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TRANSITION RADIATION IN THE PRE-WAVE ZONE: AN APPROACH TO SOLUTION

V.A.Verzilov
INFN, Laboratori Nazionali di Frascati, P.O. Box 13, I-00044 Frascati, Italy

Abstract

An analytic approach to solution to the pre-wave zone problem is suggested for transition radiation produced by relativistic charged particles at the normal incidence on the infinite boundary between a metal and vacuum.

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1 Introduction

Generally, radiation electromagnetic fields show very different properties at close and far distances from their sources: each case is characterized by how the distance relates to a wavelength and source dimensions [1]. An observation angle may be involved in definitions as well. Fields have the simplest form in the wave zone (also known as the far-field zone), at distances R very large compared to both the wavelength and dimensions of the radiating system. In this spatial domain radiation can be asymptotically represented by an outgoing spherical wave of the type $f(\theta)\mathrm{e}^{\mathrm{i}kR}/R$, where $f(\theta)$ is a function of the observation angle θ and k is a wavevector. As an observer moves from the wave zone towards a source, he finds that field characteristics change significantly. This occurs because source dimensions come into play.

In Ref.[2], considering a problem of diffraction radiation from an infinite half-plane, it was found that the asymptotic (i.e., wave-zone) expression for the field was valid at distances exceeding a certain value of the order of the so-called formation length. This was true for both the forward direction and that of the specular reflection. For edge radiation from a bending magnet, due to the diffraction-limited source size, the far-field approximation requires an observer distance of $R > \lambda\gamma^2$ [3], which is of the same order of magnitude as the formation length at relativistic energies.

In the previous work [4], the author examined the wave zone condition for transition radiation (TR) in the ultrarelativistic regime and found that it sets in at distances such that $R \gg \lambda\gamma^2$. Radiation characteristics at smaller R were shown to be strongly dependent on the observer distance and measurably distinct from those in the wave zone.

Sometimes, the region $R \leq \lambda\gamma^2$ is thought about as the near-field one. This, however, may lead to an ambiguity since, classically, near-fields are characterized by a short-range action $kR \ll 1$ [5]. Instead, the author uses somewhat an awkward term ”pre-wave zone” to specify an intermediate region between the near-field zone (in its classical definition) and the wave zone.

While, Ref. [4] mainly focused on physical arguments, the present paper is intended to provide the necessary mathematical supplement.

2 Formulation of the problem. The wave zone condition

Consider TR generated by a relativistic particle with a charge q and a velocity $v \to c$ when it crosses, along a normal, an infinite boundary $z = 0$ between vacuum and a perfectly conducting metal.

It is essential for the following to note that the wave zone location is determined by dimensions of the electromagnetic field of a moving particle. In fact, while the particle itself can certainly be considered a point, its Coulomb field Fourier-component, involved in radiation at a wavelength λ, occupies a finite space, outer border of which scales in the transverse (to the particle trajectory) plane roughly as $\lambda\gamma$. Since, ultimately, the source of TR is atomic electron currents induced on the metallic boundary by the incident particle Coulomb field, its size is that of the field. From this it follows that the wave zone is located at the same distance for both backward and forward TR. In the forward direction, however, one should consider interference of the radiation field with the Coulomb field of the particle, since they remain overlapped over a certain period of time. In order to single out the radiation part only, we limit ourselves to the case of backward TR.

At relativistic energies it is sufficient to consider transverse components of the particle field [6]:

$$E^q_{x,y}(z,\varkappa,\omega) = -\frac{4\pi \mathrm{i} q}{v}\frac{\varkappa_{x,y}}{\varkappa^2+\alpha^2}\mathrm{e}^{-\mathrm{i}z\omega/v}\,, \tag{1}$$

where $\alpha = \omega/v\gamma$, ω is the radiation frequency, $\varkappa_x$ and $\varkappa_y$ are x and y projections of the transverse component $\varkappa$ of the wave vector. The radiation field is obtained by satisfying the boundary condition

