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N=(4,4), 2D SUPERGRAVITY IN *SU(2)* x *SU(2)* HARMONIC SUPERSPACE

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Abstract

We work out the basics of conformal *N=(4,4)*, 2D supergravity in the *N=(4,4)*, *2D* analytic harmonic superspace with two independent sets of harmonic variables. We define the relevant most general analytic superspace diffeomorphism group and show that in the flat limit it goes over into the "large" *N=(4,4)*, *2D* superconformal group. The basic objects of the supergravity considered are analytic vielbeins covariantizing two analyticity-preserving harmonic derivatives. For self-consistency they should be constrained in a certain way. We solve the constraints and show that the remaining irreducible field content in a WZ gauge amounts to a new short *N=(4,4)* Weyl supermultiplet. As in the previously known cases, it involves no auxiliary fields and the number of remaining components in it coincides with the number of residual gauge invariances. We discuss various truncations of this "master" conformal supergravity group and its compensations via couplings to *N=(4,4)* superconformal matter multiplets. Besides recovering the standard minimal off-shell *N=(4,4)* conformal and Poincaré supergravity multiplets, we find, at the linearized level, several new off-shell gauge representations.

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1 Introduction

For building up self-consistent string models with $N = (4,4)$ worldsheet supersymmetry (SUSY) it is of primary importance to explore in full the structure of the relevant worldsheet conformal supergravity (SG), both on and off shell, as well as its couplings to $N = (4,4)$, $2D$ superconformal sigma models. In components and in the standard $N = (4,4)$, $2D$ superspace these issues were addressed in refs. [1]-[10]. Recently, there was a revival of interest to $N = (4,4)$ superconformal $2D$ theories caused by the fact that they describe the low-energy limits of some string theory compactifications (see. e.g., [11]-[13]). This makes it urgent to revert to the problem of finding out an adequate superspace description of $N = (4,4)$ SG and listing all possible versions of the latter.

Here we present the basics of conformal $N = (4,4)$, $2D$ SG in the analytic harmonic $SU(2) \times SU(2)$ superspace [14] with two independent sets of harmonic variables (for the left and right light-cone sectors). This kind of harmonic superspace is indispensable for the off-shell description of $N = (4,4)$ supersymmetric torsionful sigma models, with all supersymmetries being manifest. Our construction in its starting points follows the analogous one for conformal SG in the analytic subspace of $N = 2$, $4D$ harmonic superspace [15]-[17], but eventually we find a few essential differences from the latter theory. These differences amount to a number of novel features of our construction compared to the existing approaches to $N = (4,4)$, $2D$ SG.

First, in the $SU(2) \times SU(2)$ harmonic superspace *three* different SG groups containing local $SU(2)_L \times SU(2)_R$ symmetry can be defined (L and R stand for the left- and right-handed $2D$ light-cone sectors). Two of them have as the rigid limits two different infinite-dimensional $N = 4$, $SU(2)$ superconformal groups [18] the realization of which in the flat harmonic superspace was given in [14]. A closure of these two rigid superconformal groups is the large $N = 4$, $2D$ superconformal group with the $SO(4) \times U(1)$ affine Kac-Moody group as internal symmetry (in each of two $2D$ light-cone sectors) [19]-[22]. The most general SG group which can be defined in the analytic $SU(2) \times SU(2)$ harmonic superspace yields in the flat limit just this large $N = (4,4)$ superconformal group. The corresponding SG can be treated as a "master theory" producing two $N = (4,4)$, $SU(2)$ SG theories as its proper truncations. Another, more elegant way of getting $N = (4,4)$, $SU(2)$ SG theories from the master $N = (4,4)$ SG is to couple the latter to appropriate harmonic superfield compensators. We explicitly demonstrate how one of $N = (4,4)$, $SU(2)$ SG groups can be recovered using this compensation procedure. The relevant compensator is one of the $SU(2) \times SU(2)$ harmonic superfields defined in [23] (it contains $(32+32)$ off-shell components and generalizes the so-called nonlinear supermultiplet [24]). It should be stressed that the most characteristic feature of the master $N = (4,4)$ SG group is the presence of *four* local $SU(2)$ symmetries (corresponding to gauging left and right $SO(4)$) and two local $U(1)$ symmetries (corresponding to gauging left and right $U(1)$). The versions of off-shell conformal $N = (4,4)$ SG known until now contained at most two local $SU(2)$ symmetries and no local $U(1)$ symmetries at all.

One more difference from the $N = 2$, $4D$ case stems from the presence of two independent sets of $SU(2)$ harmonic variables in the $SU(2) \times SU(2)$ harmonic superspace. This peculiarity gives rise, on the one hand, to the property that the relevant groups of analytic superdiffeomorphisms are more powerful than their $N = 2$, $4D$ counterpart, in the sense

that they allow to gauge away more fields from the basic geometric objects of the theory, analytic vielbeins which covariantize two analyticity-preserving harmonic derivatives. On the other hand, prior to any gauge fixing, we are led to impose the constraints on the analytic vielbeins reflecting the commutativity of two independent analyticity-preserving harmonic derivatives in the flat case. The constraints and the original SG gauge group together work in such a way that in the WZ gauge we are left with no auxiliary fields at all. Besides, the number of gauge fields coincides with that of the remaining independent gauge parameters in the left and right sectors. Thus, the analytic vielbeins in the considered case actually describe a sum of two pure gauge Weyl multiplets. This sum can be naturally called the $N = (4,4)$ Beltrami-Weyl (BW) multiplet (the $SU(2)$ or $SO(4) \times U(1)$ one, depending on from which superdiffeomorphism group one proceeds). For the $N = (4,4)$, $SU(2)$ case our results agree with those of Schoutens [3], who constructed the corresponding SG in the component approach by directly gauging the product of left and right $N = 4$, $SU(2)$ superconformal groups. The standard conformal $N = (4,4)$ SG group corresponds to gauging the maximal finite-dimensional subgroup $SU(1,1|2) \times SU(1,1|2)$ of this product [1, 2], [4]-[6] and also gives rise to Weyl multiplet containing no off-shell degrees of freedom. A novel point is that this phenomenon of the one-to-one correspondence between the gauge fields and residual gauge invariances is continued as well to the more general case of $N = (4,4)$, $SO(4) \times U(1)$ SG group. The supermultiplet of what is usually referred to as "the minimal off-shell Poincaré $N = 4$ SG" [5, 6] arises after coupling $N = 4$, $SU(2)$ BW multiplet to a compensating superfield which represents one of twisted chiral multiplets in the analytic harmonic $SU(2) \times SU(2)$ superspace. Thus the minimal $N = (4,4)$ SG representation corresponds to the two successive compensations: firstly, the $N = (4,4)$, $SO(4) \times U(1)$ SG group is compensated down to its $N = (4,4)$, $SU(2)$ subgroup by using some special harmonic compensator and, secondly, this subgroup is further compensated down to the group corresponding to the minimal representation via coupling to a twisted $N = (4,4)$ supermultiplet. The existence of a dual formulation of the twisted multiplet with an infinite number of auxiliary fields [14] implies the existence of new off-shell version of $N = (4,4)$ Poincaré SG with an *infinite* number of auxiliary fields.

In the present paper we do not aim to present the whole formalism of $N = (4,4)$ SG in harmonic superspace. We concentrate on describing the analytic superspace geometry of the $SU(2)$ and $SO(4) \times U(1)$, $N = (4,4)$ BW supermultiplets: define the relevant groups, the analyticity-preserving harmonic derivatives and the covariant constraints on the latter, and show that after choosing appropriate WZ gauges and solving the constraints we are left with the needed irreducible field contents. We present the invariant couplings of $SO(4) \times U(1)$, $N = (4,4)$ BW multiplet to the compensating $N = (4,4)$ multiplets, such that the residual gauge freedom is just one of the $N = (4,4)$, $SU(2)$ SG groups. Then we extend this coupling to include an arbitrary number of self-interacting twisted multiplets. We also show, at the linearized level, how to extract another $N = (4,4)$, $SU(2)$ SG group from the $N = (4,4)$, $SO(4) \times U(1)$ one. We discuss various truncations and schemes of compensation of $N = (4,4)$, $SO(4) \times U(1)$ SG down to its $N = (4,4)$, $SU(2)$ superconformal descendants and, further, to different versions of Poincaré SG. A few novel possibilities are found. More detailed considerations with passing to component actions, etc, will be given elsewhere.

2 Flat $SU(2) \times SU(2)$ analytic harmonic superspace

To proceed, we need some facts about the flat analytic harmonic $SU(2) \times SU(2)$ superspace. In our notation we will basically follow ref. [14] with minor deviations.

This superspace is spanned by the following set of coordinates

$$\mathbf{A}^{(1+2,1+2|2,2)} = (z^{++}, z^{--}, \theta^{(1,0)\,\underline{k}\,+}, \theta^{(0,1)\,\underline{b}\,-}, u_i^{(\pm 1,0)},\ v_a^{(0,\pm 1)}) \equiv (\zeta^\mu, u_i^{(\pm 1,0)},\ v_a^{(0,\pm 1)})\ . \quad (2.1)$$

Here, the $+, -$ indices of the z and θ coordinates are the left and right light-cone $SO(1,1)$ ones, while $i, \underline{k}, a, \underline{b}$ are doublet indices of four commuting $SU(2)$ groups which constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of $N = (4,4)$, $2D$ Poincaré superalgebra. In what follows we will omit the light-cone indices of Grassmann coordinates, keeping in mind that the indices $\underline{i}$ and $\underline{a}$ are always accompanied by the indices $+$ and $-$. The harmonic part of $\mathbf{A}^{(1+2,1+2|2,2)}$ is parametrized by two independent sets of harmonic variables $u_i^{(\pm 1,0)}$, $v_a^{(0,\pm 1)}$, each associated with one of the $SU(2)$ factors of $SO(4)_L$ and $SO(4)_R$, respectively (we denote these "harmonized" $SU(2)$ groups as $SU(2)_L$ and $SU(2)_R$):

$$u^{(1,0)\,i} u_i^{(-1,0)} = 1, \quad v^{(0,1)\,a} v_a^{(0,-1)} = 1\ . \quad (2.2)$$

The harmonics u and v, as well as the left and right odd coordinates, carry two independent $U(1)$ charges "$(n,0)$", "$(0,m)$" which are assumed to be strictly conserved (like in the $N = 2$, $4D$ harmonic superspace approach [15]). This requirement restricts u and v to parametrize 2-spheres $SU(2)_L/U(1)_L$ and $SU(2)_R/U(1)_R$. The superfields given on $\mathbf{A}^{(1+2,1+2|2,2)}$ (*analytic* $N = (4,4)$ superfields), $\Phi^{(p,q)}(\zeta, u, v)$, are also labelled by a pair of such $U(1)$ charges "(p,q)" and are assumed to admit expansions in the double harmonic series on the above 2-spheres. It should be stressed that the "harmonized" subgroups $SU(2)_L$, $SU(2)_R$ and the two remaining $SU(2)$ factors of $SO(4)_L$, $SO(4)_R$ are realized in essentially different ways. Namely, the "harmonized" $SU(2)$ symmetries are hidden, in the sense that they manifest themselves only in the existence of the double harmonic series; on the other hand, two extra $SU(2)$ symmetries are explicit, as they rotate the underlined doublet indices of the analytic Grassmann coordinates and the related indices of component fields in the θ expansion of $\Phi^{(p,q)}$. Note that the latter in general can carry indices of any linear representation of these explicit $SU(2)$ symmetries.

The analytic superspace (2.1) is real with respect to the generalized involution "$\sim$" which is the product of ordinary complex conjugation and an antipodal map of the 2-spheres $SU(2)_L/U(1)_L$ and $SU(2)_R/U(1)_R$

$$(\widetilde{\theta^{(1,0)}\,\underline{i}}) = \theta_{\underline{i}}^{(1,0)}\ , \quad (\widetilde{u^{(\pm 1,0)\,i}}) = -u_i^{(\pm 1,0)}\ , \quad (2.3)$$

(and similarly for $\theta^{(0,1)\,\underline{a}}, v_a^{(0,\pm 1)}$). The analytic superfields $\Phi^{(p,q)}$ can be chosen real with respect to this involution, provided $|p+q| = 2n$

$$(\widetilde{\Psi^{(p,q)}}) = \Psi^{(p,q)}\ , |p+q| = 2n\ . \quad (2.4)$$

In what follows we will need the fact of the existence of the mutually commuting sets of harmonic derivatives $D^{(2,0)}$, $D_u^{(0,0)} \equiv D_u^0$ and $D^{(0,2)}$, $D_v^{(0,0)} \equiv D_v^0$ which preserve

$N = (4,4)$ Grassmann harmonic analyticity, i.e. yield an analytic superfield when acting on some analytic superfield. They are given by the expressions

$$D^{(2,0)} = \partial^{(2,0)} + i(\theta^{(1,0)})^2\partial_{++} \ , \ \ D^{(0,2)} = \partial^{(0,2)} + i(\theta^{(0,1)})^2\partial_{--} \tag{2.5}$$

$$D^0_u = \partial^0_u + \theta^{(1,0)\,\underline{i}}\frac{\partial}{\partial\theta^{(1,0)\,\underline{i}}} \ , \ \ D^0_v = \partial^0_v + \theta^{(0,1)\,\underline{a}}\frac{\partial}{\partial\theta^{(0,1)\,\underline{a}}} \ , \tag{2.6}$$

$$\left[\, D^0_u, D^{(2,0)} \,\right] = 2\, D^{(2,0)} \ , \ \ \left[\, D^0_v, D^{(0,2)} \,\right] = 2\, D^{(0,2)} \ . \tag{2.7}$$

Here $\partial_{\pm\pm} = \partial/\partial z^{\pm\pm}$ and

$$\partial^{(2,0)} = u^{(1,0)\,i}\frac{\partial}{\partial u^{(-1,0)\,i}} \ , \ \ \partial^0_u = u^{(1,0)\,i}\frac{\partial}{\partial u^{(1,0)\,i}} - u^{(-1,0)\,i}\frac{\partial}{\partial u^{(-1,0)\,i}} \ , \tag{2.8}$$

(the same formulas are valid for $\partial^{(0,2)}$ and ∂^0_v with the change $u \to v$). The operators D^0_u, D^0_v count the $U(1)$ charges of the analytic superfields

$$D^0_u\,\Phi^{(p,q)}(\zeta,u,v) = p\,\Phi^{(p,q)}(\zeta,u,v) \ , \ \ D^0_v\,\Phi^{(p,q)}(\zeta,u,v) = q\,\Phi^{(p,q)}(\zeta,u,v) \ . \tag{2.9}$$

In the analytic superspace (2.1) one can realize two different infinite-dimensional groups of superconformal transformations. Each group consists of two commuting light-cone branches, the left and right ones, having as the algebra the classical $N = 4$, $SU(2)$ superconformal algebra (SCA) [18]. Without loss of generality we can specialize, e.g., to the left sector. It turns out that the form of the relevant superconformal transformations is basically specified by the transformation law of the analyticity-preserving covariant harmonic derivative $D^{(2,0)}$ (or $D^{(0,2)}$ in the right sector).

