



## ENVELOPE SOLITONS VERSUS SOLITONS

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### Abstract

A theory involving a correspondence between envelope solitonlike solutions of the generalized nonlinear Schrödinger equation (GNLSE) and solitonlike solutions of the generalized Korteweg-de Vries equation (GKdVE) is developed within the context of the Madelung's fluid description (fluid counterpart description of the GNLSE). This correspondence, which, under suitable constraints, can be made invertible, seems to be very helpful for finding one family of solutions (whether envelope solitonlike solutions of the GNLSE or solitonlike solutions of the GKdVE) starting from the knowledge of the other family of solution (whether solitonlike solutions of the GKdVE or envelope solitonlike solutions of the GNLSE). The theory is successfully applied to a wide classes of both modified nonlinear Schrödinger equation (MNLSE) and modified Korteweg-de Vries equation (MKdVE), for which bright and gray/dark solitonlike solutions are found. In particular, bright and gray/dark solitary waves are determined for the MNLSE with a quartic nonlinear potential in the modulus of the wavefunction (i.e.  $q_1|\Psi|^2 + q_2|\Psi|^4$ ) as well as for the associated MKdVE. Furthermore, the well known bright and gray/dark envelope solitons of the cubic NLSE and the corresponding solitons of the associated standard KdVE are easily recovered from the present theory. Remarkably, this approach opens up the possibility to transfer all the know how concerning the instability criteria for solitonlike solutions of the MKdVE to the instability theory of envelope solitonlike solutions of the MNLSE.

PACS: 52.35.Mw; 05.45.Yv; 42.65; 67.57.Jj

Published by SIS-Pubblicazioni  
Laboratori Nazionali di Frascati

In this paper, a correspondence between a wide family of generalized nonlinear Schrödinger equations (GNLSEs) and a wide family of generalized Korteweg-de Vries equations (GKdVEs) is constructed in such a way that stationary-profile solutions of the latter are the squared modulus of stationary-profile envelope solutions of the former. This is done by extending a recent work [1] that has considered a similar correspondence involving the cubic nonlinear Schrödinger equation (NLSE) and the standard Korteweg-de Vries equation (KdVE).

From one side, the investigation is carried out considering the following GNLSE for a complex wavefunction  $\Psi$ :

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - U[|\Psi|^2] \Psi = 0 \quad , \quad (1)$$

where the real constant  $\alpha$  accounts for the dispersive/diffractive effects and  $U[|\Psi|^2]$  is a real functional of  $|\Psi|^2$ ; from the other side, the following GKdVE for the real function  $u$  is considered:

$$a \frac{\partial u}{\partial s} - G[u] \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0 \quad , \quad (2)$$

where  $a$  and  $\nu$  are real constants, and  $G[u]$  is a real functional of  $u$ <sup>1</sup>. In both equations,  $x$  is the 1-D configurational space coordinate and  $s$  is the timelike coordinate. In particular, in this paper, special attention will be devoted to a correspondence between the special case of (1) when  $U[|\Psi|^2] = q_0 |\Psi|^{2\beta}$ ,  $q_0$  and  $\beta$  being real and positive real numbers, respectively, namely

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - q_0 |\Psi|^{2\beta} \Psi = 0 \quad , \quad (3)$$

and the special case of (2) when  $G[u] = p_0 u^\gamma$ ,  $p_0$  and  $\gamma$  being real and positive real numbers, respectively, namely

$$a \frac{\partial u}{\partial s} - p_0 u^\gamma \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0 \quad . \quad (4)$$

According to other authors [2]-[5], let us define (3) as modified nonlinear Schrödinger

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<sup>1</sup>It is easy to see that, without loss of generality, we have chosen, in Eq. (2), a positive coefficient of the third-derivative term. Actually, the coefficient  $a$  in principle can be positive or negative in such a way that the dispersion coefficient of (2), i.e.  $\mu \equiv \nu^2/4a$ , can be positive or negative, as well

equation (MNLSE) and (4) as modified Korteweg-de Vries equation (MKdVE).

Our goals are: (i) to construct a correspondence between (1) and (2) in such a way that each positive stationary-profile solution of (2), namely  $u(x, s) = u(\xi)$  with  $\xi = x - u_0 s$  ( $u_0$  being a real dimensionless constant velocity), is the squared modulus of a stationary-profile envelope solution of (1) having the form

$$\Psi(x, s) = \sqrt{\rho(\xi)} \exp[i\Theta(x, s)] \quad , \quad (5)$$

with  $\Theta$  real function and  $\rho$  real and positive function such that  $u(\xi) = |\Psi(x, s)|^2 = \rho(\xi)$ , and consider explicitly the conditions that guarantee the existence of solitonlike solutions; (ii) to apply the above correspondence to the case of (3) and (4) and find for them solitonlike solutions, generalizing the set of solitonlike solutions given already in literature [6,7]; (iii) to apply, in turn, the results of (ii) to find envelope solitons of (1) in the case

$$U(|\Psi|^2) = q_1 |\Psi|^2 + q_2 |\Psi|^4 \quad . \quad (6)$$

The framework in which the above goals will be reached is the same as in Ref. [1], namely the Madelung's fluid [8] picture of the GNLSE (1).