$$E^q_{x,y} + E^r_{x,y} = 0\,. \tag{2}$$

In the cylindric system of coordinates $E^r_{x,y}$ are expressed through the integral over $\varkappa$

$$E^r_{x,y}(z,\rho,\omega) = -\frac{2q}{v}n_{x,y}\int_0^\infty \frac{\varkappa^2\mathrm{d}\varkappa}{\varkappa^2+\alpha^2}J_1(\varkappa\rho)\mathrm{e}^{\mathrm{i}z\sqrt{k^2-\varkappa^2}}\,, \tag{3}$$

where $\mathbf{n}$ is the unit vector lying in the x,y- plane and directed from the z-axis to the observation point with a radial coordinate ρ, and J_1 is the Bessel function of the first kind. The spatial-spectral distribution of TR, that is the radiation power per unit of the frequency and per unit of the transversal area can be written in the form

$$\frac{\mathrm{d}^2W}{\mathrm{d}\omega\mathrm{d}\mathbf{u}} = \frac{q^2}{\pi^2 c}|\Phi(u,w,\gamma)|^2\,, \tag{4}$$

where dimensionless variables $u = k\rho$ and $w = kz$ are introduced and the function Φ is given by

$$\Phi(u,w,\gamma) = \int_0^\infty \frac{t^2\mathrm{d}t}{t^2+(\beta\gamma)^{-2}}J_1(ut)\mathrm{e}^{\mathrm{i}w\sqrt{1-t^2}}\,. \tag{5}$$

Equation (5) provides a solution to the problem under consideration which is valid at an arbitrary distance w from the origin. The asymptotic $w \to \infty$ expression for the integral as well as limits of its applicability can be found in the context of the stationary phase formalism [7]. In terms of the method, the value of the integral at large w is essentially determined by the contribution from the close vicinity of the stationary point $t_s \approx u/w$, that yields to

$$\Phi(u,w,\gamma) \underset{w\to\infty}{\to} -\frac{\mathrm{e}^{\mathrm{i}\sqrt{u^2+w^2}}}{\sqrt{u^2+w^2}} \frac{uw}{u^2+\frac{u^2+w^2}{(\beta\gamma)^2}} \,. \tag{6}$$

However, for Eq.(6) to hold true, the stationary point must be separated from the poles of the fraction at $t = \pm\mathrm{i}/\beta\gamma$ (i) and the lower boundary of the integration domain $t = 0$ (ii) by a distance large enough compared to a region of the steepest descent $|t - t_s| \sim 1/\sqrt{w}$. The two requirements can be expressed quantitatively with following inequalities

$$w \gg \frac{1}{t_s^2 + (\beta\gamma)^{-2}} \tag{7}$$

and

$$w \gg \frac{1}{t_s^2}\,, \tag{8}$$

correspondingly. These conditions (basically the second one) impose limits to applicability of the asymptotic expression for Φ and, thus, outline the low-w boundary of the wave zone.

At relativistic energies, since $t_s \sim \gamma^{-1} \ll 1$, the wave zone condition may also be written in the form

$$z \gg \lambda\gamma^2 \,. \tag{9}$$

However, one should always bear in mind that Eq.(9) defines rather a "characteristic" parameter, while the actual wave zone boundary is a function of observer angles and, in particular, moves rapidly to infinity when these angles become very small ($t_s \to 0$).

If the wave-zone condition is not fulfilled, that occurs in the pre-wave zone, Eq.(6) is no more valid and a generic solution of Eq.(5) is to be found. In the next section an approximate solution to the general problem is developed, which can be considered as adequate to deal with a majority of practical situations in the relativistic regime $\gamma \gg 1$.