The basic distinguishing feature of the first group is that it does not touch the harmonics

$$\delta_I\, u^{(\pm1,0)}_i = 0 \ . \tag{2.10}$$

Its realization in $\mathbf{A}^{(1+2,1+2|2,2)}$ is completely fixed by the requirement that $D^{(2,0)}$ is invariant

$$\delta_I\, D^{(2,0)} = 0 \ . \tag{2.11}$$

The second superconformal group has the same Lie bracket structure as the first one, but it acts on *all* the left coordinates of $\mathbf{A}^{(1+2,1+2|2,2)}$, including the harmonic ones $u^{(\pm1,0)}$. We give here only the generic form of transformations of harmonics and the derivative $D^{(2,0)}$ [14]

$$\begin{aligned} \delta_{II}\, u^{(1,0)}_i &= \Lambda^{(2,0)}_I(z^{++},\theta^{(1,0)},u)u^{(-1,0)}_i \ , \ \delta_{II}\, u^{(-1,0)}_i = 0 \\ \delta_{II}\, D^{(2,0)} &= -\Lambda^{(2,0)}D^0_u \ , \ \ \Lambda^{(2,0)} = D^{(2,0)}\Lambda_L \ , \ \ D^{(2,0)}\Lambda^{(2,0)} = 0 \ . \end{aligned} \tag{2.12}$$

The main difference between these two $N = 4$, $SU(2)$ superconformal groups lies in the realization of their affine $SU(2)$ subgroups: the second one acts on the indices i, j and affects both the Grassmann and harmonic coordinates, while the first one acts only on the underlined indices and so affects only θ's. These groups do not commute; their closure is the "large" $N = 4$, $SO(4) \times U(1)$ group [18, 20, 21, 22]. For our further purposes it will be important that the latter involves an extra $U(1)$ affine (Kac-Moody) symmetry with

the dimensionless holomorphic parameter $\lambda_L(z^{++})$ (or $\lambda_R(z^{--})$ in the right sector). It is realized, e.g., on $u^{(1,0)\,i}$ as [14]

$$\delta_{U(1)} u^{(1,0)\,i} = (\, D^{(2,0)}\lambda_L(z)\,)\, u^{(-1,0)\,i} = i(\theta^{(0,1)})^2 \partial_{++}\lambda_L(z)\, u^{(-1,0)\,i}\ . \tag{2.13}$$

The "large" superconformal algebra corresponds to the most general solution [25] of the constraints on $\Lambda^{(2,0)}$ in eq. (2.12), while two of its $SU(2)$ subalgebras (SCA-I and SCA-II in what follows) are singled out by some additional conditions. Here we will not present the explicit form of the coordinate transformations of all these superconformal groups (see [14, 23] for details), since we will recover them as flat limits of the appropriate SG groups in the next Sections. Notice the following important property: both $N = 4$, $SU(2)$ superconformal groups, and hence their closure, leave invariant the analytic superspace integration measure $\mu^{(-2,-2)} = d^2z d^2\theta^{(1,0)} d^2\theta^{(0,1)}[du][dv]$:

$$\delta_I\ \mu^{(-2,-2)} = \delta_{II}\ \mu^{(-2,-2)} = 0\ . \tag{2.14}$$

The last topic of this introductory Section is the harmonic superspace description of some important $N = (4,4)$ multiplets. We start with one of the possible $N = (4,4)$ twisted chiral multiplets [26, 27], namely, the one having a simple description in $SU(2) \times SU(2)$ harmonic analytic superspace. It is represented by a real analytic $(4,4)$ superfield $q^{(1,1)}(\zeta, u, v)$ subject to the constraints

$$D^{(2,0)} q^{(1,1)} = D^{(0,2)} q^{(1,1)} = 0\ . \tag{2.15}$$

They leave in $q^{(1,1)}$ $8+8$ independent components [14], just the off-shell field content of $N = (4,4)$ twisted multiplet. The superfield $q^{(1,1)}$ is scalar with respect to the first $N = 4$, $SU(2)$ superconformal group but it is transformed with the weight 1 under the second one (this is necessary for preserving the constraints (2.15))

$$\delta_I\ q^{(1,1)} = 0\ , \quad \delta_{II}\ q^{(1,1)} = \Lambda_L\, q^{(1,1)} \tag{2.16}$$

(the transformations from the right-handed branches are similar). The physical dimension components of $q^{(1,1)}$ (four dimension 0 bosons and eight dimension 1/2 fermions) behave in different ways under these two kinds of $N = (4,4)$, $SU(2)$ transformations. In particular, the $SU(2)$ affine transformations from the first superconformal group act only on fermions, while those from the second group act both on bosons and fermions. The physical bosonic fields are naturally combined, with respect to the latter transformations and their right-handed counterparts, into a 2×2 matrix $q^{ia}(z^{++}, z^{--})$ on which the left (right) conformal $SU(2)$ acts as a left (right) multiplication . So the purely $SU(2)$ part of q^{ia} represents the coset $SU(2)_L \times SU(2)_R/SU(2)_{diag}$, and it is not too surprising that the $q^{(1,1)}$ action invariant under the second superconformal group is none other than $N = (4,4)$ extension of the $SU(2)$ WZW sigma model action. Indeed, it is just the $N = 4$, $SU(2) \times U(1)$ WZW sigma model action of ref. [27, 21, 22, 28, 29, 30]. The $SU(2) \times SU(2)$ analytic superspace form of this action reads [14]

$$S_{wzw} = -\frac{1}{4\gamma^2} \int \mu^{(-2,-2)}\ \hat{q}^{(1,1)} \hat{q}^{(1,1)} \left(\frac{1}{(1+X)X} - \frac{\ln(1+X)}{X^2} \right)\ , \tag{2.17}$$

where

$$\hat{q}^{(1,1)} = q^{(1,1)} - c^{(1,1)} \ , \ X = c^{(-1,-1)}\hat{q}^{(1,1)} \ , \ c^{(\pm 1,\pm 1)} = c^{ia}u_i^{(\pm 1,0)}v_a^{(0,\pm 1)} \ , \ c^{ia}c_{ia} = 2 \ , \quad (2.18)$$

and γ is a dimensionless sigma model coupling constant. Despite the presence of an extra quartet constant c^{ia} in the analytic superfield Lagrangian, the action actually does not depend on c^{ia} [14].

We wish to stress that the action (2.17) is unique (up to adding full harmonic derivatives) in the sense that it is the only possible action of a single superfield $q^{(1,1)}$ invariant under the second $N = (4,4)$, $SU(2)$ superconformal group. As we will see later, in the curved case the superfield $q^{(1,1)}$ serves as a compensator which breaks the appropriate $N = (4,4)$, $SU(2)$ SG group (having as the rigid limit the second $N = (4,4)$, $SU(2)$ superconformal group) down to the supergroup of minimal $N = (4,4)$, $2D$ SG [5].

As for the first superconformal group, an arbitrary action of the superfield $q^{(1,1)}$,

$$S_q = \int \mu^{(-2,-2)} \ \mathcal{L}^{(2,2)}(q^{(1,1)\,M}(\zeta, u, v), \ u, \ v) \ , \quad (2.19)$$

is invariant with respect to it. As a consequence of this property, the particular $q^{(1,1)}$ action (2.17) is invariant under both $N = (4,4)$, $SU(2)$ superconformal groups and, hence, under their closure, i.e. the "large" $N = (4,4)$, $SO(4) \times U(1)$ superconformal group. Note that $q^{(1,1)}$ transforms under the left affine $U(1)$ transformations (2.13) as

$$\delta_{U(1)}q^{(1,1)} = \lambda_L(z^{++})\, q^{(1,1)} \quad (2.20)$$

(and analogously under their right-handed counterparts). The full transformation law of $q^{(1,1)}$ under the left "large" group looks like the second law in eq. (2.16), with $\Lambda_L = \lambda_L(z^{++}) + \lambda^{(ik)}(z^{++})u_i^{(1,0)}u_k^{(-1,0)} + ...$ Further details will be given in Sect. 4. It is worth mentioning that the general action (2.19) always yelds the sigma model *with torsion* in the sector of physical bosons, just of the same kind as in the $N = (4,4)$ supersymmetric subclass of general $N = (2,2)$ chiral and twisted chiral superfield sigma models explored in [26]. The actions of other matter multiplets in $SU(2) \times SU(2)$ harmonic superspace reveal the same characteristic feature. This is the radical difference of the considered case from the dimensionally-reduced off-shell sigma model actions of hypermultiplets in the standard harmonic superspace with one set of the $SU(2)$ harmonic variables [15]: for physical bosons they yield the *torsionless* hyper-Kähler sigma model actions.

Note that there exist other types of twisted $N = 4$ multiplets, with the same number of off-shell components, but with different realizations of various $SU(2)$ factors of the full $SO(4)_L \times SO(4)_R$ automorphism group of rigid $N = (4,4)$, $2D$ SUSY [31, 32]. Respectively, the above two $N = (4,4)$, $SU(2)$ superconformal groups are realized in different ways on these multiplets. In particular, there exists a sort of twisted multiplet on which the first and second superconformal groups act in the way just opposite to their action on $q^{(1,1)}$. [1]

[1] In [7, 31] such a multiplet is called TM-I as opposed to $q^{(1,1)}$ which is TM-II in this classification. Such a classification makes sense with respect to a fixed $N = (4,4)$, $SU(2)$ SCA: if the conformal $SU(2)$ groups act both on the physical bosons and fermions, one deals with TM-II, whereas if they act only on fermions, one faces TM-I. Conversely, $q^{(1,1)}$ is TM-I with respect to the first of the two $N = (4,4)$, $SU(2)$ SCAs defined above, but it is TM-II with respect to the second one.

The $SU(2) \times SU(2)$ harmonic superspace description of these complementary twisted multiplets [32] is somewhat more complicated. Nevertheless, all of them can be coupled to the $N = (4,4)$ Beltrami-Weyl SG multiplets to be defined below and so can serve as compensators. We are planning to present these couplings in a future work.

Finally, we mention one more analytic $SU(2) \times SU(2)$ harmonic supermultiplet which will be used in Sec. 5 as a compensator reducing the $N = (4,4)$, $SO(4) \times U(1)$ SG group to one of its $N = (4,4)$, $SU(2)$ subgroups. It is represented by a pair of analytic superfields $N^{(2,0)}$, $N^{(0,2)}$ satisfying the constraints [23]

$$\begin{aligned} D^{(2,0)}N^{(2,0)} + N^{(2,0)}N^{(2,0)} &= 0, \;\; D^{(0,2)}N^{(0,2)} + N^{(0,2)}N^{(0,2)} = 0, \\ D^{(2,0)}N^{(0,2)} - D^{(0,2)}N^{(2,0)} &= 0 \; . \end{aligned} \tag{2.21}$$

These constraints are analogous to those defining the so-called nonlinear supermultiplet [24] in the $N = 2$, $4D$ harmonic superspace (the latter goes into $N = (4,4)$, $SU(2)_{diag}$ harmonic superspace upon reduction to $2D$). They are obviously covariant under the first $N = (4,4)$, $SU(2)$ superconformal group, if $N^{(2,0)}, N^{(0,2)}$ are assumed to transform as scalars with respect to it. They are also covariant under the second group, provided $N^{(2,0)}, N^{(0,2)}$ transform according to

$$\delta_{II} N^{(2,0)} = \Lambda^{(2,0)}, \quad \delta_{II} N^{(0,2)} = \Lambda^{(0,2)} \; . \tag{2.22}$$

The simplest invariant action (with the correct sign of the kinetic terms of the physical fields) is as follows:

$$S_N \sim - \int \mu^{(-2,-2)} \, N^{(2,0)} N^{(0,2)} \; . \tag{2.23}$$

To see that it is invariant (up to surface terms) under (2.22), one should take into account the invariance of the analytic superspace integration measure and the properties

$$\Lambda^{(2,0)} = D^{(2,0)}\Lambda_L, \;\; \Lambda^{(0,2)} = D^{(0,2)}\Lambda_R, \;\; D^{(2,0)}\Lambda_R = D^{(0,2)}\Lambda_L = 0 \; . \tag{2.24}$$

The pair $N^{(2,0)}, N^{(0,2)}$ describes $32+32$ off-shell degrees of freedom and is dual-equivalent to four $q^{(1,1)}$ superfields [23].

Having the multiplet $N^{(2,0)}, N^{(0,2)}$, one can define further consistent non-linear multiplets $G^{(2,0)}, G^{(0,2)}$ which are zero-weight scalars under both $N = (4,4)$, $SU(2)$ superconformal groups

$$\delta_{I,II} \, G^{(2,0)} = \delta_{I,II} \, G^{(0,2)} = 0 \; . \tag{2.25}$$

The corresponding constraints (covariant with respect to both superconformal groups) are a slight modification of (2.21)

$$\begin{aligned} (D^{(2,0)} + 2N^{(2,0)})G^{(2,0)} + \alpha \, G^{(2,0)}G^{(2,0)} &= 0 \; , \\ (D^{(0,2)} + 2N^{(0,2)})G^{(0,2)} + \alpha \, G^{(0,2)}G^{(0,2)} &= 0 \; , \\ D^{(2,0)}G^{(0,2)} - D^{(0,2)}G^{(2,0)} &= 0 \; , \end{aligned} \tag{2.26}$$

where α is an arbitrary dimensionless parameter (it can be equal to zero). All such representations comprise $32 + 32$ off-shell degrees of freedom. Their Lagrangians are bilinears like in (2.23).

3 Curved $SU(2) \times SU(2)$ analytic superspace and N=(4,4) Beltrami-Weyl multiplet

By analogy with the $N = 2$, $4D$ case [16, 17] we assume that the fundamental group of $N = (4,4)$, $2D$ conformal supergravity is represented by the following diffeomorphisms of the analytic harmonic $SU(2) \times SU(2)$ superspace

$$\delta\zeta^\mu = \Lambda^\mu(\zeta,u,v),\ \delta u_i^{(1,0)} = \Lambda^{(2,0)}(\zeta,u,v)u_i^{(-1,0)},\ \delta v_a^{(0,1)} = \Lambda^{(0,2)}(\zeta,u,v)v_a^{(0,-1)},$$
$$\delta u_i^{(-1,0)} = \delta v_a^{(0,-1)} = 0\ . \tag{3.1}$$

Here $\zeta^\mu = (z^{++}, z^{--}, \theta^{(1,0)\,\underline{k}\,+}, \theta^{(0,1)\,\underline{b}\,-})$ as in (2.1) and the gauge parameters Λ^μ, $\Lambda^{(2,0)}$, $\Lambda^{(0,2)}$ are arbitrary functions over the *whole* harmonic analytic superspace $\mathbf{A}^{(1+2,1+2|2,2)}$. These transformation laws preserve the defining relations of harmonic variables (2.2) and the reality of $\mathbf{A}^{(1+2,1+2|2,2)}$ with respect to the "$\sim$ " conjugation. The analyticity-preserving harmonic derivatives $D^{(2,0)}$ and $D^{(0,2)}$ are covariantized by introducing appropriate analytic vielbeins

$$\begin{aligned}
D^{(2,0)} \Rightarrow \nabla^{(2,0)} &= D^{(2,0)} + H^{(2,0)\,\mu}\partial_\mu + H^{(4,0)}\partial^{(-2,0)} + H^{(2,2)}\partial^{(0,-2)} \\
&\equiv D^{(2,0)} + H^{(2,0)\,M}\partial_M\ , \\
D^{(0,2)} \Rightarrow \nabla^{(0,2)} &= D^{(0,2)} + H^{(0,2)\,\mu}\partial_\mu + \tilde{H}^{(2,2)}\partial^{(-2,0)} + H^{(0,4)}\partial^{(0,-2)} \\
&\equiv D^{(0,2)} + H^{(0,2)\,M}\partial_M\ ,
\end{aligned} \tag{3.2}$$

where we used the notation

$$M = (\mu, (2,0), (0,2)), \quad \partial_M = (\partial_\mu, \partial^{(-2,0)}, \partial^{(0,-2)}),$$
$$\partial^{(-2,0)} = u^{(-1,0)\,i}\frac{\partial}{\partial u^{(1,0)\,i}}, \quad \partial^{(0,-2)} = v^{(0,-1)\,a}\frac{\partial}{\partial v^{(0,1)\,a}} \tag{3.3}$$

and separated the flat parts of the vielbein components in front of ∂_{++} in $\nabla^{(2,0)}$ and ∂_{--} in $\nabla^{(0,2)}$. In eqs. (3.2) all the vielbeins are analytic $N = (4,4)$, $2D$ superfields,

$$H^{(2,0)\,M} = H^{(2,0)\,M}(\zeta,u,v)\ ,\ \ H^{(0,2)\,M} = H^{(0,2)\,M}(\zeta,u,v)\ .$$

The flat limit is achieved by putting them equal to zero. The $U(1)$ charge-counting operators D_u^0 and D_v^0 retain their flat form (2.6).

Again in analogy with refs. [16, 17], we postulate for $\nabla^{(2,0)}$, $\nabla^{(0,2)}$ the following transformation law under the $N = (4,4)$ SG group (3.1)

$$\delta\nabla^{(2,0)} = -\Lambda^{(2,0)}D_u^0\ ,\quad \delta\nabla^{(0,2)} = -\Lambda^{(0,2)}D_v^0\ , \tag{3.4}$$

whence

$$\begin{aligned}
\delta H^{(2,0)\,++} &= \nabla^{(2,0)}\Lambda^{++} - 2i\Lambda^{(1,0)}\theta^{(1,0)}\ ,\ \ \delta H^{(2,0)\,--} = \nabla^{(2,0)}\Lambda^{--}\ , \\
\delta H^{(3,0)\,\underline{i}} &= \nabla^{(2,0)}\Lambda^{(1,0)\,\underline{i}} - \Lambda^{(2,0)}\theta^{(1,0)\,\underline{i}}\ ,\ \ \delta H^{(2,1)\,\underline{a}} = \nabla^{(2,0)}\Lambda^{(0,1)\,\underline{a}}\ , \\
\delta H^{(4,0)} &= \nabla^{(2,0)}\Lambda^{(2,0)}\ ,\ \ \delta H^{(2,2)} = \nabla^{(2,0)}\Lambda^{(0,2)}\ ,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\delta H^{(0,2)\,++} &= \nabla^{(0,2)}\Lambda^{++}\ ,\ \ \delta H^{(0,2)\,--} = \nabla^{(0,2)}\Lambda^{--} - 2i\Lambda^{(0,1)}\theta^{(0,1)}\ , \\
\delta H^{(1,2)\,\underline{i}} &= \nabla^{(0,2)}\Lambda^{(1,0)\,\underline{i}}\ ,\ \ \delta H^{(0,3)\,\underline{a}} = \nabla^{(0,2)}\Lambda^{(0,1)\,\underline{a}} - \Lambda^{(0,2)}\theta^{(0,1)\,\underline{a}}\ , \\
\delta\tilde{H}^{(2,2)} &= \nabla^{(0,2)}\Lambda^{(2,0)}\ ,\ \ \delta H^{(0,4)} = \nabla^{(0,2)}\Lambda^{(0,2)}\ .
\end{aligned} \tag{3.6}$$

From now on, the similarity with the $N = 2$, $4D$ construction ceases to be literal and the specificity of the $N = (4, 4)$ case comes into play.