As it is well known, once  $\Psi(x, s) = \sqrt{\rho(x, s)} \exp[i\Theta(x, s)/\alpha]$  is assumed, the (1) can be cast as a closed system of nonlinear coupled equations (Madelung's fluid [8]): continuity + motion equation. However, according to Ref.[1], this system can be, in turn, reduced to the following closed system of coupled equations:

$$\frac{\partial \rho}{\partial s} + \frac{\partial}{\partial x}(\rho V) = 0, \quad (7)$$

$$-\left(\frac{\partial V}{\partial s}\right) \rho + V \frac{\partial \rho}{\partial s} + 2 \left[ c_0(s) - \int \left(\frac{\partial V}{\partial s}\right) dx \right] \frac{\partial \rho}{\partial x} - \left( \frac{\partial U}{\partial x} \rho + 2 U \frac{\partial \rho}{\partial x} \right) + \frac{\alpha^2}{4} \frac{\partial^3 \rho}{\partial x^3} = 0 \quad , \quad (8)$$

where  $V$  is the current velocity, given by

$$V(x, s) = \frac{\partial \Theta(x, s)}{\partial x}, \quad (9)$$

and  $c_0(s)$  is an arbitrary real function of  $s$ .

## 2 Correspondence between solitons and envelope solitons

Let us denote with  $\mathcal{E} = \{\Psi\}$  the set of all the envelope solutions (5) of (1), i.e. the set of all the stationary-profile envelope solutions of the GNLSE. Let us also denote

with  $\mathcal{S} = \{u(\xi) \geq 0\}$  the set of all non-negative stationary-profile solutions of the GKdVE (2).

In order to construct the above correspondence between  $\mathcal{E}$  and  $\mathcal{S}$ , we observe that if  $\Psi \in \mathcal{E}$ , thus  $\rho$  and  $V$  have the form  $\rho = \rho(\xi)$  and  $V = V(\xi)$ , respectively.

Under the above hypothesis, it is easy to see that: (a).  $c_0(s)$  becomes constant (so that, let us put  $c_0(s) \equiv c_0$ ); (b). continuity equation (7) becomes:

$$u_0 \frac{d\rho}{d\xi} = \frac{d}{d\xi} (\rho V) \quad , \quad (10)$$

which integrated gives:

$$V(\xi) = u_0 + \frac{A_0}{\rho(\xi)} \quad , \quad (11)$$

where  $A_0$  is an arbitrary constant. By combining (8) and (10), we easily have:

$$(u_0^2 + 2c_0) \frac{d\rho}{d\xi} - \mathcal{I}[\rho] \frac{d\rho}{d\xi} + \frac{\alpha^2}{4} \frac{d^3\rho}{d\xi^3} = 0 \quad , \quad (12)$$

where the functional  $\mathcal{I}[\rho]$  is defined as:

$$\mathcal{I}[\rho] = \rho \frac{dU[\rho]}{d\rho} + 2U[\rho] \quad . \quad (13)$$

On the other hand, for stationary-profile solution  $u = u(\xi)$ , Eq. (2) becomes:

$$-u_0 a \frac{du}{d\xi} - G[u] \frac{du}{d\xi} + \frac{\nu^2}{4} \frac{d^3u}{d\xi^3} = 0 \quad . \quad (14)$$

Consequently, (12) and (14) have the same solutions, if the same boundary conditions are taken for them and provided that their coefficients are respectively proportional. In particular, it follows that  $u(\xi)$  is a non-negative stationary-profile solution of the following GKdVE ( $u_0 \neq 0$ ):

$$-\frac{u_0^2 + 2c_0}{u_0} \frac{\partial \rho}{\partial s} - \mathcal{I}[\rho] \frac{\partial \rho}{\partial x} + \frac{\alpha^2}{4} \frac{\partial^3 \rho}{\partial x^3} = 0 \quad , \quad (15)$$

where the following identities have been used:

$$a \equiv - (u_0^2 + 2c_0) / u_0 \quad , \quad G[u] \equiv \mathcal{I}[u] \quad , \quad \nu \equiv \alpha \quad . \quad (16)$$

Thus, it results that, starting from Eq. (1), we have constructed the following correspondence:

$$\begin{aligned} \mathcal{F} & : \quad \Psi \in \mathcal{E} \rightarrow u \in \mathcal{S} \quad , \\ u & = \quad \mathcal{F}[\Psi] = |\Psi|^2 = \rho(\xi) \quad . \end{aligned} \quad (17)$$

$\mathcal{F}$  associates a stationary-profile envelope solution of (1) to a stationary-profile solution of the associated GKdVE (15). In particular, it may associate an envelope solitonlike solution of (1) with a solitonlike solution of (2).