3 An approach to solution

From the above analysis of Eq.(5), it follows, that for $w \gg 1$, regardless of integration limits, the value of the integral is actually determined by the range $\Delta t \sim 1/\sqrt{w}$ around the point $t_s \sim u/w$. Thus, as long as radiation is observed at points $u \ll w$ and distances

$w \gg 1$, we can consider $t \ll 1$ and make use of $\sqrt{1-t^2} \approx 1 - t^2/2$. The first condition implies small observation angles $\theta \ll 1$, the usual approximation made at relativistic energies. The second one is equivalent to $z \gg \lambda$, thus imposing a limit to observation distances. For convenience, the integral still goes to infinity, since this has no effect on its value. Upon introducing a new variable $y = \beta\gamma t$, the function Φ can be written as

$$\Phi \approx \frac{\mathrm{e}^{\mathrm{i}w}}{\beta\gamma} \int_0^\infty \frac{y^2 \mathrm{d}y}{y^2+1}\, J_1(\sigma\, y)\, \mathrm{e}^{-\mathrm{i}\mu y^2}\,, \tag{10}$$

where $\sigma = u/\beta\gamma$ and $\mu = w/2(\beta\gamma)^2$.

In Ref. [4] the solution of Eq.(10) was found by using a series expansion for the Bessel function followed by a straightforward integration term by term. Here, however, first we find a new expression for Φ. To this end let us introduce a function

$$\Phi^\star = \beta\gamma\Phi \mathrm{e}^{-\mathrm{i}(w+\mu)} \tag{11}$$

and differentiate it with respect to μ

$$\frac{\partial\Phi^\star}{\partial\mu} = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\mu} \int_0^\infty y^2 J_1(\sigma\, y)\, \mathrm{e}^{-\mathrm{i}\mu y^2}\, \mathrm{d}y\,. \tag{12}$$

The integral in the right hand side can be found [9], whereupon Eq. (12) reads

$$\frac{\partial\Phi^\star}{\partial\mu} = \mathrm{i}\frac{\sigma}{4\mu^2}\mathrm{e}^{\mathrm{i}(\frac{\sigma^2}{4\mu}-\mu)}\,. \tag{13}$$

Solving the differential equation gives

$$\Phi^\star(\mu) = \Phi^\star(0) + \mathrm{i}\frac{\sigma}{4}\int_0^\mu \mathrm{e}^{\mathrm{i}(\frac{\sigma^2}{4z}-z)}\frac{\mathrm{d}z}{z^2}\,. \tag{14}$$

Though $\Phi^\star(0)$ is obtained from Eq.(11) and Eq.(10) by setting $\mu = 0$:

$$\Phi^\star(0) = \int_0^\infty \frac{y^2\mathrm{d}y}{y^2+1}\, J_1(\sigma\, y) = K_1(\sigma)\,, \tag{15}$$

where K_1 is the modified Bessel function, one should bear in mind that, by virtue of the approximations made, the final result

$$\Phi \approx \frac{\mathrm{e}^{\mathrm{i}(w+\mu)}}{\beta\gamma}\left\{K_1(\sigma) + \mathrm{i}\frac{\sigma}{4}\int_0^\mu \mathrm{e}^{\mathrm{i}(\frac{\sigma^2}{4z}-z)}\frac{\mathrm{d}z}{z^2}\right\} \tag{16}$$

is valid at $\mu \gg 1/2(\beta\gamma)^2$.

We first make sure that the previously obtained result [4] can be reproduced on the basis of Eq.(16). With this in mind, we split the integral in the right hand side of Eq.16 in two ones as follows

$$\int_0^\mu \mathrm{e}^{\mathrm{i}(\frac{\sigma^2}{4z}-z)}\frac{\mathrm{d}z}{z^2} = \int_0^\infty - \int_\mu^\infty . \tag{17}$$