First of all, we wish to generalize the notion of the twisted analytic superfield $q^{(1,1)}$ to the curved case and hence need to find a correct generalization of the defining constraints (2.15) and the superconformal transformation laws (2.16). As we have started with the most general diffeomorphism group of the analytic superspace, we expect it to yield, in the flat limit, the product of the left and right "large" $SO(4) \times U(1)$ superconformal groups, including their $U(1)$ affine subgroups with the parameters $\lambda_L(z^{++})$, $\lambda_R(z^{--})$. However, a close inspection of the analytic superfield gauge parameters $\Lambda^\mu(\zeta, u, v)$, $\Lambda^{(2,0)}(\zeta, u, v)$ and $\Lambda^{(0,2)}(\zeta, u, v)$ shows that there is no place in them for such dimensionless parameters (these can appear only with their z derivatives). To generalize the transformation laws of $q^{(1,1)}$ (2.16), (2.20) to the curved case, we are led to introduce two extra *independent* analytic gauge functions

$$\Lambda_L(\zeta, u, v) = \lambda_L(z^{++}, z^{--}) + \dots\ , \quad \Lambda_R(\zeta, u, v) = \lambda_R(z^{++}, z^{--}) + \dots$$

and to ascribe the following transformation laws to $q^{(1,1)}$

$$\delta q^{(1,1)} = (\Lambda_L + \Lambda_R) q^{(1,1)}\ . \tag{3.7}$$

We call these transformations the "$U(1)$ weight" ones, to distinguish them from the harmonic $U(1)$ phase transformations. We normalize the left and right $U(1)$ weights J_L and J_R as

$$J_L q^{(1,1)} = J_R q^{(1,1)} = q^{(1,1)}\ . \tag{3.8}$$

At this stage, the $U(1)$ weight analytic parameters Λ_L, Λ_R are entirely unrelated to those of the coordinate transformations.

Such a relation naturally comes out, as a result of choosing the appropriate transformation law for the $U(1)$ weight-covariantized harmonic derivatives and fixing a proper gauge.

We covariantize $\nabla^{(2,0)}$, $\nabla^{(0,2)}$ by introducing four analytic superfield $U(1)$ connections $H_L^{(2,0)}(\zeta, u, v)$, $H_R^{(2,0)}(\zeta, u, v)$, $H_L^{(0,2)}(\zeta, u, v)$, $H_R^{(0,2)}(\zeta, u, v)$

$$\begin{aligned} \nabla^{(2,0)} \ &\Rightarrow\ \mathcal{D}^{(2,0)} = \nabla^{(2,0)} + H_L^{(2,0)} J_L + H_R^{(2,0)} J_R \\ \nabla^{(0,2)} \ &\Rightarrow\ \mathcal{D}^{(0,2)} = \nabla^{(0,2)} + H_L^{(0,2)} J_L + H_R^{(0,2)} J_R\ , \end{aligned} \tag{3.9}$$

and postulate the following transformation laws for $\mathcal{D}^{(2,0)}$, $\mathcal{D}^{(0,2)}$

$$\begin{aligned} \delta\mathcal{D}^{(2,0)} &= -\Lambda^{(2,0)}(D_u^0 - J_L) - \nabla^{(2,0)}\Lambda_L\ J_L - \nabla^{(2,0)}\Lambda_R\ J_R\ , \\ \delta\mathcal{D}^{(0,2)} &= -\Lambda^{(0,2)}\ (D_v^0 - J_R) - \nabla^{(0,2)}\Lambda_L\ J_L - \nabla^{(0,2)}\Lambda_R\ J_R\ . \end{aligned} \tag{3.10}$$

The transformation laws of the vielbeins in $\nabla^{(2,0)}$, $\nabla^{(0,2)}$ do not change, while the newly introduced $U(1)$ connections are transformed as

$$\begin{aligned} \delta H_L^{(2,0)} &= \Lambda^{(2,0)} - \nabla^{(2,0)}\Lambda_L\ , \ \ \delta H_R^{(2,0)} \ = \ -\nabla^{(2,0)}\Lambda_R\ , \\ \delta H_L^{(0,2)} &= -\nabla^{(0,2)}\Lambda_L\ , \ \ \delta H_R^{(0,2)} \ = \ \Lambda^{(0,2)} - \nabla^{(0,2)}\Lambda_R\ . \end{aligned} \tag{3.11}$$

The $\mathcal{D}^{(2,0)}$ and $\mathcal{D}^{(0,2)}$ derivatives of the analytic superfield $\Phi^{(p,q)}$, with the left and right $U(1)$ weights equal to l and r, are transformed as follows:

$$\begin{aligned}\delta\mathcal{D}^{(2,0)}\Phi^{(p,q)} &= -\Lambda^{(2,0)}(p-l)\Phi^{(p,q)} + (l\Lambda_L + r\Lambda_R)\mathcal{D}^{(2,0)}\Phi^{(p,q)} \ , \\ \delta\mathcal{D}^{(0,2)}\Phi^{(p,q)} &= -\Lambda^{(0,2)}(q-r)\Phi^{(p,q)} + (l\Lambda_L + r\Lambda_R)\mathcal{D}^{(0,2)}\Phi^{(p,q)} \ . \end{aligned} \tag{3.12}$$

We see that only provided $p = l$, $q = r$, these derivatives are actually covariant, i.e. they transform as the superfield $\Phi^{(p,q)}$ itself. But this is precisely what happens for $q^{(1,1)}$, which possesses $J_L = J_R = 1$. Therefore, as the appropriate curved generalization of the constraints (2.15), we choose the following ones:

$$\begin{aligned}\mathcal{D}^{(2,0)}q^{(1,1)} &= (\nabla^{(2,0)} + H_L^{(2,0)} + H_R^{(2,0)})q^{(1,1)} = 0 \ , \\ \mathcal{D}^{(0,2)}q^{(1,1)} &= (\nabla^{(0,2)} + H_L^{(0,2)} + H_R^{(0,2)})q^{(1,1)} = 0 \ . \end{aligned} \tag{3.13}$$

Before going further, let us adduce some reasoning in favor of the choice of the transformation laws of $\mathcal{D}^{(2,0)}$, $\mathcal{D}^{(0,2)}$ in the form (3.10). The primary reason for this choice is the desire to relate the coordinate transformations with the $U(1)$ weight transformations, so as to eventually ensure a correct flat limit. Indeed, from eqs. (3.11) it follows that the connections $H_L^{(2,0)}$, $H_R^{(0,2)}$ can be entirely gauged away, thereby establishing the sought relation

$$H_L^{(2,0)} = H_R^{(0,2)} = 0 \Rightarrow \Lambda^{(2,0)} = \nabla^{(2,0)}\Lambda_L \ , \quad \Lambda^{(0,2)} = \nabla^{(0,2)}\Lambda_R \ . \tag{3.14}$$

In what follows we will frequently stick to this gauge. One more argument why we should assume (3.10) is based on an analogy with the harmonic space description of quaternionic manifolds in [33]. There, the analyticity-preserving harmonic derivative in the analytic basis necessarily involves an analytic connection ϕ^{++} associated with the so called "$Sp(1)$ weight". Its transformation law literally mimics that of $H_L^{(2,0)}$, $H_R^{(0,2)}$, so it is natural to assume that the $U(1)$ weights J_L, J_R and the associated analytic superfield parameters Λ_L and Λ_R are direct analogs of the just mentioned $Sp(1)$ weight and the related analytic parameter inherent to the quaternionic manifolds [2]. Of course, the most direct way to justify the transformation law (3.10) would be to deduce it proceeding from the appropriate constraints in the standard $N = (4,4)$ superspace. An alternative way is to show that it leads to a self-consistent SG theory, still in the framework of the analytic superspace. This is just what we are going to demonstrate.

An important consequence of the presence of two independent harmonic constraints in the definition of the twisted superfield $q^{(1,1)}$, eqs. (3.13), is the integrability condition

$$[\mathcal{D}^{(2,0)}, \mathcal{D}^{(0,2)}]q^{(1,1)} = 0 \ . \tag{3.15}$$

It is easy to see that the direct generalization of the flat condition $[D^{(2,0)}, D^{(0,2)}] = 0$, namely,

$$[\mathcal{D}^{(2,0)}, \mathcal{D}^{(0,2)}] = 0 \ ,$$

is not covariant under (3.10). The covariant version of this constraint is as follows:

$$[\mathcal{D}^{(2,0)}, \mathcal{D}^{(0,2)}] = -H^{(2,2)}(D_v^0 - J_R) + \tilde{H}^{(2,2)}(D_u^0 - J_L) \ . \tag{3.16}$$

[2] A deep analogy between the description of quaternionic manifolds in the harmonic space and that of conformal $N = 2$, $4D$ SG in the harmonic superspace was pointed out in [33].

It is evident that eq. (3.15) is automatically satisfied as a consequence of (3.16) and (3.8). This constraint implies

$$\begin{aligned}\nabla^{(2,0)}H_L^{(0,2)} - \nabla^{(0,2)}H_L^{(2,0)} + \tilde{H}^{(2,2)} &= 0 \ , \\ \nabla^{(2,0)}H_R^{(0,2)} - \nabla^{(0,2)}H_R^{(2,0)} - H^{(2,2)} &= 0 \end{aligned} \tag{3.17}$$

and

$$[\nabla^{(2,0)}, \nabla^{(0,2)}] = -H^{(2,2)}D_v^0 + \tilde{H}^{(2,2)}D_u^0 \ . \tag{3.18}$$

From the latter relation one deduces the constraints on the analytic vielbeins

$$\begin{aligned}\nabla^{(2,0)}H^{(0,2)\,++} - \nabla^{(0,2)}H^{(2,0)\,++} - 2iH^{(1,2)}\theta^{(1,0)} &= 0 \ , \\ \nabla^{(2,0)}H^{(0,2)\,--} - \nabla^{(0,2)}H^{(2,0)\,--} + 2iH^{(2,1)}\theta^{(0,1)} &= 0 \ , \\ \nabla^{(2,0)}H^{(1,2)\,\underline{i}} - \nabla^{(0,2)}H^{(3,0)\,\underline{i}} - \tilde{H}^{(2,2)}\theta^{(1,0)\,\underline{i}} &= 0 \ , \\ \nabla^{(2,0)}H^{(0,3)\,\underline{a}} - \nabla^{(0,2)}H^{(2,1)\,\underline{a}} + H^{(2,2)}\theta^{(0,1)\,\underline{a}} &= 0 \ , \\ \nabla^{(2,0)}H^{(0,4)} - \nabla^{(0,2)}H^{(2,2)} &= 0 \ , \\ \nabla^{(2,0)}\tilde{H}^{(2,2)} - \nabla^{(0,2)}H^{(4,0)} &= 0 \ . \end{aligned} \tag{3.19}$$

Thus we see that in the $N = (4,4)$, $SU(2) \times SU(2)$ case the analytic vielbeins and $U(1)$ connections covariantizing $D^{(2,0)}$, $D^{(0,2)}$ are necessarily constrained. This is the crucial difference from the formulation of $N = 2$, $4D$ conformal SG in the standard harmonic superspace [16, 17], where the analogous quantities are unconstrained analytic superfields, i.e. the prepotentials of the theory. Of course, this peculiarity is a direct consequence of the presence of two independent sets of harmonic variables in the considered case.

For the time being, we do not know how to solve (3.17), (3.19) via unconstrained superfield prepotentials. To single out the irreducible field representation carried by vielbeins and $U(1)$ connections, we keep to another strategy. Namely, we use the initial gauge freedom to gauge away from these objects as many components as possible, then substitute the resulting expressions into the constraints and solve the latter in this WZ-type gauge. Eventually, it turns out that the solution exists, is unique and is not reduced to a pure gauge. The superfield constraints prove to be purely kinematic: indeed, they do not imply any differential conditions, nor equations of motion, for the remaining fields. At present we are aware of the full nonlinear solution of these constraints. Here, we limit ourselves to the linearized level. This is quite sufficient for revealing the irreducible field contents of the SG theory under consideration.

In the present case, one can choose the WZ gauge in several different ways, the basic criterion for one or another choice being the desire to simplify the constraints (3.17), (3.19) as much as possible. As a first step, we choose the gauge (3.14) and the following additional ones

$$H^{(2,0)\,++} = H^{(0,2)\,--} = H^{(3,0)\,\underline{i}} = H^{(0,3)\,\underline{a}} = 0 \ , \tag{3.20}$$

$$H^{(4,0)} = H^{(0,4)} = 0 \ . \tag{3.21}$$

These gauges restrict in a certain way the original gauge parameters. At the considered linearized level, (3.20) and (3.21) give rise to the following relations:

$$D^{(2,0)}\Lambda^{++} - 2i\Lambda^{(1,0)}\theta^{(1,0)} = 0 \ , \ D^{(2,0)}\Lambda^{(1,0)\,\underline{i}} - D^{(2,0)}\Lambda_L\,\theta^{(1,0)\,\underline{i}} = 0 \ ,$$

$$
\begin{aligned}
D^{(0,2)}\Lambda^{--} - 2i\Lambda^{(0,1)}\theta^{(0,1)} &= 0\ ,\ D^{(0,2)}\Lambda^{(0,1)\,\underline{a}} - D^{(0,2)}\Lambda_R\,\theta^{(0,1)\,\underline{a}} = 0\ , \\
(D^{(2,0)})^2\Lambda_L &= (D^{(0,2)})^2\Lambda_R = 0\ ,
\end{aligned}
\tag{3.22}
$$

which strictly fix the u or v dependence of the relevant parameters (depending on which derivative, i.e. either $D^{(2,0)}$ or $D^{(0,2)}$, enters the given relation). After this, there still remains a freedom associated with the surviving harmonic dependence. This freedom can be used to further gauge away some of the components in the double harmonic expansion of the remaining vielbeins $H^{(2,0)\,--}$, $H^{(0,2)\,++}$, $H^{(2,1)\,\underline{a}}$, $H^{(1,2)\,\underline{i}}$ and the $U(1)$ connections $H^{(2,0)}_R$, $H^{(0,2)}_L$. At this stage, the u and v dependence of all analytic superfield gauge parameters is completely fixed and we are left with a finite set of the component parameters. However, in the vielbeins and connections one still finds a non-trivial harmonic dependence which is entirely fixed only after imposing the constraints. The final expressions for the vielbeins, connections and superfield gauge parameters at the linearized level are as follows:

$$
\begin{aligned}
H^{(2,0)\,--} &= i(\theta^{(1,0)})^2\{h^{--}_{++} - 2i\theta^{(0,1)}_{\underline{a}}h^{-\,a\underline{a}}_{++}v^{(0,-1)}_a - i(\theta^{(0,1)})^2h^{(ab)}_{++}v^{(0,-1)}_a v^{(0,-1)}_b\}\ , \\
H^{(2,1)\,\underline{a}} &= i(\theta^{(1,0)})^2\{h^{-\,a\underline{a}}_{++}v^{(0,1)}_a + \theta^{(0,1)\,\underline{b}}[h^{(\underline{a}}_{++\,\underline{b})} + \frac{1}{2}\delta^{\underline{a}}_{\underline{b}}(\partial_{--}h^{--}_{++} - 2h^{(ab)}_{++}v^{(0,1)}_a v^{(0,-1)}_b)] \\
&\quad +(\theta^{(0,1)})^2(\frac{1}{2}t^{b\underline{a}}_{++-} - i\partial_{--}h^{-\,b\underline{a}}_{++})v^{(0,-1)}_b\}\ , \\
H^{(2,0)}_R &= i(\theta^{(1,0)})^2\{h_{++} + h^{(ab)}_{++}v^{(0,1)}_a v^{(0,-1)}_b - \theta^{(0,1)}_{\underline{a}}t^{b\underline{a}}_{++-} \\
&\quad - i(\theta^{(0,1)})^2\partial_{--}h^{(ab)}_{++}v^{(0,-1)}_a v^{(0,-1)}_b\}\ ,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\Lambda^{--} &= \lambda^{--} - 2i\theta^{(0,1)}_{\underline{a}}\lambda^{-\,a\underline{a}}v^{(0,-1)}_a + i(\theta^{(0,1)})^2\lambda^{(ab)}v^{(0,-1)}_a v^{(0,-1)}_b\ , \\
\Lambda^{(0,1)\,\underline{a}} &= \lambda^{-\,a\underline{a}}v^{(0,1)}_a + \theta^{(0,1)\,\underline{b}}[\lambda^{(\underline{a}}_{\underline{b})} + \frac{1}{2}\delta^{\underline{a}}_{\underline{b}}(\partial_{--}\lambda^{--} + 2\lambda^{(ab)}v^{(0,1)}_a v^{(0,-1)}_b)] \\
&\quad -(\theta^{(0,1)})^2(i\partial_{--}\lambda^{-\,a\underline{a}} + \frac{1}{2}\beta^{a\underline{a}}_{-})v^{(0,-1)}_a\ , \\
\Lambda_R &= \lambda_R + \lambda^{(ab)}v^{(0,-1)}_a v^{(0,1)}_b - \theta^{(0,1)}_{\underline{a}}\beta^{a\underline{a}}_{-}v^{(0,-1)}_a \\
&\quad - i(\theta^{(0,1)})^2\partial_{--}\lambda^{(ab)}v^{(0,-1)}_a v^{(0,-1)}_b\ ,
\end{aligned}
\tag{3.24}
$$

and $H^{(0,2)\,++}$, $H^{(1,2)\,\underline{i}}$, $H^{(0,2)}_L$, Λ^{++}, $\Lambda^{(1,0)\,\underline{i}}$, Λ_L can be obtained from these expressions via the substitutions $+ \leftrightarrow -$, $\theta^{(1,0)\,\underline{i}} \leftrightarrow \theta^{(0,1)\,\underline{a}}$, $u \leftrightarrow v$, $i, \underline{i} \leftrightarrow a, \underline{a}$. In (3.23), (3.24) all the component fields and gauge parameters are functions of z^{++}, z^{--} and we have explicitly indicated their $2D$ space-time indices. Note that in the chosen gauge the diagonal components of the world-sheet zweibein h^{++}_{++}, h^{--}_{--} equal unity and the parameters of two independent Weyl rescalings of $\theta^{(1,0)\,\underline{i}}$, $\theta^{(0,1)\,\underline{a}}$ are fixed to be $\partial_{++}\lambda^{++}$, $\partial_{--}\lambda^{--}$, so the difference between the world and tangent indices of the involved fields actually disappears. Actually, we have used all the gauge symmetries with pure shifts in their transformation laws for gauging away the corresponding field components (rescalings are just of this kind). We ended up only with the transformations starting with z-derivatives of gauge parameters.

