.4). The crucial difference between (5.6) (or (5.8)) and (5.9) is that the latter involves fields having standard transformation properties under the conformal group $SO(2,4)$, while in (5.6) the latter is realized as the group of isometry of AdS_5, with transformations depending on $|X|$. The group $SO(6)$ has the same realization in both representations as the isometry group of 5-sphere S^5.

6 Discussion

In this paper we have found a new kind of holographic relation between field theories possessing spontaneously broken conformal symmetry in d-dimensional Minkowski space and the codimension-$(n+1)$ branes in $\mathrm{AdS}_{(d+1)} \times X^n$ type backgrounds in the static gauge (with the sphere S^n as a particular case of X^n). This relation takes place already at the classical level and transforms the dilaton Goldstone field associated with the spontaneous breaking of scale invariance into the transverse (or radial) brane co-ordinate completing the d-dimensional brane worldvolume to the full $\mathrm{AdS}_{(d+1)}$ manifold. It does not touch the X^n-valued part of transverse coordinates which are described by a kind of nonlinear sigma model action in both representations. The conformally invariant minimal actions in Minkowski space including the dilaton are transformed into the highly nonlinear actions given on the AdS brane worldvolume and involving, as their essential part, couplings to the extrinsic curvature of the brane. Conversely, the standard worldvolume AdS brane effective actions prove to be equivalent to some non-polynomial conformally invariant actions in the Minkowski space. This map is one-to-one (at least, classically) for the conformal actions containing no dilaton potential and for brane actions with the vanishing vacuum energy. The geometric origin of this map can be revealed most clearly within the nonlinear realization description of AdS branes [19] which generalizes the analogous description of branes in the flat backgrounds [16, 17, 18]. In particular, it turns out that the standard realization of the conformal group in the Minkowski space and its transverse brane coordinate-dependent realization as the $\mathrm{AdS}_{(d+1)}$ isometry group in the solvable-subgroup parametrization of $\mathrm{AdS}_{(d+1)}$ are simply two alternative ways of presenting symmetry of the same system.

As the most interesting subjects for further study we mention the generalization of the above relationship to the case of AdS superbranes and, respectively, superconformal symmetries, as well as the understanding of how it can be promoted to the quantum case.

Since the appropriate framework for the bosonic case is provided by nonlinear realizations of conformal groups, we expect that the generalization to the supersymmetry case can be fulfilled most naturally within the PBGS (Partial Breaking of Global Supersymmetry) approach to superbranes (see [29] and refs. therein). In the given context the PBGS approach amounts to describing AdS superbranes in terms of superfield nonlinear realizations of the appropriate superconformal group, with half of supersymmetries (special conformal supersymmetries) being nonlinearly realized and the rest providing manifest linear invariances of the corresponding actions. The superanalog of the map (4.1) should relate different coset superspaces of superconformal groups: those where these groups are realized in the standard way, i.e. with the superspace coordinates transforming through themselves without any mixing with the Goldstone superfields (see, e.g. [30, 31]), and those where the transformation laws of superspace coordinates essentially involve the Goldstone superfields, like the modified bosonic transformations (3.10). The second type of realizations should be relevant to the PBGS superbrane actions with superextensions of AdS$\times S$ manifolds as the target supermanifolds for which the appropriate superconformal groups define superisometries. An example of the worldvolume superfield PBGS action for AdS superbranes, that of the AdS$_4$ supermembrane, was recently constructed in [19]. The relevant Goldstone superfield-dependent realization of the corresponding superisometry group $OSp(1|4)$ ($N = 1$, $d = 3$ superconformal group) on the $N = 1, d = 3$ worldvolume superspace coordinates was explicitly found.

As for generalizing the map (4.1) to the quantum case, one should firstly understand how to treat the field dependence of the change of space-time coordinates in (4.1) in this case. Since the fields q and Φ will not longer commute with their derivatives, it seems that the transformed coordinates should also be non-commuting. To keep (4.1) invertible, for consistency one should require both coordinate sets $\{y^\mu\}$ and $\{x^\nu\}$ to be non-commuting. This could provide a link with the non-commutative geometry.

We shall finish with a few further comments on possible implications of the holographic map (4.1).