The first integral of the two was considered in Ref.[8] and is

$$I_0 = \frac{4\mathrm{i}}{\sigma}K_1(\sigma)\,. \tag{18}$$

By expanding $\mathrm{e}^{\mathrm{i}\sigma^2/4z}$ in a power series and using the known formula for the incomplete gamma function [9]:

$$\int_\mu^\infty \frac{\mathrm{e}^{\mathrm{i}\alpha x}}{x^{n+1}}\mathrm{d}x = (-\mathrm{i}\alpha)^n\Gamma(-n,-\mathrm{i}\alpha\mu)\,, \tag{19}$$

the second integral can be evaluated in the form of an infinite sum

$$I_\mu = \mathrm{i}\sum_{n=0}^{\infty}(-1)^n\left(\frac{\sigma}{2}\right)^{2n}\frac{\Gamma(-n-1,\mathrm{i}\mu)}{n!}\,. \tag{20}$$

Collecting Eq.(17), Eq.(18) and Eq.(20) in Eq.(16), we arrive basically at the same formula for Φ as in Ref. [4]:

$$\Phi \approx \frac{\mathrm{e}^{\mathrm{i}(w+\mu)}}{\beta\gamma}\sum_{n=0}^{\infty}(-1)^n\frac{\sigma^{2n+1}}{2^{2n+2}}\frac{\Gamma(-n-1,\mathrm{i}\mu)}{n!}\,. \tag{21}$$

After that, we note that Eq.(16) allows also to get an alternative solution for Φ. This time we expand the exponential $\mathrm{e}^{-\mathrm{i}z}$ in a power series immediately in the integral and perform integration by making use of a new variable $x = 1/z$ and Eq.(19). This gives following results for the integral

$$\int_0^\mu \mathrm{e}^{\mathrm{i}(\frac{\sigma^2}{4z}-z)}\frac{\mathrm{d}z}{z^2} = -\mathrm{i}\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!}\left(\frac{\sigma}{2}\right)^{2n-2}\Gamma\left(1-n,-\frac{\mathrm{i}\sigma^2}{4\mu}\right) \tag{22}$$

and the function Φ

$$\Phi \approx \frac{\mathrm{e}^{\mathrm{i}(w+\mu)}}{\beta\gamma}\left\{K_1(\sigma)+\sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!}\frac{\sigma^{2n-1}}{2^{2n}}\Gamma\left(1-n,-\frac{\mathrm{i}\sigma^2}{4\mu}\right)\right\}\,. \tag{23}$$

Compared to Eq.(21), the last formula has definitely certain advantages. The most important one is that the series in brackets converges very rapidly at $\mu < 1$.

In this regard, it is worthwhile to note that Eq.(9) can be rewritten as

$$\mu \gg 1\,. \tag{24}$$

Therefore, the condition $\mu \leq 1$ points at the pre-wave zone, for which Eq.(23) is able to provide simple approximate formulae by retaining just a limited number of terms in the sum. For this purpose it is convenient also to exploit a known recurrent relation for incomplete gamma functions

$$\Gamma(-n,x) = \frac{1}{n}\left[\frac{e^{-x}}{x^n} - \Gamma(1-n,x)\right]\,, n>0\,, \tag{25}$$

with two initializing equations

$$\Gamma(0,x) = -\mathrm{Ei}(-x) \tag{26}$$

and

$$\Gamma(1,x) = \mathrm{e}^{-x}\,, \tag{27}$$

where $\mathrm{Ei}(x)$ is the exponential integral function. Thus, e.g., for the first two terms in the sum, Eq.(23) is reduced to

$$\Phi \approx \frac{\mathrm{e}^{\mathrm{i}(w+\mu)}}{\beta\gamma}\left\{K_1(\sigma) - \frac{1}{\sigma}\mathrm{e}^{\frac{\mathrm{i}\sigma^2}{4\mu}} - \frac{\sigma}{4}Ei\left(\frac{\mathrm{i}\sigma^2}{4\mu}\right)\right\}\,. \tag{28}$$