Looking at the above expressions we observe that the irreducible content of the original set of analytic vielbeins and connections includes only gauge fields: the two components

of the world-sheet zweibein h^{++}_{--}, h^{--}_{++}, the left and right gravitino components $h^{+\ \underline{ii}}_{--}, h^{-\ \underline{aa}}_{++}$, the left and right components of the $SO(4)_L \times U(1)_L$ and $SO(4)_R \times U(1)_R$ gauge connections $h^{(ij)}_{--}, h^{(\underline{ij})}_{--}, h_{--}$ and $h^{(ab)}_{++}, h^{(\underline{ab})}_{++}, h_{++}$, as well as the left and right components of the "conformal gravitino" $t^{\underline{ii}}_{--+}$, $t^{\underline{ab}}_{++-}$, with a total of $(16+16)$ independent components. The remaining gauge freedom involves just the same number of gauge parameters, so locally all these gauge fields can be gauged away, though such a gauge is inadmissible globally (e.g., after coupling this multiplet to the $N=(4,4)$ string fields, the zweibein components should produce two Virasoro constraints). Therefore it is natural to call the obtained gauge multiplet, with no off-shell degrees of freedom, the " $N=(4,4)$, $SO(4)\times U(1)$ Beltrami-Weyl (BW) multiplet". We shall see later that it admits truncations to two different $N=(4,4)$, $SU(2)$ ones. We will also show that the off-shell (8+8) "minimal $N=4$, $2D$ SG multiplet" [5, 6] naturally comes out as the result of coupling one of the $N=(4,4)$, $SU(2)$ BW multiplets to one kind of twisted $N=(4,4)$ multiplet treated as a compensator.

Actually, in order to be able to construct manifestly invariant superfield couplings of $N=(4,4)$ BW multiplets to $N=(4,4)$ matter, we need one more ingredient. This is an analytic density which should transform so as to cancel the transformation of the analytic superspace integration measure $\mu^{(-2,-2)}$. Indeed, as distinct from the flat superspace superconformal groups, the full local group (3.1) does not leave $\mu^{(-2,-2)}$ invariant:

$$\delta\mu^{(-2,-2)} = ((-1)^{P(\mu)}\partial_\mu\Lambda^\mu + \partial^{(-2,0)}\Lambda^{(2,0)} + \partial^{(0,-2)}\Lambda^{(0,2)})\ \mu^{(-2,-2)} \equiv \tilde{\Lambda}\ \mu^{(-2,-2)}\ , \quad (3.25)$$

where $P(\mu)$ is 0 for bosonic and 1 for fermionic indices.

Defining the objects

$$\Gamma^{(2,0)} = (-1)^{P(M)}\partial_M H^{(2,0)\ M}\ ,\ \ \Gamma^{(0,2)} = (-1)^{P(M)}\partial_M H^{(0,2)\ M}\ , \quad (3.26)$$

one finds them to transform as

$$\delta\Gamma^{(2,0)} = \nabla^{(2,0)}\tilde{\Lambda}\ ,\ \ \delta\Gamma^{(0,2)} = \nabla^{(0,2)}\tilde{\Lambda} \quad (3.27)$$

and to satisfy, as a consequence of the constraints (3.19), the condition

$$\nabla^{(2,0)}\Gamma^{(0,2)} - \nabla^{(0,2)}\Gamma^{(2,0)} = 0\ . \quad (3.28)$$

It is easy to show that (3.28) implies

$$\Gamma^{(2,0)} = \nabla^{(2,0)}\Sigma(\zeta,u,v)\ ,\ \ \Gamma^{(0,2)} = \nabla^{(0,2)}\Sigma(\zeta,u,v)\ . \quad (3.29)$$

Again, with making use of the constraints (3.19), $\Sigma(\zeta,u,v)$ can be expressed in terms of the original BW multiplet (up to an unessential additive constant) [3] and shown to transform as

$$\delta\Sigma = \tilde{\Lambda}\ . \quad (3.30)$$

Hence the quantity

$$\Omega \equiv e^{-\Sigma}\ ,\ \ \delta\Omega = -\tilde{\Lambda}\,\Omega \quad (3.31)$$

[3] To the zeroth order in the θ's and the first order in the fields, one has $\Sigma = const + (h^{++}_{++} + h^{--}_{--}) + ...$

is the sought object, compensating for the non-invariance of the measure. In what follows we will need only the property

$$(\nabla^{(2,0)} + \Gamma^{(2,0)}) \Omega = 0 \ , \quad (\nabla^{(0,2)} + \Gamma^{(0,2)}) \Omega = 0 \ . \tag{3.32}$$

In particular, due to this property, one can still integrate by parts with respect to the *covariantized* harmonic derivatives. Indeed, for any analytic function $F(\zeta, u, v)$, the integral

$$\int \mu^{(-2,-2)} \Omega \nabla^{(2,0)} F(\zeta, u, v) \ ,$$

up to full *ordinary* derivatives, reduces to

$$- \int \mu^{(-2,-2)} (\nabla^{(2,0)} + \Gamma^{(2,0)}) \Omega \ F(\zeta, u, v) = 0$$

(the same is true for $\nabla^{(0,2)}$).

4 Various limits and truncations

Inspecting the residual symmetry parameters (3.24), one observes that after constraining their z dependence, in such a way that the left (right) parameters are functions solely of $z^{++}(z^{--})$,

$$\begin{aligned} \partial_{--} \Lambda^{++} &= \partial_{++} \Lambda^{(1,0)\, \underline{i}} = \partial_{--} \Lambda_L = 0 \ , \\ \partial_{++} \Lambda^{--} &= \partial_{++} \Lambda^{(0,1)\, \underline{a}} = \partial_{++} \Lambda_R = 0 \ , \end{aligned} \tag{4.1}$$

they constitute the direct sum of two "large" $N = (4,4)$, $SO(4) \times U(1)$ superconformal algebras [18, 20, 21, 22]. To see this, one should study the Lie brackets of the transformations (3.1) into which these restricted parameters expanded in series in $z^{\pm\pm}$ are substituted. Then, e.g., for the right branch, one finds that the expansion of $\lambda^{--}(z^{--})$ produces a Virasoro subsector, that of $\lambda^{(\underline{ab})}(z^{--})$, $\lambda^{(ab)}(z^{--})$ yields two affine $SU(2)$ subalgebras, and that of $\lambda^{-a\underline{b}}(z^{--})$, $\beta_-^{a\underline{b}}(z^{--})$ corresponds to the two types of SUSY generators present in this SCA, i.e. the canonical generators (they involve, in particular, the $N = 4$ Poincaré SUSY and the special conformal SUSY generators) and the non-canonical ones. [4] The affine $U(1)$ parameters contained in $\lambda_R(z^{--})$ appear in the closure of the canonical and non-canonical SUSY transformations (actually, the rigid $U(1)$ parameter $\lambda_R(z^{--})|_{z=0}$ never appears in the closure on the superspace coordinates, but it does appear when one considers the closure on the superfield $q^{(1,1)}$ with the transformation law (3.7)). It is also easy to check that these restricted superparameters coincide with those appearing in the realizations of these $N = 4$ SCAs in the flat $SU(2) \times SU(2)$ harmonic superspace [14, 32].

Thus, we found that the original $N = (4,4)$ SG group (3.1), (3.5), (3.6), (3.7), (3.11) contains the direct sum of two $N = 4$, $SO(4) \times U(1)$ SCAs as the essential invariance subalgebra of the residual gauge freedom associated with the superparameters (3.24) (and

[4] Strictly speaking, such expansions define that part of $N = 4$ SCA which is regular at the origin. Just such subalgebras of the left and right $N = 4$, $SU(2)$ SCAs were gauged in the component approach of ref. [3].

their left counterparts). It should be stressed that it is an invariance of the full nonlinear theory, not only of the linearized approximation (3.23). Indeed, it could be recovered from the general harmonic vielbein transformation laws (3.5), (3.6), (3.11), as the maximal subgroup preserving the flat limit

$$H^{(2,0)\,M} = H^{(0,2)\,M} = H^{(2,0)}_{L,R} = H^{(0,2)}_{L,R} = 0 \ . \tag{4.2}$$

Thus, the analytic superdiffeomorphism group of Sect. 3 can be regarded as the local, gauged version of this maximal rigid $N = (4,4)$ superconformal group, with the BW multiplet defined by eq. (3.23) (and by its left counterpart) as the corresponding gauge multiplet. Presumably, the latter can be alternatively recovered via direct gauging of this SCA following the procedure of ref. [3]. The $SU(2) \times SU(2)$ harmonic superspace approach allows one to relate it to the fundamental objects of the analytic superspace geometry, the analytic harmonic vielbeins $H^{(2,0)M}, H^{(0,2)M}$ and the analytic $U(1)$ connections $H^{(2,0)}_{L,R}, H^{(0,2)}_{L,R}$.

Since $N = (4,4)$, $SO(4) \times U(1)$ SCA contains as its infinite-dimensional subalgebras two $N = (4,4)$, $SU(2)$ SCAs (SCA-I and SCA-II), it is natural to expect that its local extension also contains two smaller $N = (4,4)$ SG groups having these superconformal symmetries as the maximal "rigid" subgroups. They can naturally be called the $N = (4,4)$, $SU(2)$ SG-I and SG-II groups. They should come out as appropriate truncations of (3.1), (3.5), (3.6), (3.11) implemented through imposing certain constraints on the group parameters. The analytic harmonic vielbeins comprising the relevant shortened BW multiplets should then arise upon setting certain relations among the original analytic vielbeins, in a way covariant under the truncated SG group.

One obvious truncation of the original group and vielbeins is as follows:

$$\Lambda^{(2,0)} = \Lambda^{(0,2)} = \Lambda_L = \Lambda_R = 0 \ , \tag{4.3}$$

$$H^{(4,0)} = H^{(0,4)} = H^{(2,2)} = \tilde{H}^{(2,2)} = H^{(2,0)}_{L,R} = H^{(0,2)}_{L,R} = 0 \ . \tag{4.4}$$

The resulting group is the group of general analytic diffeomorphisms of the coordinates ζ^μ, with the inert harmonics

$$\delta\zeta^\mu = \Lambda^\mu(\zeta, u, v), \qquad \delta u = \delta v = 0 \ . \tag{4.5}$$

The corresponding covariant harmonic derivatives read

$$\nabla^{(2,0)} = D^{(2,0)} + H^{(2,0)\,\mu}\partial_\mu \ , \ \nabla^{(0,2)} = D^{(0,2)} + H^{(0,2)\,\mu}\partial_\mu \ . \tag{4.6}$$

The transformation laws of these derivatives and vielbeins, as well as the constraints the latter should satisfy, directly follow from those given in the previous Section, after taking into account the constraints (4.3), (4.4). Note that the harmonic derivatives now are inert,

$$\delta\,\nabla^{(2,0)} = \delta\,\nabla^{(0,2)} = 0 \ ,$$

and the integrability condition (3.18) becomes

$$[\nabla^{(2,0)}, \nabla^{(0,2)}] = 0 \quad \Rightarrow \tag{4.7}$$

$$\nabla^{(2,0)} H^{(0,2)\,++} - \nabla^{(0,2)} H^{(2,0)\,++} - 2iH^{(1,2)}\theta^{(1,0)} = 0 \ ,$$
$$\nabla^{(2,0)} H^{(0,2)--} - \nabla^{(0,2)} H^{(2,0)--} + 2iH^{(2,1)}\theta^{(0,1)} = 0 \ ,$$
$$\nabla^{(2,0)} H^{(1,2)\,\underline{i}} - \nabla^{(0,2)} H^{(3,0)\underline{i}} = 0 \ ,$$
$$\nabla^{(2,0)} H^{(0,3)\,\underline{a}} - \nabla^{(0,2)} H^{(2,1)\,\underline{a}} = 0 \ . \qquad (4.8)$$

Comparing the above truncated transformations with those of the first rigid superconformal $N = 4$, $SU(2)$ group (eqs. (2.11), (2.10)), one can suspect that the truncated SG group corresponds to gauging just this SCA-I. This is indeed the case. One can again choose the gauges

$$H^{(2,0)\,++} = H^{(0,2)\,--} = H^{(0,3)\,\underline{a}} = H^{(3,0)\,\underline{i}} \ , \qquad (4.9)$$

as in (3.20), and repeat all the steps which led us to the irreducible field representation (3.23) and the residual gauge freedom (3.24). For the truncated SG case we finally get, at the linearized level,

$$\begin{aligned} H^{(2,0)\,--} &= i(\theta^{(1,0)})^2\{h^{--}_{++} - 2i\theta^{(0,1)}_{\underline{a}} h^{-\,a\underline{a}}_{++} v^{(0,-1)}_a\} \ , \\ H^{(2,1)\,\underline{a}} &= i(\theta^{(1,0)})^2\{h^{-\,a\underline{a}}_{++} v^{(0,1)}_a + \theta^{(0,1)\,\underline{b}}[h^{(\underline{a}}_{++\,\underline{b})} + \frac{1}{2}\delta^{\underline{a}}_{\underline{b}}\,\partial_{--} h^{--}_{++}] \\ &\quad - i(\theta^{(0,1)})^2 \partial_{--} h^{-\,b\underline{a}}_{++} v^{(0,-1)}_b\} \ , \end{aligned} \qquad (4.10)$$

$$\begin{aligned} \Lambda^{--} &= \lambda^{--} - 2i\theta^{(0,1)}_{\underline{a}} \lambda^{-\,a\underline{a}} v^{(0,-1)}_a \ , \\ \Lambda^{(0,1)\,\underline{a}} &= \lambda^{-\,a\underline{a}} v^{(0,1)}_a + \theta^{(0,1)\,\underline{b}}[\lambda^{(\underline{a}}_{\underline{b})} + \frac{1}{2}\delta^{\underline{a}}_{\underline{b}}\,\partial_{--}\lambda^{--}] - i(\theta^{(0,1)})^2 \partial_{--}\lambda^{-\,a\underline{a}} v^{(0,-1)}_a \end{aligned} \qquad (4.11)$$

and analogous relations for the left vielbeins and parameters. We observe that the same can be obtained simply by setting

$$H^{(2,0)}_R = 0 \ , \ \ \Lambda_R = 0, \qquad H^{(0,2)}_L = 0 \ , \ \ \Lambda_L = 0$$

in the relations (3.23), (3.24) (and their left counterparts). Thus we end up with the BW multiplet h^{--}_{++}, h^{++}_{--}, $h^{-\,a\underline{a}}_{++}, h^{+\,i\underline{i}}_{--}$, $h^{(\underline{ab})}_{++}, h^{(\underline{ik})}_{--}$ the field content of which basically coincides with that of the $N = (4,4)$, $SU(2)$ gauge multiplet found by Schoutens [3] (a slight difference comes from the fact that, on the way to this field representation, we have already gauge-fixed some local symmetries with pure shifts in the relevant gauge parameters, in particular, the local $2D$ Lorentz and scale invariances by setting $h^{++}_{++} = h^{--}_{--} = 1$). The residual gauge group has the parameters $\lambda^{--}, \lambda^{++}$ (local translations), $\lambda^{-\,a\underline{a}}, \lambda^{+\,i\underline{i}}$ (local supertranslations), $\lambda^{(\underline{ab})}, \lambda^{(\underline{ik})}$ (right and left $SU(2)$ groups). The number of these gauge invariances coincides with that of the gauge fields, so that the $N = (4,4)$, $SU(2)$ BW multiplet (BW-I in what follows) contains no off-shell components like its parental $N = (4,4)$, $SO(4) \times U(1)$ BW multiplet. Once again, the maximal subgroup of (4.5) preserving the flat limit

$$H^{(2,0)\,\mu} = H^{(0,2)\,\mu} = 0$$

is just the $N = (4,4)$, $SU(2)$ SCA-I. It is singled out by imposing the light-cone chirality conditions on the parameters of the residual gauge group.