In the AdS/CFT context the actions of standard conformal field theories are usually treated as a the $R \to 0$ (or low-velocity) approximation of the AdS brane effective worldvolume actions. For instance, the $U(1)$ part of the $\mathcal{N} = 4$ $SU(2)$ SYM action in the Coulomb branch can be recovered as the $R \to 0$ limit of the abelian D3-brane action on AdS$_5 \times S^5$. Indeed, for the bosonic part of the latter, eq. (5.6), we have

$$S_5 \sim \int d^4x \left[\frac{1}{2}\partial^\mu X^i \partial_\mu X^i - \frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + O(R)\right] .$$

In this limit the field-dependent conformal transformations (3.10), (3.15) are reduced to the standard ones which are characteristic of the field theory actions (in (3.10) one needs to rescale $q \to Rq$ to approach this limit in an unambiguous way).

The existence of the holographic map (4.1) suggests a different view of the relationship between the conformal field theory actions and the worldvolume actions of AdS superbranes. As we saw, any conformal field theory action in the branch with spontaneously broken conformal symmetry, after singling out the dilaton field, can be rewritten in terms of the AdS brane variables, with the field-modified conformal transformations defining the relevant symmetry. This relationship exists at any finite and non-vanishing AdS radius $R = 1/m$. We observed, however, that the AdS images of conformal field theory dilaton actions do not coincide with the standard NG type brane actions, but are given by the expressions of the type (4.5), (4.6) which essentially include powers of extrinsic curvature of the brane.[3] Besides, the AdS images of other fields do not appear under the square root as, e.g. in the standard AdS$_5 \times S^5$ D3-brane action (5.6), but have the form

[3]An interesting exception [32] is the $d = 1$ case of conformal mechanics where (4.6) coincides, up to a full derivative, with the $d = 1$ case of (3.16).

(4.13), (4.14) where all nonlinearities are solely due to the AdS brane transverse coordinate $q(x)$ and its derivatives. It is interesting to further explore this surprising 'brane' representation of (super)conformal field theories, especially in the quantum domain, and to better understand the role of couplings to extrinsic curvature which are unavoidable in this representation. In this connection, let us recall that a string with 'rigidity', i.e. with extrinsic curvature terms added to the action, was considered as a candidate for the QCD string [25] (see also [26, 27]). We also notice that the higher-derivative corrections to the minimal worldvolume superbrane actions are κ-invariant extensions of the extrinsic curvature terms (see [33] and refs. therein).

Besides addressing the obvious problem of studying $\mathrm{AdS}_5 \times S^5$ brane representation of the full $\mathcal{N} = 4$, $d = 4$ SYM action (both in the component and superfield approaches), it would be instructive to investigate analogous representations of the actions of some superconformal theories in lower dimensions, e.g. the action of $\mathcal{N} = (4, 4)$, $d = 2$ WZW sigma model [22] which was mentioned in the end of Sect. 2. Since its bosonic sector in the standard (conformal) basis includes the dilaton and the $S^3 \sim SU(2) \times SU(2)/SU(2)$ coset fields, it should admit a representation in terms of variables of superstring on $\mathrm{AdS}_3 \times S^3$.

One more possible implication of the holographic AdS/CFT map is as follows. As was already mentioned, the worldvolume action of some probe superbrane in the $\mathrm{AdS}_n \times S^m$ type background (obtained as a solution of the appropriate supergravity) is expected to be recovered on the CFT side as a sum of the leading (and subleading) terms in the loop expansion of the low-energy quantum effective action of the related (super)conformal field theory taken in a phase with spontaneously broken (super)conformal symmetry [10, 11, 1]. If the quantum field theory is arranged to respect non-anomalous rigid symmetries of the classical theory, it is reasonable to assume that there exists a formulation of its quantum effective action (e.g., in the appropriate background field formalism) such that it is still invariant under the standard conformal group. Then for checking the above mentioned 'supergravity-CFT' correspondence one is led to compare the quantum effective action just with the conformal basis form of the corresponding superbrane worldvolume action, i.e. with expressions like (5.4), (5.9). In the context of the correspondence between the Coulomb branch of $\mathcal{N} = 4$ SYM and abelian D3-branes on $\mathrm{AdS}_5 \times S^5$ this reasoning implies that the scalar field sector of the $\mathcal{N} = 4$ SYM quantum effective action should be of the form (5.9) rather than (5.6) or (5.8). The latter expressions are to be recovered only after performing the holographic transformation (4.1). As a rule, the correspondence discussed is checked for the gauge field sector only, by setting scalar fields to be constants [12]. From (5.9) and (5.10) it is seen that in this approximation $\Phi = mq$, and (5.9) actually coincides with (5.8) or (5.6). It would be of interest to explore the structure of the scalar field sector of the low-energy $N = 4$ SYM effective action beyond this constant field approximation and compare it with (5.9).[4]

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[4]One should restore the omitted 'magnetic' 5-form Chern-Simons term in (5.9) while checking this.

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