4 Asymptotes and convergence

Finally, let us consider the asymptotic behaviour of Eq.(21) and Eq.(23) at $\mu \to \infty$ and compare it with the exact asymptotic formula Eq.(6). For the sake of convenience, in this section we make use of a variable $x = u\beta\gamma/w$ and a function $\tilde{\Phi} = w\Phi/\beta\gamma$. Note, that x has a meaning of tangent of the observation angle measured in units of $(\beta\gamma)^{-1}$. With new notations we obtain from Eq.(21)

$$\tilde{\Phi} = \mathrm{e}^{\mathrm{i}(w+\mu)}\sum_{n=0}^{\infty}(-1)^n x^{2n+1}\mu^{2n+2}\frac{\Gamma(-n,-1,\mathrm{i}\mu)}{n!}\,. \tag{29}$$

Using the asymptotic representation for the incomplete gamma function [9]

$$\Gamma(-n,x) \underset{x\to\infty}{\to} \frac{(-1)^n}{n!}\mathrm{e}^{-x}\sum_{m=n}^{\infty}(-1)^m\frac{m!}{x^{m+1}} \tag{30}$$

and after some algebra we can write the asymptotic expression for $\tilde{\Phi}$ in the following form

$$\tilde{\Phi} \underset{\mu\to\infty}{\to} -e^{i(w+\mu x^2)}\left\{x - x^3 + x^5 - x^7 + ... + O\left(\frac{a(x)}{\mu}\right)\right\}\,, x > 0\,. \tag{31}$$

Similarly we find for Eq.(23)

$$\tilde{\Phi} = \mathrm{e}^{\mathrm{i}(w+\mu)}\left\{2\mu K_1(2x\mu) + \sum_{n=0}^{\infty}\frac{(-1)^{n-1}}{n!}x^{2n-1}\mu^{2n}\Gamma(1-n, -\mathrm{i}x^2\mu)\right\}\,. \tag{32}$$

In this case to proceed with asymptotes one must require $x\mu \to \infty$ and $x^2\mu \to \infty$. Then we can write

$$\begin{aligned}\tilde{\Phi} \underset{\mu\to\infty}{\to} & \; e^{i(w+\mu)}\left\{\sqrt{\frac{\pi\mu}{x}}e^{-2x\mu}\right. \\ & \left. - e^{i\mu(x^2-1)}\left\{\frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \frac{1}{x^7} + \ldots + O\left(\frac{b(1/x)}{\mu}\right)\right\}\right\}\,, x > 0\,,\end{aligned} \tag{33}$$

where the first term in brackets goes asymptotically to zero. Functions $a(x)$ and $b(1/x)$ in Eq.(31) and Eq.(33) are simple polynomials of their arguments. As a formal sign of consistency, one should mention the particular status of the point $x = 0$ at which asymptotes of both Eq.(29) and Eq.(32) become ”undetermined”. This was explained in the Section 2 by the fact that the wave zone boundary moves to infinity as the observation angle tends to zero.

As for Eq.(6), after a transition is made to the new variables and in limits $\beta\gamma \gg 1$ and $x/\beta\gamma \ll 1$, it becomes

$$\tilde{\Phi} \underset{\mu\to\infty}{\to} -\mathrm{e}^{\mathrm{i}(w+\mu x^2)}\frac{x}{x^2+1} \tag{34}$$

Curiously enough to note that Eq.(31) converges to Eq.(34) only for $x < 1$, while Eq.(33) does it for $x > 1$. Such convergence selectivity with respect to the value $x = 1$ is an obvious artifact, a result which one could expect, however, because of the fact that there does not exists a power series for the fraction in Eq.(34) converging for both $x < 1$ and $x > 1$, simultaneously.

The analytic study of convergence of infinite series in Eq.(21) and Eq.(23) in the general case of arbitrary μ implies serious computational difficulties and has not been done yet. Instead, known criteria were applied numerically and indicate that there are no reasons to doubt convergence of both formula for finite μ's. From the point of view of numerical computations, the series in Eq.(21) shows slow convergence for $x^2\mu \gg 1$ due to the growth of a number of terms to be taken into account. In Eq.(23) this occurs at $\mu \gg 1$. Again, Eq.(23) is advantageous, since it converges at the same rate for the full range of x.

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