While specializing to the $N = (4,4)$, $SU(2)$ SG-I group, we may retain the standard defining constraint for the twisted superfield $q^{(1,1)}$,

$$\nabla^{(2,0)} q^{(1,1)} = \nabla^{(0,2)} q^{(1,1)} = 0 \tag{4.12}$$

(because of the commutativity property (4.7)), and the zero-weight scalar transformation rule

$$q^{(1,1)\prime}(\zeta', u, v) = q^{(1,1)}(\zeta, u, v) \ . \tag{4.13}$$

So, with respect to this SG-I group, $q^{(1,1)}$ is what is called TM-I in [7, 31] because its physical bosonic fields $q^{ia}(z)$ are not affected by the local $SU(2)$ symmetries (on the contrary, the auxiliary fields $F^{\underline{ia}}$ are transformed). Thus, the general rigidly supersymmetric $q^{(1,1)}$ action (2.19) can be straightforwardly extended to the locally supersymmetric one

$$S_q^I = \int \mu^{(-2,-2)} \, \hat{\Omega} \, \mathcal{L}^{(2,2)}(q^{(1,1)M}, u, v) \ , \tag{4.14}$$

where the density $\hat{\Omega}$ is still defined by eqs. (3.29), (3.31), with the truncation conditions (4.4) taken into account. In components and with the auxiliary fields eliminated, it gives the general locally supersymmetric $N = (4,4)$ sigma-model of ref. [4] which is a modification of the sigma-model action of ref. [2] by torsion terms in the sector of the physical bosons. For the rigid $q^{(1,1)}$ action (2.19), the general torsionful off-shell component action was presented in [14]. The action (4.14) yields a locally supersymmetric version of the latter. In Appendix we present, as an example, the component form of a very simple particular case of (4.14).

What about the second $N = (4,4)$, $SU(2)$ SCA, with respect to which $q^{(1,1)}$ is TM-II? How to extract the relevant $N = (4,4)$ SG group from the original "master" SG group? It is easy to answer these questions at the linearized level. The answer is prompted by the known realization of the $N = (4,4)$, $SU(2)$ SCA-II in the $SU(2) \times SU(2)$ harmonic analytic superspace [14, 23]. In order to have this SCA as the maximal symmetry after imposing the light-cone chiral constraints (4.1), one must seek for restrictions on the residual gauge superparameters (3.24) and their left counterparts such, that: i) the $U(1)$ parameters $\lambda_{L.R}$ are identified with $\partial_{\pm\pm}\lambda^{\pm\pm}$; ii) the affine $SU(2)$ parameters $\lambda^{(\underline{ab})}, \lambda^{(\underline{ik})}$ are eliminated. The unique possibility to obey these requirements, still leaving the "true" $SU(2)$ parameters $\lambda^{(ab)}, \lambda^{(ik)}$ unconstrained, is to impose the following relations:

$$\frac{\partial \Lambda^{(1,0)\,\underline{i}}}{\partial \theta^{(1,0)\,\underline{k}}} = \delta^{\underline{i}}_{\underline{k}} \, (\Lambda_L + \partial_{++}\Lambda^{++}) \ , \quad \frac{\partial \Lambda^{(0,1)\,\underline{a}}}{\partial \theta^{(0,1)\,\underline{b}}} = \delta^{\underline{a}}_{\underline{b}} \, (\Lambda_R + \partial_{--}\Lambda^{--}) \ , \tag{4.15}$$

whence

$$\begin{aligned} &\lambda^{(\underline{ab})} = \lambda^{(\underline{ik})} = 0 \ , \quad \lambda_L = -\frac{1}{2}\,\partial_{++}\lambda^{++} \ , \ \lambda_R = -\frac{1}{2}\,\partial_{--}\lambda^{--} \ , \\ &\beta_+^{i\underline{k}} = -2i\partial_{++}\lambda^{+\,i\underline{k}} \ , \quad \beta_-^{a\underline{b}} = -2i\partial_{--}\lambda^{-\,a\underline{b}} \ . \end{aligned} \tag{4.16}$$

It is easy to explicitly check that the superparameters (3.24) (and their left counterparts) restricted in this way indeed span the sought $N = (4,4)$, $SU(2)$ SCA-II after imposing the chirality conditions (4.1). Then, at the linearized level, it is a consistent truncation

to set equal to zero those combinations of the analytic vielbeins, which are not shifted under the subgroup singled out by eqs.(4.15):

$$\frac{\partial H^{(1.2)\,\underline{i}}}{\partial\theta^{(1,0)\,\underline{k}}} = \delta^{\underline{i}}_{\underline{k}}\,(\partial_{++}H^{(0,2)\,++} - H_L^{(0,2)})\ ,\ \frac{\partial H^{(2,1)\,\underline{a}}}{\partial\theta^{(0,1)\,\underline{b}}} = \delta^{\underline{a}}_{\underline{b}}\,(\partial_{--}H^{(2,0)\,--} - H_R^{(2,0)})\ . \quad (4.17)$$

These relations amount to the following linearized constraints on the gauge fields:

$$\begin{aligned} &h^{(\underline{ab})}_{++} = h^{(\underline{ik})}_{--} = 0\ ,\quad h_{--} = \frac{1}{2}\,\partial_{++}h^{++}_{--}\ ,\ h_{++} = \frac{1}{2}\,\partial_{--}h^{--}_{++}\ ,\\ &t^{\underline{ii}}_{--+} = 2i\partial_{++}h^{+\,\underline{ii}}_{--}\ ,\ \ t^{\underline{aa}}_{++-} = 2i\partial_{--}h^{-\,\underline{aa}}_{++}\ . \end{aligned} \quad (4.18)$$

They leave us with the representation h^{--}_{++}, h^{++}_{--}, $h^{+\,\underline{ik}}_{--}$, $h^{-\,\underline{ab}}_{++}$, $h^{(ik)}_{--}, h^{(ab)}_{++}$, which is again a $N=(4,4)$, $SU(2)$ BW multiplet, but with another chiral pair of $SU(2)$ gauge fields, compared to (4.10). We call it the $N=(4,4)$, $SU(2)$ BW-II multiplet. For completeness, we explicitly quote here the counterparts of (4.10), (4.11) for the considered case

$$\begin{aligned} H^{(2,0)\,--} &= i(\theta^{(1,0)})^2\{h^{--}_{++} - 2i\theta^{(0,1)}_{\underline{a}}h^{-\,\underline{aa}}_{++}v^{(0,-1)}_a - i(\theta^{(0,1)})^2h^{(ab)}_{++}v^{(0,-1)}_a v^{(0,-1)}_b\}\ ,\\ H^{(2,1)\,\underline{a}} &= i(\theta^{(1,0)})^2\{h^{-\,\underline{aa}}_{++}v^{(0,1)}_a + \frac{1}{2}\,\theta^{(0,1)\,\underline{a}}(\partial_{--}h^{--}_{++} - 2h^{(ab)}_{++}v^{(0,1)}_a v^{(0,-1)}_b)\}\ ,\\ H^{(2,0)}_R &= i(\theta^{(1,0)})^2\{\frac{1}{2}\,\partial_{--}h^{--}_{++} + h^{(ab)}_{++}v^{(0,1)}_a v^{(0,-1)}_b - 2i\,\theta^{(0,1)}_{\underline{a}}\partial_{--}h^{-\,\underline{ba}}_{++}\\ &\quad - i(\theta^{(0,1)})^2\partial_{--}h^{(ab)}_{++}v^{(0,-1)}_a v^{(0,-1)}_b\}\ , \end{aligned} \quad (4.19)$$

$$\begin{aligned} \Lambda^{--} &= \lambda^{--} - 2i\theta^{(0,1)}_{\underline{a}}\lambda^{-\,\underline{aa}}v^{(0,-1)}_a + i(\theta^{(0,1)})^2\lambda^{(ab)}v^{(0,-1)}_a v^{(0,-1)}_b\ ,\\ \Lambda^{(0,1)\,\underline{a}} &= \lambda^{-\,\underline{aa}}v^{(0,1)}_a + \frac{1}{2}\,\theta^{(0,1)\,\underline{a}}(\partial_{--}\lambda^{--} + 2\lambda^{(ab)}v^{(0,1)}_a v^{(0,-1)}_b)\ ,\\ \Lambda_R &= -\frac{1}{2}\,\partial_{--}\lambda^{--} + \lambda^{(ab)}v^{(0,-1)}_a v^{(0,1)}_b + 2i\,\theta^{(0,1)}_{\underline{a}}\partial_{--}\lambda^{-\,\underline{aa}}v^{(0,-1)}_a\\ &\quad - i(\theta^{(0,1)})^2\partial_{--}\lambda^{(ab)}v^{(0,-1)}_a v^{(0,-1)}_b\ . \end{aligned} \quad (4.20)$$

The left objects are obtained via the same substitutions as in the previous cases.

For the time being, we do not know how to go beyond the linearized level in this important case. It seems that it is more fruitful to descend to the above shortened versions of the BW multiplets (and further to the Poincaré SG), using a more convenient approach based on the concept of superconformal compensation.

5 Superconformal matter couplings

The basic idea of the compensation approach (see, e.g., [34]) is to start from the pure superconformal SG and then to couple to it, in a superconformally covariant way, appropriate matter multiplets with inhomogeneous (Goldstone type) transformation laws with respect to certain (super)conformal symmetries. Then, by properly fixing gauges (normally, in such a way that all inhomogeneously transforming components are fully gauged away), one gets as a net result the theory with a smaller number of local symmetries

and supersymmetries, i.e. a sort of Poincaré SG. The auxiliary fields of the compensating superfield become in this gauge auxiliary fields of the relevant Poincaré SG gauge multiplet. If, from the beginning, a few matter superfields coupled to a given conformal SG are included, being one of them a compensator, we end up with the theory of the remaining matter multiplets in a Poincaré SG background. In this way, one can derive various Poincaré-type supergravities (with all, or a part of, the original conformal symmetries compensated for), different off-shell SG multiplets (depending on the choice of compensator), etc.

We believe that the $N = (4,4)$, $SO(4) \times U(1)$ SG group defined in Sect. 3 is the maximal, "master" $N = (4,4)$, $2D$ conformal SG group. Then, the relevant gauge multiplet, $N = (4,4)$, $SO(4) \times U(1)$ BW multiplet, is the "master" multiplet from which all other known $N = (4,4)$ SG multiplets should follow by the appropriate compensating procedure. To list all possibilities, we need to know all possible superconformal rigid off-shell matter multiplets which can be defined in $SU(2) \times SU(2)$ harmonic superspace, their off-shell actions, and the locally superconformal extensions of the latter. As it was already noticed earlier, not all known types of twisted superfields (and their variant representations) admit a simple formulation in $SU(2) \times SU(2)$ analytic harmonic superspace [32]. There also exists some other off-shell representation with finite number of auxiliary fields, whose $SU(2) \times SU(2)$ harmonic superspace description (if existing) is not known, as yet. In what follows, we shall deal with the superconformal off-shell matter multiplets which admit a description in terms of analytic $SU(2) \times SU(2)$ harmonic superfields and which were reviewed in Sect. 2. These are the nonlinear multiplets $N^{(2,0)}, N^{(0,2)}$, $G^{(2,0)}, G^{(0,2)}$ and the twisted chiral multiplets $q^{(1,1)}$ which can be either TM-I or TM-II, depending on the superconformal $N = (4,4)$, $SU(2)$ group with respect to which one studies their transformation properties. We shall show that some of these superfields can be used to compensate the "master" $N = (4,4)$ conformal SG group down to its $N = (4,4)$, $SU(2)$ subgroups and, further, to the Poincaré SG groups, including the group of minimal off-shell SG of refs. [5, 6].

We start with a local extension of the set $N^{(2,0)}, N^{(0,2)}$. The rigid superconformal transformation laws of this multiplet (2.22) naturally generalize to the whole "master" $N = (4,4)$ SG group as

$$\delta N^{(2,0)} = \Lambda^{(2,0)} \ , \quad \delta N^{(0,2)} = \Lambda^{(0,2)} \ , \tag{5.1}$$

where the transformation parameters are now the general analytic superfunctions introduced in (3.1). The defining constraints (2.21) are covariantized as follows:

$$\begin{aligned} &(a) \quad \nabla^{(2,0)} N^{(2,0)} + N^{(2,0)} N^{(2,0)} = H^{(4,0)} \ , \ \nabla^{(0,2)} N^{(0,2)} + N^{(0,2)} N^{(0,2)} = H^{(0,4)} \ , \\ &(b) \quad \nabla^{(2,0)} N^{(0,2)} - \nabla^{(0,2)} N^{(2,0)} = H^{(2,2)} - \tilde{H}^{(2,2)} \end{aligned} \tag{5.2}$$

(for a similar covariantization of the standard nonlinear multiplet in the conventional harmonic superspace, see [17, 23]). It is obvious that the N-multiplet can be used to fully compensate all gauge invariances contained in $\Lambda^{(2,0)}, \Lambda^{(0,2)}$, including two chiral $SU(2)$ symmetries acting on the harmonic variables. One can achieve this purpose, choosing the gauge

$$N^{(2,0)} = N^{(0,2)} = 0 \quad \Rightarrow \quad (a)\, H^{(0,4)} = H^{(4,0)} = 0 \ , \ \ (b)\, H^{(2,2)} - \tilde{H}^{(2,2)} = 0 \ . \tag{5.3}$$

Prior to any gauge-fixing, it is instructive to fully elaborate on the corollaries of the constraints (5.2). For the quantities

$$Q^{(2,0)} \equiv N^{(2,0)} - H_L^{(2,0)} - H_R^{(2,0)} \ , \quad Q^{(0,2)} \equiv N^{(0,2)} - H_L^{(0,2)} - H_R^{(0,2)} \ , \tag{5.4}$$

eq. (5.2*b*), combined with eqs. (3.17) implies the following constraint:

$$\nabla^{(2,0)} Q^{(0,2)} - \nabla^{(0,2)} Q^{(2,0)} = 0 \quad \Rightarrow \quad Q^{(2,0)} = \nabla^{(2,0)} \Phi \ , \ Q^{(0,2)} = \nabla^{(0,2)} \Phi \ , \tag{5.5}$$

where $\Phi = \Phi(\zeta, u, v)$ is a new analytic compensating superfield. Recalling the transformation properties (3.11), (5.1), we see that

$$\delta Q^{(2,0)} = \nabla^{(2,0)} (\Lambda_L + \Lambda_R) \ , \ \ \delta Q^{(0,2)} = \nabla^{(0,2)} (\Lambda_L + \Lambda_R) \ \ \Rightarrow \ \delta \Phi = \Lambda_L + \Lambda_R \ . \tag{5.6}$$

Hence, the newly introduced analytic object Φ can be fully gauged away using the analytic gauge parameter $\Lambda_L + \Lambda_R$

$$\Phi = 0 \quad \Rightarrow \quad \Lambda_L = -\Lambda_R \equiv \Lambda \ . \tag{5.7}$$

As a corollary of this choice, the following relations occur:

$$Q^{(2,0)} = Q^{(0,2)} = 0 \quad \Rightarrow \quad N^{(2,0)} = H_L^{(2,0)} + H_R^{(2,0)} \ , \ \ N^{(0,2)} = H_L^{(0,2)} + H_R^{(0,2)} \ . \tag{5.8}$$

At this stage, it is time to fix the gauge freedom associated with the superparameters $\Lambda^{(2,0)}, \Lambda^{(0,2)}$, by imposing the gauge (5.3). As a result of this gauge choice, the original "master" $N = (4,4)$ SG group (3.1) proves to be compensated just down to its $N = (4,4)$, $SU(2)$ SG-I subgroup (4.5). Eqs. (5.8), in this gauge, imply

$$H_L^{(2,0)} = -H_R^{(2,0)} \equiv H^{(2,0)} \ , \quad H_L^{(0,2)} = -H_R^{(0,2)} \equiv H^{(0,2)} \ , \tag{5.9}$$

$$\delta H^{(2,0)} = \nabla^{(2,0)} \Lambda \ , \ \delta H^{(0,2)} = \nabla^{(0,2)} \Lambda \ . \tag{5.10}$$

As a consequence of these relations and the gauge choice (5.7), the transformation law (3.7) of the twisted multiplet $q^{(1,1)}$ in the "master" SG group, as well as its defining constraints (3.13), are reduced to those covariant under the $N = (4,4)$, $SU(2)$ SG group, i.e. (4.13) and (4.12). Nevertheless, the resulting theory is not yet identical to what we have got after truncation in Sect. 4. Indeed, the gauge-fixed covariant derivatives $\nabla^{(2,0)}, \nabla^{(0,2)}$ differ from those defined by eq. (4.6)

$$\nabla^{(2,0)} = D^{(2,0)} + H^{(2,0)\,\mu} \partial_\mu + H^{(2,2)} \partial^{(0,-2)} \ ,$$

$$\nabla^{(0,2)} = D^{(0,2)} + H^{(0,2)\,\mu} \partial_\mu + H^{(2,2)} \partial^{(-2,0)} \ , \tag{5.11}$$

$$H^{(2,2)} = \nabla^{(2,0)} H^{(0,2)} - \nabla^{(0,2)} H^{(2,0)} \ . \tag{5.12}$$

Though $H^{(2,2)}$ as well as $H^{(2,0)}$, $H^{(0,2)}$ transform as scalars under the remaining $N = (4,4)$, $SU(2)$ SG-I group, the harmonic partial derivatives $\partial^{(0,-2)}, \partial^{(-2,0)}$ are not covariant, due to the presence of a non-trivial u, v dependence in the group parameters in (4.5). As a result, $H^{(2,2)}$ appears in the transformation laws of the vielbeins $H^{(2,0)\,\mu}, H^{(0,2)\,\mu}$. The constraints (3.19) also do not go into the set (4.8), due to the presence of $H^{(2,2)}$. For

this object, the original constraints (3.19) imply the following ones (recall that $H^{(4,0)} = H^{(0,4)} = 0$ in the gauge (5.3)):

$$\nabla^{(2,0)} H^{(2,2)} = \nabla^{(0,2)} H^{(2,2)} = 0 \ . \tag{5.13}$$

This peculiarity comes out only at the nonlinear level. The linearized analysis goes as before and shows that $H^{(2,0)\,\mu}, H^{(0,2)\,\mu}$, in the present case, carry the same set of fields forming the $N = (4,4),\ SU(2)$ BW-I multiplet. In other words, after fixing appropriate conformal gauges in the locally superconformal system of the original "master" BW multiplet and the compensator multiplet $N^{(2,0)}, N^{(0,2)}$, we are left with a smaller $N = (4,4),\ SU(2)$ BW-I multiplet and an extra off-shell multiplet. The latter is carried by the superfields $H^{(2,0)}, H^{(0,2)}$ which exhibit the gauge freedom (5.10) with an extra analytic gauge parameter $\Lambda(\zeta, u, v)$ and satisfy the constraints (5.13). This extended representation is not fully reducible, in the sense that the additional gauge superfields $H^{(2,0)}, H^{(0,2)}$ are scalars with respect to the conformal $N = (4,4),\ SU(2)$ SG-I group (4.5) while the SG-I transformation laws of the analytic vielbeins $H^{(2,0)\,\mu}, H^{(0,2)\,\mu}$ include these extra superfields.

Thus, we have found the previously unknown off-shell $N = (4,4)$ SG gauge multiplet. In the WZ gauge and at the linearized level, its part coming from the analytic vielbeins is the same BW-I gauge fields representation which was described in Sect. 4 and which has no off-shell degrees of freedom (the linearized structure (4.10) in this case is slightly modified by the fields from $H^{(2,0)}, H^{(0,2)}$, because of the presence of $H^{(2,2)}$ in the constraints on $H^{(0,2)\,\mu}, H^{(2,0)\,\mu}$). To examine the off-shell content of $H^{(2,0)}, H^{(0,2)}$, we have chosen an appropriate WZ gauge with respect to the parameter $\Lambda(\zeta, u, v)$, so as to kill as much component fields in the $\theta,\ u, v$ expansions of these superfields as possible, and inserted the result into the linearized form of the constraints (5.13). Solving the latter (it does not put any field on shell), we have eventually found $(32 + 32)$ independent off-shell components listed below (the numerals in the parentheses on the right to the fields denote the "engineering" dimension and the number of independent real components, respectively):

$$\begin{aligned} &\underline{\text{bosons}}: \quad (h_{++}, h_{--})\ (1,1)\ ,\ \ l^{(ab)}_{++}\ (1,3)\ ,\ \ l^{(ik)}_{--}\ (1,3)\ ,\ \ l^{ia}_{\underline{ia}}\ (1,16)\ ,\ \ l^{(ik)(ab)}\ (0,9)\ , \\ &\underline{\text{fermions}}: \quad l^{b}_{\underline{a}\,+}\ (3/2,4)\ ,\ \ l^{i}_{\underline{k}\,-}\ (3/2,4)\ ,\ \ l^{(ik)\,a}_{\underline{b}}\ (1/2,12)\ ,\ \ l^{(ab)\,i}_{\underline{k}}\ (1/2,12)\ . \end{aligned} \tag{5.14}$$

The fields $h_{\pm\pm}$ are gauge fields for a $U(1)$ with the gauge parameter $\lambda(z)$ which is the first component in $\Lambda(\zeta, u, v)$. This $U(1)$ is the only residual gauge symmetry of the given WZ gauge. The fields $l^{(ab)}_{++}$, $l^{(ik)}_{--}$ and $l^{b}_{\underline{a}\,+}$, $l^{i}_{\underline{k}\,+}$ are "former" gauge fields for the symmetries with the parameters $\lambda^{(ab)}, \lambda^{(ik)}$ and $\beta^{b}_{\underline{a}\,-}$, $\beta^{i}_{\underline{k}\,+}$ in the "master" BW multiplet (eqs. (3.23), (3.24) and their left counterparts). Now these local symmetries have been entirely compensated by the appropriate compensating fields from $N^{(2,0)}, N^{(0,2)}$. Note that the residual gauge group $U(1)$ is the diagonal in the product of two chiral gauge $U(1)$ groups realized on the "master" BW multiplet; the rest of these $U(1)$ symmetries has been compensated by a dimension-0, $SO(4)$ singlet field present in $N^{(2,0)}, N^{(0,2)}$ [23] (this is just the first component of the compensator Φ introduced in (5.5)). The biggest flat limit symmetry of the extended gauge multiplet ($N = (4,4),\ SU(2)$ BW-I together with (5.14)) is $N = (4,4),\ SU(2)$ SCA-I augmented with an extra rigid $U(1)$ symmetry. Note that in

the matter couplings we shall discuss in this Section, the superfields $H^{(2,0)}, H^{(0,2)}$ always appear only through their analytic superfield strength $H^{(2,2)}$ containing, in particular, the field strength of the $U(1)$ gauge field $h_{\pm\pm}$. In other words, the residual local $U(1)$ group is hidden, and for the time being we do not see in which situations it could become active.

A comment is to the point here. In principle, we could completely eliminate the extra multiplet by treating it as pure gauge. This possibility corresponds to adding additional constraints to the set (5.2)

$$H^{(2,2)} = \nabla^{(2,0)} N^{(0,2)} \ , \quad \tilde{H}^{(2,2)} = \nabla^{(0,2)} N^{(2,0)} \ . \tag{5.15}$$

These constraints are manifestly covariant and compatible both with (5.2) and (3.19).The same reasoning which led us to eqs. (5.5), (5.9) implies that in the gauge (5.3) the pairs $H_L^{(2,0)}, H_L^{(0,2)}$ and $H_R^{(2,0)}, H_R^{(0,2)}$ become pure gauge, with respect to the $U(1)$ gauge groups with parameters Λ_L and Λ_R. Hence, they can be gauged away, fully compensating this gauge freedom. As the result, the $N = (4,4)$, $SU(2)$ SG-I group and the BW-I multiplet are finally reproduced. A deviation from the standard compensation point of view is that, after imposing (5.15), the compensators $N^{(2,0)}, N^{(0,2)}$ cease to have a flat off-shell limit (when all vielbeins are put equal to zero): the resulting modified set of constraints proves to be too restrictive, it puts these superfield on shell [23]. On the other hand, one can view the relations (5.2) and (5.15) merely as the covariant definition of particular harmonic vielbeins $H^{(0,4)}, H^{(4,0)}, H^{(2,2)}, \tilde{H}^{(2,2)}$, such that it provides a covariant way to make some gauge fields in the "master" BW multiplet purely longitudinal and, so, globally removable by fixing appropriate gauges. Indeed, from the standpoint of the linearized WZ representation (3.23) for the "master" BW multiplet, these relations mean that all gauge fields except those comprising the $N = (4,4)$, $SU(2)$ BW-I multiplet are postulated to be pure gauge.

Let us now turn to the issue of constructing matter actions invariant under the "master" conformal SG group.

We start by seeking for the appropriate generalization of the N-action (2.23). Somewhat surprisingly, it cannot be straightforwardly promoted to an invariant of the local superconformal group. The best we have reached, in our attempts to covariantize (2.23), is the action

$$S_N^{loc} = -\int \mu^{(-2,-2)}\, \Omega \left(Q^{(2,0)} Q^{(0,2)} + 2 Q^{(2,0)} H_L^{(0,2)} + 2 Q^{(0,2)} H_R^{(2,0)} + 2 H_L^{(0,2)} H_R^{(2,0)} \right) , \tag{5.16}$$

where $Q^{(2,0)}, Q^{(0,2)}$ are defined in (5.4). It is shifted, up to surface terms, by the expression

$$\delta\, S_N^{loc} = -2 \int \mu^{(-2,-2)}\, \Omega \ \left(H_L^{(0,2)} \nabla^{(2,0)} \Lambda_L + H_R^{(2,0)} \nabla^{(0,2)} \Lambda_R \right) \ , \tag{5.17}$$

which cannot be further cancelled in any way.

On the other hand, it is possible to construct invariant actions for the second type of superconformally invariant (32+32) nonlinear multiplet defined by (2.25), (2.26). The constraints (2.26) admit a direct covariantization

$$\begin{aligned} &(\nabla^{(2,0)} + 2N^{(2,0)}) G^{(2,0)} + \alpha\, G^{(2,0)} G^{(2,0)} = 0 \ , \\ &(\nabla^{(0,2)} + 2N^{(0,2)}) G^{(0,2)} + \alpha\, G^{(0,2)} G^{(0,2)} = 0 \ , \\ &\nabla^{(2,0)} G^{(0,2)} - \nabla^{(0,2)} G^{(2,0)} = 0 \ . \end{aligned} \tag{5.18}$$

Indeed, it is easy to check their covariance under (3.1) provided that the superfields G transform as scalars: $\delta\, G^{(2,0)} = \delta\, G^{(0,2)} = 0$. Then the simplest manifestly invariant action of $G^{(2,0)}, G^{(0,2)}$ in the background of the $N = (4,4)$ "master" conformal SG fields and compensators $N^{(2,0)}, N^{(0,2)}$ is given by

$$S_G^{loc} = -\int \mu^{(-2,-2)}\, \Omega\, G^{(2,0)} G^{(0,2)} \ . \tag{5.19}$$

Another possibility to construct an invariant off-shell action for the pair of compensators $N^{(2,0)}, N^{(0,2)}$ is to take as the relevant Lagrangian density the constraints (5.2) with the appropriate analytic Lagrange multipliers $\omega^{(-2,2)}$, $\omega^{(2,-2)}$, ω. Just an action of this kind describes the standard nonlinear multiplet coupled to a conformal $N = 2,\ 4D$ SG in the conventional harmonic superspace [16, 17]. Its $SU(2) \times SU(2)$ analogue would also have no propagating degrees of freedom and, before varying with respect to Lagrange multipliers, contain an infinite number of auxiliary fields. This possibility requires a thorough analysis and we postpone discussing it to the future.

It is worth noting that there are no problems with extending the flat superspace actions of $N^{(2,0)}, N^{(0,2)}$ and $G^{(2,0)}, G^{(0,2)}$ to invariants of the $N = (4,4)$ SG-I group. The relevant constraints are obtained from the flat ones (2.21), (2.26) by the replacements $D^{(2,0)}, D^{(0,2)} \ \to\ \nabla^{(2,0)}, \nabla^{(0,2)}$, where $\nabla^{(2,0)}, \nabla^{(0,2)}$ are given by eqs. (4.6), and the locally supersymmetric actions are obtained via the replacement $\mu^{(-2,-2)} \ \to\ \mu^{(-2,-2)}\, \hat{\Omega}$ in the flat superspace ones.

Let us now switch over to the twisted multiplets. We already constructed in Sect. 4 a locally supersymmetric $q^{(1,1)}$ action (4.14) invariant under the $N = (4,4),\ SU(2)$ SG-I group. An important question is how to construct the $q^{(1,1)}$ actions invariant under the full "master" $N = (4,4)$ group (3.1). The main difficulty here is related to the non-trivial transformation law (3.7) of $q^{(1,1)}$ in this group.

The simplest way to construct such a coupling is to consider $q^{(1,1)}$ together with the compensators $N^{(2,0)}, N^{(0,2)}$. In this case, due to the existence of the analytic scalar compensator Φ which is shifted by the sum $\Lambda_L + \Lambda_R$ (eq. (5.5)), one can redefine any $q^{(1,1)}$ with the transformation law (3.7) in such a way that it will transform as a scalar under the "master" SG group

$$q^{(1,1)}(\zeta, u, v) \quad \Rightarrow \quad \tilde{q}^{(1,1)} = e^{\Phi}\, q^{(1,1)} \ , \ \ \tilde{q}^{(1,1)\prime}(\zeta', u', v') = \tilde{q}^{(1,1)}(z, u, v) \ . \tag{5.20}$$

The constraints (3.13) become

$$(\nabla^{(2,0)} + N^{(2,0)})\tilde{q}^{(1,1)} = 0 \ , \quad (\nabla^{(0,2)} + N^{(0,2)})\tilde{q}^{(1,1)} = 0 \ . \tag{5.21}$$

Their covariance is evident. The general invariant action is similar to (4.14)

$$\tilde{S}_q^I = \int \mu^{(-2,-2)}\ \Omega\ \mathcal{L}^{(2,2)}(\tilde{q}^{(1,1)}, \tilde{u}, \tilde{v}) \ , \tag{5.22}$$

where [5]

$$\begin{aligned} &\tilde{u}^{(1,0)} = u^{(1,0)} - N^{(2,0)} u^{(-1,0)} \ , \quad \tilde{u}^{(-1,0)} = u^{(-1,0)} \ , \\ &\tilde{v}^{(0,1)} = v^{(0,1)} - N^{(0,2)} v^{(0,-1)} \ , \quad \tilde{v}^{(-1,0)} = v^{(0,-1)} \ . \end{aligned} \tag{5.23}$$

[5] Within the conventional harmonic superspace, the necessity of analogous redefinitions of the harmonics explicitly appearing in the action of hypermultiplets coupled to conformal $N = 2,\ 4D$ SG was firstly shown in [17].

In the gauge (5.3) the action (5.22) coincides with (4.14) modulo a modification of both the covariant harmonic derivatives and the constraints on the analytic vielbeins due to the presence of the $U(1)$ gauge multiplet $H^{(2,0)}, H^{(0,2)}$. The effect of this modification is two-fold: first, the constraints defining $q^{(1,1)}$ are obscured by this extra multiplet and, second, the density Ω differs from $\tilde{\Omega}$ in (4.14) owing to the presence of the extra multiplet in the constraints for the analytic vielbeins. It would be interesting to see what is the precise impact of this modification on the component sigma-model action as compared to the action (4.14) which includes the $N = (4,4)$, $SU(2)$ BW-I multiplet without any additional SG fields.

Since there exists the unique $N = (4,4)$ WZW $q^{(1,1)}$ action (2.17) invariant under the full rigid $N = (4,4)$, $SO(4) \times U(1)$ superconformal symmetry, it is natural to seek for its direct coupling to the "master" $N = (4,4)$ BW multiplet without adding any extra compensators. If such a coupling can be set up, $q^{(1,1)}$ can be regarded, like $N^{(2,0)}, N^{(0,2)}$, as a compensator extending the master BW multiplet to some SG multiplet with a smaller number of gauge symmetries and gauge fields. The corresponding SG group should be some subgroup of the master $N = (4,4)$ SG group. Indeed, the shifted superfield $\hat{q}^{(1,1)}$ defined in (2.18) transforms inhomogeneously under (3.1), (3.7)

$$\delta\, \hat{q}^{(1,1)} = (\Lambda_L + \Lambda_R)(\hat{q}^{(1,1)} + c^{(1,1)}) - \Lambda^{(2,0)} c^{(-1,1)} - \Lambda^{(0,2)} c^{(1,-1)} \ , \tag{5.24}$$

and hence it can be employed as a compensator.

Unfortunately, we do not have yet any general recipe how to construct such a locally supersymmetric extension of (2.17). The main difficulty stems from the fact that the analytic superfield density in (2.17) is not a tensor: it is shifted by full harmonic derivatives under the rigid superconformal $SO(4) \times U(1)$ transformation. The most straightforward approach is to restore the full action order by order in the SG superfields, and this is what we shall undertake.

First, we make the replacement

$$\mu^{(-2,-2)} \quad \Rightarrow \quad \mu^{(-2,-2)}\, \Omega$$

in (2.17) in order to be able to integrate by parts with respect to $\nabla^{(2,0)}$, $\nabla^{(0,2)}$ (recall the discussion at the end of Sect. 3). We do not fix beforehand any gauges including (3.14). Thus we represent the sought S^{loc}_{wzw} as a series in powers of the SG superfields

$$S^{loc}_{wzw} = S_{(0)} + S_{(1)} + S_{(2)} + ... = -\frac{1}{4\gamma^2} \int \mu^{(-2,-2)}\, \Omega\, [\mathcal{L}^{(2,2)}_{(0)} + \mathcal{L}^{(2,2)}_{(1)} + \mathcal{L}^{(2,2)}_{(2)} + ...] \ , \tag{5.25}$$

where $\mathcal{L}^{(2,2)}_{(0)}$ is just the density in (2.17). Then, using the formula

$$\delta\, S_{(0)} = \frac{1}{4\gamma^2} \int \mu^{(-2,-2)}\, \Omega \left(\hat{q}^{(1,1)} \delta\, \hat{q}^{(1,1)}\, \frac{1}{(1+X)^2} \right) \ ,$$

it is rather straightforward to restore the first correction term in (5.25):

$$\begin{aligned} S_{(1)} \ = \ & \frac{1}{4\gamma^2} \int \mu^{(-2,-2)}\, \Omega \left(\hat{q}^{(1,1)}\, \frac{1}{(1+X)^2} \left[c^{(-1,1)} H^{(2,0)}_L + c^{(1,-1)} H^{(0,2)}_R \right.\right. \\ & \left.\left. - \, (c^{(1,-1)} H^{(0,2)}_L + c^{(-1,1)} H^{(2,0)}_R)(2+X) \right] \right) \ . \end{aligned} \tag{5.26}$$

A problem is met at the next step, when trying to calculate the second term. Including from the beginning all possible appropriate structures, we finally found that almost all structures appearing in the first-order variation of $S_{(0)} + S_{(1)}$ can be cancelled by the zeroth-order variation of $S_{(2)}$. Only one term cannot be cancelled. It looks just the same as the term (5.17) appearing in the variation of the non-invariant $N^{(2,0)}, N^{(0,2)}$ action (5.16)

$$\delta\left(S_{(0)} + S_{(1)} + S_{(2)}\right) = -\frac{1}{4\gamma^2}\int \mu^{(-2,-2)}\,\Omega\,\left[2H_L^{(0,2)}\nabla^{(2,0)}\Lambda_L + 2H_R^{(2,0)}\nabla^{(0,2)}\Lambda_R\right]\ . \quad (5.27)$$

The origin of this anomaly can be inferred from the results of ref. [35] where the problem of gauging isometries of bosonic sigma models with torsion was studied. As was shown there, in the case of group manifold WZW model associated with a group G it is impossible to construct an action in which the *full* $G \times G$ symmetry of the rigid WZW action would be gauged (without adding extra copies of WZW fields). One can only gauge either the left, or right, or diagonal subgroups of $G \times G$. The bosonic sector of the above rigid $q^{(1,1)}$ action is just the $SU(2)_L \times SU(2)_R/SU(2)_{diag}$ WZW action, while the "master" $N = (4,4)$ SG group implies gauging both $SU(2)_L$ and $SU(2)_R$ symmetries. Thus, in view of the argument just adduced, a direct coupling of the WZW $q^{(1,1)}$ action (2.17) to the "master" BW multiplet does not exist and the "classical anomaly" (5.27) is just a manifestation of this fact. The unremovable piece in the gauge variation (5.17) is of the same origin, because the $N^{(2,0)}, N^{(0,2)}$ action (2.23) also contains the $SU(2)_L \times SU(2)_R$ WZW model in its bosonic sector. The same reasoning implies the non-existence of similar straightforward $N = (4,4)$ SG-II group-invariant extensions of (2.17), (2.23), since this SG group still includes gauge $SU(2)_L, SU(2)_R$ symmetries which act on the physical bosons of $q^{(1,1)}$ ($SU(2)$ WZW fields). Note that no problems of this sort arise while promoting (2.17) to an invariant of the $N = (4,4)$ SG-I gauge group, or to that of the "master" SG group with making use of the $N^{(2,0)}, N^{(0,2)}$ compensators at the intermediate step: such locally supersymmetric $q^{(1,1)}$ actions are particular cases of (4.14), (5.22).

Thus the construction of direct couplings of $N = (4,4)$ WZW action (2.17) or the $N^{(2,0)}, N^{(0,2)}$ action (2.23) to the "master" conformal $N = (4,4)$ SG or $N = (4,4)$ SG-II is a non-trivial problem. It seems that the unique possibility to arrange such couplings is to consider a few copies of the superfields $q^{(1,1)}$, $N^{(2,0)}, N^{(0,2)}$. Then one can construct invariant actions as sums of the individual actions of the type (5.16), (5.25), taking some of them with the wrong sign so as to cancel out the non-vanishing variations like (5.17), (5.27) coming from different actions. This is possible just because these anomalous variations involve only SG gauge fields.

The simplest possibility is to consider a pair of nonlinear multiplets, $N_1^{(2,0)}, N_1^{(0,2)}$ and $N_2^{(2,0)}, N_2^{(0,2)}$, each set being subjected to the constraints (5.2). Then the difference of two actions (5.16)

$$S^{loc}_{N_1 N_2} = S^{loc}_{N_1} - S^{loc}_{N_2} = -\int \mu^{(-2,-2)}\,\Omega\,\Big[N_1^{(2,0)}N_1^{(0,2)} - N_2^{(2,0)}N_2^{(0,2)}$$
$$+ (N_1^{(2,0)} - N_2^{(2,0)})(H_L^{(0,2)} - H_R^{(0,2)}) - (N_1^{(0,2)} - N_2^{(0,2)})(H_L^{(2,0)} - H_R^{(2,0)})\Big] \quad (5.28)$$

can be easily checked to be invariant under the "master" SG group. Each of these multiplets, or their sum $N_1^{(2,0)} + N_2^{(2,0)}, N_1^{(0,2)} + N_2^{(0,2)}$ can be chosen as compensators reducing

the "master" SG group to SG-I group via gauge-fixings like (5.3) (and, further, (5.7)). Note that the gauge-invariant combinations $\tilde{N}^{(2,0)} = N_1^{(2,0)} - N_2^{(2,0)}, \tilde{N}^{(0,2)} = N_1^{(0,2)} - N_2^{(0,2)}$ obey the constraints

$$\left[\nabla^{(2,0)} + (N_1^{(2,0)} + N_2^{(2,0)})\right] \tilde{N}^{(2,0)} = 0\,, \ \left[\nabla^{(0,2)} + (N_1^{(0,2)} + N_2^{(0,2)})\right] \tilde{N}^{(0,2)} = 0\,,$$
$$\nabla^{(2,0)}\tilde{N}^{(0,2)} - \nabla^{(0,2)}\tilde{N}^{(2,0)} = 0\ , \tag{5.29}$$

which are recognized as the $\alpha = 0$ version of (5.18). So one can from the beginning add the invariant piece

$$\sim\ \tilde{N}^{(2,0)}\tilde{N}^{(0,2)} \tag{5.30}$$

to the Lagrangian density in (5.28). Finally, e.g., in the gauges

$$N_1^{(0,2)} + N_2^{(0,2)} = N_1^{(2,0)} + N_2^{(2,0)} = 0$$

and (5.7), we are left with the action of the (32 + 32) matter multiplet $\tilde{N}^{(2,0)}, \tilde{N}^{(0,2)}$ in the background of the BW-I multiplet augmented with the $U(1)$ multiplet (5.14). The invariant couplings of an arbitrary number of $q^{(1,1)}$ multiplets can be arranged with the help of the compensator $N_1^{(2,0)} + N_2^{(2,0)}, N_1^{(0,2)} + N_2^{(0,2)}$ as explained above, and added to the N_1, N_2 action.

Another possibility to set up a direct coupling to the "master" $N = (4,4)$ conformal SG is to take the difference of the "almost-covariant" N -superfields action (5.16) and the $q^{(1,1)}$ action (5.25)

$$S^{loc}_{qN} = S^{loc}_{wzw} - \frac{1}{4\gamma^2}\, S^{loc}_{N}\ . \tag{5.31}$$

In analogy with the $N^{(2,0)}, N^{(0,2)}$ action (5.16), it is natural to assume that the only unremovable term in the "master" SG group variation of (5.25) is given by (5.27), while all higher-order variations can be cancelled by inserting into the Lagrangian density the appropriate higher-order structures composed out of the analytic vielbeins and the superfield $q^{(1,1)}$. This still has to be proved (it would be desirable to find out the geometric principle behind such a recursion procedure). [6] If such an "almost-covariant" $q^{(1,1)}$ action exists, the action (5.31) is invariant like (5.28), and in the gauges (5.3), (5.7) we arrive at the SG-I group-invariant action of the (8+8) multiplet $q^{(1,1)}$ in the background consisting of the BW-I gauge multiplet and the extra multiplet (5.14). Adding other $q^{(1,1)}$ superfields in a way covariant under "master" SG group can be accomplished, like in the previous case, with making use of the compensators $N^{(2,0)}, N^{(0,2)}$.

An essentially new situation comes out if one directly couples $q^{(1,1)}$ superfields to BW multiplet, without using N-superfields. For this purpose one should take at least two different WZW $q^{(1,1)}$ superfields with the same transformation law (5.24) (although with different sets of constants c^{ia}, generally speaking). Under the assumptions that the "almost-covariant" $q^{(1,1)}$ action S^{loc}_{wzw} exists to all orders in SG fields and that its non-invariance is given only by the variation (5.27), the fully invariant action could be constructed as

$$S^{loc}_{q_1q_2} = S^{loc}_{wzw_1} - S^{loc}_{wzw_2}\ . \tag{5.32}$$

[6] The $N = (4,4)$, $2D$ WZW - SG couplings were earlier constructed using $N = 1$ superfields and the conventional $N = 4$ superfields in [10, 8, 9].

More generally, one can take a sum of n such actions and choose the coefficients in such a way that the anomaly variations (5.27) coming from different items in the sum are cancelled out. [7] Clearly, at least one of such actions should enter with a "wrong" sign, presumably indicating that the relevant $q^{(1,1)}$ is a sort of "Liouville coordinate" [10]. In view of the inhomogeneous nature of the transformation law (5.24), one of the $q^{(1,1)}$ superfields will play the role of a compensator.

As it was already mentioned, for the time being we are not aware of the full nonlinear structure of the "almost-covariant" $q^{(1,1)}$ actions and even of the complete proof of their existence. Nevertheless, taking for granted that such actions can be constructed, let us inspect which kind of compensation of the "master" SG group can be achieved with the help of $q^{(1,1)}$. It will be enough to perform this analysis at the linearized level.

We shall start from the linearized WZ gauge content of BW gauge multiplet (3.23) and the corresponding form (3.24) of the residual symmetry. At the linearized level, the $q^{(1,1)}$ constraints (3.13) read (for the shifted superfield $\hat{q}^{(1,1)} = q^{(1,1)} - c^{(1,1)}$)

$$\begin{aligned} D^{(2,0)}\hat{q}^{(1,1)} &= c^{(1,-1)}\, D^{(0,2)}H_R^{(2,0)} - c^{(1,1)}\, H_R^{(2,0)}\ , \\ D^{(0,2)}\hat{q}^{(1,1)} &= c^{(-1,1)}\, D^{(2,0)}H_L^{(0,2)} - c^{(1,1)}\, H_L^{(0,2)}\ . \end{aligned} \tag{5.33}$$

They imply

$$\begin{aligned} \hat{q}^{(1,1)} \;=\;& \hat{q}^{ia}(z)u_i^{(1,0)}v_a^{(0,1)} + \theta^{(1,0)\,\underline{i}}\,\psi^a_{+\,\underline{i}}(z)\,v_a^{(0,1)} + \theta^{(0,1)\,\underline{a}}\,\chi^i_{-\,\underline{a}}(z)\,u_i^{(1,0)} \\ &+ i\theta^{(1,0)\,\underline{i}}\,\theta^{(0,1)\,\underline{a}}\,F_{\underline{ia}}(z) + \ \dots\ , \end{aligned} \tag{5.34}$$

where dots stand for the terms involving the BW multiplet gauge fields and derivatives of the explicitly written physical dimension fields of $q^{(1,1)}$. The purely shift part of the transformation (5.24) (we need only the latter for our linearized analysis)

$$\delta\,\hat{q}^{(1,1)} = c^{(1,1)}\ (\Lambda_L + \Lambda_R) - c^{(-1,1)}\,D^{(2,0)}\Lambda_L - c^{(1,-1)}\,D^{(0,2)}\Lambda_R \tag{5.35}$$

amounts to the following transformations of the fields:

$$\begin{aligned} &\delta\,\hat{q}^{ia} = c^{ia}\,(\lambda_L + \lambda_R) - c_j^{\ a}\,\lambda_L^{(ji)} - c^i_{\ b}\,\lambda_R^{(ba)}\ , \\ &\delta\,\psi^a_{+\,\underline{i}} = -c_i^{\ a}\,\beta^i_{+\,\underline{i}}\ , \quad \delta\,\chi^i_{-\,\underline{a}} = -c^i_{\ a}\,\beta^a_{-\,\underline{a}}\ , \\ &\delta\,F_{\underline{ia}} = 0\ . \end{aligned} \tag{5.36}$$

One observes that all the physical dimension fields can be gauged away by appropriate gauge parameters

$$\hat{q}^{ia} = 0\ \Rightarrow\ (a)\ \lambda_L = -\lambda_R \equiv \lambda\ ,\ \ (b)\ \lambda_L^{(ij)} = \frac{1}{c^2}\,\lambda_R^{(ab)}\,c^i_{\ a}c^j_{\ b} \equiv \lambda^{(ij)}\ , \tag{5.37}$$

$$\psi^a_{\underline{i}} = \chi^i_{\underline{a}} = 0 \quad \Rightarrow \quad \beta^i_{+\,\underline{i}} = \beta^a_{-\,\underline{a}} = 0\ , \tag{5.38}$$

where $c^2 = c^{ia}c_{ia} \neq 0$. As it follows from (5.37), the product of two local $U(1)$ symmetries is compensated down to the diagonal $U(1)$, and the same occurs for the product $SU(2)_L \times$

[7] A similar trick was used in [36] for construction of a gauge-invariant WZW action with the full $G \times G$ symmetry group gauged.

$SU(2)_R$ (this results in identifying the $SU(2)$ indices of the left and right harmonics, though still does not reduce two harmonic sets to each other). Eq. (5.38) implies the full compensation of the local non-canonical supersymmetries.

As the result, in the gauge (5.37), (5.38) the irreducible off-shell gauge representation comprises the (0+0) BW-I multiplet (4.10) as a submultiplet of the "master" BW multiplet we started with, as well as a new off-shell (8+8) gauge multiplet. The latter inherits a part of its fields from the original BW multiplet, and a part from the compensating $\hat{q}^{(1,1)}$ multiplet

$$\begin{aligned} &\underline{\text{bosons}}: \quad (h_{++},\ h_{--})\ (1,1)\ , \quad (h^{(ik)}_{++},\ h^{(ik)}_{--})\ (1,3)\ , \quad F_{\underline{ia}}\ (1,4)\ , \\ &\underline{\text{fermions}}: \quad t^{i\,\underline{a}}_{++-}\ (3/2,4)\ , \quad t^{k\underline{j}}_{--+}\ (3/2,4)\ . \end{aligned} \tag{5.39}$$

Comparing it with the (8+8) "$Sp(1)$ vector multiplet" of ref. [3], we find almost full identity between the two representations, except for a minor distinction related to the fact that one bosonic degree of freedom in (5.39) is represented by the dimension 1 $U(1)$ gauge field $h_{\pm\pm}$, while in [3] it is carried over by the dimension 2 auxiliary field. It is natural to identify the latter with the curl $\partial_{++}h_{--} - \partial_{--}h_{++}$, in view of the well-known equivalence of the auxiliary scalar field and the curl of gauge vector field in two dimensions. Note that the $Sp(1)$ vector multiplet was introduced in [3] "by hand", in addition to the purely gauge SG multiplet which we call here BW-I, in order to be able to construct locally $N = (4,4)$ supersymmetric sigma models on quaternionic manifolds. In our scheme it naturally appears, along with the BW-I gauge multiplet, as a result of compensating the "master" $N = (4,4)$ SG group by the TM-II multiplet $q^{(1,1)}$. The gauge $Sp(1)$ symmetry of [3] is recognized as the diagonal in the product of $SU(2)_L$ and $SU(2)_R$ symmetries realized as isometries of the WZW bosonic fields in $q^{(1,1)}$. It would be of interest to study this correspondence at the full nonlinear level and, in particular, to inquire how to construct superconformally-invariant couplings of some other matter $q^{(1,1)}$ superfields to this field representation (different from a simple sum of the "almost-covariant" actions). Because of the presence of the $SU(2)_{diag}$ gauge fields in (5.39) which couple to the physical bosonic fields of $q^{(1,1)}$, such couplings should be very restrictive.

Let us summarize the above ways of descending from the "master" conformal BW multiplet to the BW-I multiplet.

A. The (32 + 32) field representation. This option corresponds to the use of the pure gauge nonlinear multiplet $N^{(2,0)}$, $N^{(0,2)}$ as the conformal compensator. One imposes the covariant constraints (5.2) which imply some specific form for the analytic vielbeins $H^{(4,0)}$, $H^{(0,4)}, H^{(2,2)}, \tilde{H}^{(2,2)}$. After properly fixing the gauges, one ends up with the BW-I multiplet and an additional (32+32) off-shell $U(1)$ gauge multiplet (5.14) represented by the analytic superfield strength $H^{(2,2)}$ (5.12). The general action of the TM-II superfields $q^{(1,1)}$ in the background of this representation is given by eq. (5.22). No action for the compensator $N^{(2,0)}$, $N^{(0,2)}$ itself is assumed. A version with the additional constraints (5.15) yields the pure BW-I multiplet, with no extra multiplets.

B. The (64+64) field representation. This case corresponds to assuming an invariant action for the $N^{(2,0)}$, $N^{(0,2)}$ compensator. It is constructed using two copies of such superfields (eqs. (5.28), (5.30)). Only one set from this pair is the genuine compensator.

As in the previous case, after gauging this compensator away, one ends up with the BW-I multiplet and the (32+32) multiplet (5.14). One more (32+32) off-shell multiplet is $\tilde{N}^{(2,0)}$, $\tilde{N}^{(0,2)}$ (5.29),which is the remnant of the two original copies of N-multiplets.

C. The (40+40) field representation. In this scheme, in order to construct the invariant action for the compensator $N^{(2,0)}$, $N^{(0,2)}$, one uses the hypothetical "almost-covariant" action for one TM-II multiplet $q^{(1,1)}$ which is a gauged extension of the $N = (4,4)$, $SU(2)$ WZW action (2.17). The invariant action of two multiplets is given by (5.31). After gauging away the N-compensator, one is left with the (0+0) BW-I multiplet, the (32+32) $U(1)$ multiplet (5.14) and the (8+8) TM-II multiplet $q^{(1,1)}$.

D. The (16+16) field representation. This option is different from the preceding ones, as it uses $q^{(1,1)}$ as a compensator for the SG-I group. The invariant action (5.32) is given by the difference of two "almost-covariant" $q^{(1,1)}$ actions. In the gauge with all possible symmetries of the "master" SG group being compensated for, the surviving field representation consists of the BW-I multiplet, the (8+8) $SU(2)$ gauge multiplet (5.39) and the extra (8+8) TM-II multiplet $q^{(1,1)}$ which was added to set up the action (5.32).

There still remain the questions as to, how to descend to another, smaller conformal $N = (4,4)$, $SU(2)$ SG group, i.e. the SG-II group, and how to reproduce the known [5]-[9] and, perhaps, the new $N = (4,4)$ Poincaré SG multiplets, by continuing the above process of compensation.

The answer to the first question is as follows. As was already mentioned, there should be a "democracy" between different $SU(2)$ factors in the automorphism group $SO(4)_L \times SO(4)_R$ of the $N = (4,4)$, $2D$ Poincaré superalgebra. This implies the existence of "mirror" counterparts of the superconformal matter multiplets discussed so far, such that the roles of the $SU(2)$ groups acting on the doublet indices i, a and $\underline{i}, \underline{a}$ are switched. An example of such a correspondence is the TM-I multiplet [26, 27, 21], the off-shell field content of which is given by $q^{\underline{ia}}, \psi^{\underline{i}}_{k}, \chi^{\underline{a}}_{b}, F_{ia}$ that should be compared with the field content of TM-II (5.34). A similar mirror counterpart should exist for the nonlinear multiplet $N^{(2,0)}, N^{(0,2)}$. It is natural to call the latter NM-II, with respect to the $N = (4,4)$, $SU(2)$ SG-II group acting on the harmonic variables. Then, with respect to the same SG group, the mirror counterpart can be called NM-I. It seems plausible to conjecture that these mirror TM-I and NM-I multiplets (being, in fact, TM-II and NM-II with respect to the $N = (4,4)$, $SU(2)$ SG-I group), can be employed to compensate the "master" conformal SG group just down to the SG-II group, quite analogously to how TM-II and NM-II can be used for compensating the "master" group down to the SG-I group. Some of these mirror matter multiplets, in the rigid case, admit a description in the $SU(2) \times SU(2)$ harmonic superspace [32], so we can hope to find their locally supersymmetric versions, cousins of the actions considered above.

To clarify the second question, let us come back to the action (5.32) and assume that the "master" conformal SG group is reduced in it "by hand" to the SG-II one (taking for granted that a nonlinear version of the truncation conditions (4.17) exists). One of the $q^{(1,1)}$ superfields can still be used as a compensator. The linearized, purely shift part of the transformation laws of its components, under the action of the residual group (4.20),

can be obtained by the substitution of (4.16) into (5.36)

$$\delta\,\hat{q}^{ia} = -\frac{1}{2}\,c^{ia}\,(\partial_{++}\lambda^{++} + \partial_{--}\lambda^{--}) - c_j^{\ a}\,\lambda_L^{(ji)} - c^i_{\ b}\,\lambda_R^{(ba)}\ ,$$
$$\delta\,\psi^a_{+\,\underline{i}} = 2i\,c_i^{\ a}\,\partial_{++}\lambda^{+\,i}_{\underline{i}}\ , \quad \delta\,\chi^i_{-\,\underline{a}} = 2i\,c^i_{\ b}\,\partial_{--}\lambda^{-\,b}_{\underline{a}}\ ,$$
$$\delta\,F_{\underline{ia}} = 0\ . \tag{5.40}$$

One sees that the two chiral $SU(2)$ s are reduced to the diagonal $SU(2)$, like in the case (5.36), (5.37), (5.38), by gauging away the triplet part of $\hat{q}^{ia}$. However, the singlet part cannot be gauged away; it becomes just the third component of zweibein. Analogously, $\psi^a_{+\,\underline{i}}$, $\chi^i_{-\,\underline{a}}$, together with $h^{+\,\underline{ii}}_{--}, h^{-\,\underline{aa}}_{--}$ from the BW-II multiplet (4.19), are combined into the 16-component $N = (4,4)$ Poincaré SG gravitino (the indices i and a now refer to the same diagonal $SU(2)$). Eventually, bearing in mind the auxiliary field $F_{\underline{ia}}$, we end up just with the (8+8) off-shell content of the minimal $N = (4,4)$, $2D$ Poincaré SG representation [5, 6]. However, recalling that the invariant action (5.32) includes one more $q^{(1,1)}$, the total off-shell representation for this case is (16+16). This off-shell content coincides with that of the "TM $N = 4$ superstring" considered in [7].

Analogously, one can use the nonlinear multiplet NM-II as a compensator from $N = (4,4)$, $SU(2)$ SG-II down to some Poincaré SG. The resulting version involves (32+32) off-shell components; its interesting feature is that *both* conformal $SU(2)$ symmetries turn out to be fully compensated for, and $h^{(ik)}_{++}, h^{(ab)}_{--}$ in (4.19) (and in its left counterpart) cease to be gauge fields. The full off-shell content, taking into account an additional $q^{(1,1)}$ multiplet needed to construct the invariant action as in (5.31), is (40+40). This coincides with the off-shell content of the "relaxed hypermultiplet $N = 4$ superstring" of ref. [7]. It is interesting to inquire whether the latter representation is indeed identical to ours.

At last, one can start from the action (5.28) and recover a version of Poincaré SG with (64+64) off-shell fields. Once again, in this version both $SU(2)$ symmetries are fully compensated for.

In accord with the previous discussion, various mirror versions of the Poincaré SG can be obtained, starting from the $N = (4,4)$, $SU(2)$ SG-I and making use of the multiplets TM-I and NM-I as compensators. The various patterns of descent from the "master" $N = (4,4)$ SG to the SG-I described above, as well as their hypothetical mirror cousins, seem also to admit further compensations down to the $N = (4,4)$ Poincaré SG representations, along similar lines. New possibilities can arise while simultaneously using both types of matter multiplets, i.e. the types I and II, as compensators.

Finally, let us note that there exists a dual version of the rigidly supersymmetric $q^{(1,1)}$ actions, including the $N = (4,4)$ WZW one (2.17), in terms of unconstrained $SU(2) \times SU(2)$ harmonic analytic superfields with infinite numbers of auxiliary fields [14]. This should obviously generalize to the case of local SUSY, which in turn suggests the existence of new versions of Poincaré $N = (4,4)$, $2D$ SG with infinite sets of auxiliary fields.

A thorough analysis of all these possibilities can be a good program for a future study.

6 Conclusions

In this paper we constructed a new sort of $N = (4,4)$, $2D$ conformal SG gauge multiplet, i.e. the Beltrami-Weyl multiplet, starting from the group of diffeomorphisms in the $SU(2) \times SU(2)$ analytic harmonic superspace. This multiplet can be regarded as the result of gauging the most extensive rigid $N = (4,4)$ superconformal $2D$ group, i.e. the product of two light-cone copies of the infinite-dimensional "large" $SO(4) \times U(1)$, $N = 4$ superconformal group. The previously known $N = (4,4)$ conformal SG groups and the corresponding Weyl multiplets were argued to follow from the new "master" SG group and BW multiplet upon their various truncations and compensations, with making use of the appropriate superconformal matter multiplets. Also, various versions of $N = (4,4)$ Poincaré SG can be recovered.

There still remain a few important conceptual and technical points to be fully elaborated on. This concerns, before all, constructing the full nonlinear version of the "almost-covariant" $q^{(1,1)}$ action (5.25) and the nonlinear completion of the constraints (4.17), (4.15), as well as revealing the component fields structure of the locally supersymmetric superfield actions presented. An important problem is to incorporate into the present scheme mirror counterparts of the superconformal multiplets employed in this paper and to study the relevant compensation patterns. Different $N = (4,4)$ SG-matter couplings correspond to various versions of $N = (4,4)$ superstrings [7]. It would be interesting to inquire the quantum properties of the systems described here, e.g., along the lines of refs. [22], [10]. Note that the rigid $N = (4,4)$ WZW action (2.17) admits an extension to the $N = (4,4)$ WZW-Liouville one [27, 21, 30, 14], with breaking the $N = (4,4)$, $SO(4) \times U(1)$ superconformal invariance down to the type-II $N = (4,4)$, $SU(2)$ one. Such a Liouville extension plays an important role in the quantum case [10]. It is of interest to inquire whether a locally supersymmetric extension of the Liouville term can be constructed in $SU(2) \times SU(2)$ harmonic superspace.

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Appendix: A simple example of the component action

Here, just to give a feeling how the locally supersymmetric actions presented in this paper look in terms of component fields, we quote the free part of the general conformal SG-I

group-invariant action (4.14)

$$S_q^{free} = -\int \mu^{(-2,-2)}\ \hat{\Omega}\ q^{(1,1)}\, q^{(1,1)}\ . \tag{A.1}$$

It will be convenient to choose a gauge for the analytic vielbeins which is slightly different from (4.9)

$$\begin{aligned} &H^{(2,0)\,++} = H^{(3,0)\,\underline{i}} = 0\ , \quad H^{(2,0)\,--} = i(\theta^{(1,0)})^2\,\hat{h}^{--}_{++}(z,v,\theta^{(0,1)})\ , \\ &H^{(2,1)\,\underline{a}} = i(\theta^{(1,0)})^2\,\hat{h}^{(0,1)\,\underline{a}}_{++}(z,v,\theta^{(0,1)})\ . \end{aligned} \tag{A.2}$$

This gauge is also globally well-defined. To simplify the situation as soon as possible, we recall that all components of the BW-I multiplet are locally pure gauge, and we choose the additional gauge, which is admissible only locally,

$$h^{--}_{++} = h^{(\underline{ab})}_{++} = h^{-\,a\underline{a}}_{++} = 0 \tag{A.3}$$

(we could alternatively choose the left counterparts of (A.3) to vanish). It is easy to show that the *full* solution of the constraints (4.8) in this gauge is given by

$$\begin{aligned} \nabla^{(2,0)} &= \partial^{(2,0)} + i(\theta^{(1,0)})^2\,\partial_{++}\ , \quad \nabla^{(0,2)} = \partial^{(0,2)} + i(\theta^{(0,1)})^2\,\nabla_{--}\ , \\ \nabla_{--} &= \partial_{--} + \{\, h^{++}_{--} - 2i\,\theta^{(1,0)}_{\underline{i}}\, h^{+\,k\underline{i}}_{--} u^{(-1,0)}_{k}\,\}\,\partial_{++} \\ &+ \{\, h^{+\,k\underline{i}}_{--} u^{(1,0)}_{k} + \theta^{(1,0)\,\underline{k}}[h^{(\underline{i}}_{--\,\underline{k})} + \frac{1}{2}\delta^{\underline{i}}_{\underline{k}}\,\partial_{++}h^{++}_{--}] \\ &- (\theta^{(1,0)})^2\partial_{++}h^{+\,k\underline{i}}_{--} v^{(-1,0)}_{k}\,\}\frac{\partial}{\partial\theta^{(1,0)\,\underline{i}}}\ . \end{aligned} \tag{A.4}$$

It is easy to explicitly check the integrability condition

$$[\,\nabla^{(2,0)}, \nabla^{(0,2)}\,] = 0\ .$$

The residual gauge symmetry of (A.4) is given by (4.11), with all parameters being functions of only z^{--} (this is just the right $N = 4,\ SU(2)$ SCA-I), and by the left counterpart of (4.11), with the parameters still being general functions of both coordinates $z^{\pm\pm}$. It is easy to check that under this group

$$\delta\mu^{(-2,-2)} = 0\ ,$$

so one can expect $\hat{\Omega} \sim$ const in this gauge. This is indeed so, because it is easy to check that

$$\Gamma^{(2,0)} = \Gamma^{(0,2)} = 0 \tag{A.5}$$

for the vielbeins in (A.4). Then, the action (A.1) is

$$S_q^{free} = -\int \mu^{(-2,-2)}\ q^{(1,1)}\, q^{(1,1)}\ , \tag{A.6}$$

with

$$\nabla^{(2,0)} q^{(1,1)} = \nabla^{(0,2)} q^{(1,1)} = 0\ . \tag{A.7}$$

Being aware of the explicit expressions for $\nabla^{(2,0)}, \nabla^{(0,2)}$, it is easy to directly solve these constraints in terms of the physical fields of $q^{(1,1)}$ defined in (5.34) and the SG fields. This is rather straightforward, so we quote only the final form of the action. It is obtained by substituting this solution into (A.6) and integrating there over the θ's and the harmonics:

$$\begin{aligned} S_q^{free} &= \int d^2z \left\{ \partial_{++} q_{ia} \left(\hat{\nabla}_{--} q^{ia} + h_{--}^{+\,\underline{ii}} \psi_{+\,\underline{i}}^{a} \right) + \frac{i}{2} \chi_{-}^{\underline{ia}} \partial_{++} \chi_{-\,\underline{ia}} + \frac{1}{4} F^{\underline{ia}} F_{\underline{ia}} \right. \\ &+ \left. \frac{i}{2} \psi_{+}^{a\underline{i}} \left[\hat{\nabla}_{--} \psi_{+\,a\underline{i}} + (h_{--\,\underline{i})}^{(\underline{k}} + \frac{1}{2} \delta_{\underline{i}}^{\underline{k}} \partial_{++} h_{--}^{++}) \, \psi_{+\,a\underline{k}} - 2i h_{--\,\underline{i}}^{+\,\underline{k}} \partial_{++} q_{ka} \right] \right\}, \quad \text{(A.8)} \end{aligned}$$

where

$$\hat{\nabla}_{--} = \partial_{--} + h_{--}^{++} \partial_{++} \ .$$

Note that (A.8) is just the action of the $N = 4$ chiral bosons constructed in ref. [37] (up to switching the + and − light-cone indices), with the residual local $N = 4$ SUSY as the relevant Siegel symmetry.

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