It is worth to observe that, by substituting (5) directly in (1), separating real and imaginary parts, and taking into account (9) we get:

$$-\partial\Theta/\partial s = c_0 + u_0 V \quad , \quad (18)$$

which, combined with (11) becomes:

$$\Theta(x, s) = \phi_0 - (c_0 + u_0^2) s + u_0 x + A_0 \int \frac{d\xi}{\rho(\xi)} \quad , \quad (19)$$

where  $\phi_0$  is an arbitrary real constant.

Now, let  $u(\xi)$  be a positive stationary-profile solution of (2). Thus,  $u$  satisfies an equation similar to (14) and, provided that (12) and (14) have still proportional coefficients, in correspondence of the same boundary conditions,  $u$  is also solution of (12). Thus, by defining the quantity

$$\tilde{V} \equiv \frac{\partial\tilde{\Theta}(x, s)}{\partial x} \quad , \quad (20)$$

with

$$\tilde{\Theta}(x, s) = \phi_0 - (c_0 + u_0^2) s + u_0 x + A_0 \int \frac{d\xi}{u(\xi)} \quad , \quad (21)$$

it follows that  $\rho(\xi) = u(\xi)$  and  $\tilde{V}$  are solutions of the following system of coupled equations:

$$\frac{\partial u}{\partial s} + \frac{\partial}{\partial x} (u\tilde{V}) = 0 \quad , \quad (22)$$

$$-\left(\frac{\partial\tilde{V}}{\partial s}\right) u + \tilde{V} \frac{\partial u}{\partial s} + 2 \left[ c_0(s) - \int \left(\frac{\partial\tilde{V}}{\partial s}\right) dx \right] \frac{\partial u}{\partial x} - \left(\frac{\partial\mathcal{U}}{\partial x} u + 2 \mathcal{U} \frac{\partial u}{\partial x}\right) + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0 \quad , \quad (23)$$

where the functional  $\mathcal{U}$  is solution of the following differential equation:

$$u \frac{d\mathcal{U}}{du} + 2\mathcal{U} = G[u] \quad , \quad (24)$$

namely

$$\mathcal{U}[u] = \frac{1}{u^2} \left[ K_0 + \int G[u] u du \right] \quad , \quad (25)$$

where  $K_0$  is an arbitrary real constant. It follows that the complex function

$$\Psi = \sqrt{u(\xi)} \exp \left[ \frac{i}{\alpha} \tilde{\Theta}(x, s) \right] \quad (26)$$

is a stationary-profile envelope solution of the following GNLSE:

$$i\nu \frac{\partial \Psi}{\partial s} + \frac{\nu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \left[ \frac{K_0 + \int G[|\Psi|^2] |\Psi|^2 d|\Psi|^2}{|\Psi|^4} \right] \Psi = 0 \quad . \quad (27)$$

The substitution of (26) in (27) (after separating real and imaginary parts) gives the following equation for  $u$ :

$$-\frac{\nu^2}{2} \frac{d^2 u^{1/2}}{d\xi^2} + \frac{K_0}{u^{3/2}} + \frac{1}{u^{3/2}} \int G[u] u du = \left( c_0 + \frac{u_0^2}{2} \right) u^{1/2} - \frac{A_0^2}{2u^{3/2}} \quad . \quad (28)$$

It is then clear that each solution  $u(\xi)$  of (2), once substituted in (28), fixes a relationship among the constants  $K_0$ ,  $c_0$ , and  $A_0$ . This shows that these parameters are not all independent. According to (14), (21), and (26), it results that for each given  $u \in \mathcal{S}$  and for each given set of constants  $\phi_0$ ,  $c_0$ , and  $A_0$ , the modulus and the phase of  $\Psi$  are uniquely determined and, consequently, the solution of (27) is uniquely determined.

In conclusion, starting from the GKdVE (2), we have constructed the following correspondence:

$$\mathcal{H} : u \in \mathcal{S} \rightarrow \Psi \in \mathcal{E} \quad , \\ \Psi = \mathcal{H}[u] = \sqrt{u(\xi)} \exp \left\{ \frac{i}{\nu} \left[ \phi_0 - (c_0 + u_0^2) s + u_0 x + A_0 \int \frac{d\xi}{u(\xi)} \right] \right\} \quad , \quad (29)$$

which, for each given set of real constants  $\phi_0$ ,  $c_0$ , and  $A_0$ , associates a positive stationary-profile solution  $u(\xi)$  of (2) to a stationary-profile envelope solution  $\Psi(x, s)$  of (27) which is of the type (1). It is clear that, as the above parameters vary over all their accessible ranges of values,  $\mathcal{H}[u]$  describes the subset of stationary-profile envelope solutions of (27) whose squared modulus equals  $u(\xi)$ . In particular, if  $u(\xi)$  is a localized solution of (2), thus  $\mathcal{H}[u]$  describes the subset of envelope localized solutions of the associated equation (27), where  $\phi_0$  is still arbitrary and the values allowed for  $c_0$  and  $A_0$  are determined by the specific boundary conditions required for such a kind of localized solution.

An important aspect concerning the role of the boundary conditions for solitonlike solution should be now considered in connection with the search for bright solitons and gray/dark solitons.

Let  $u > 0$  be bright solitonlike solutions satisfying the following boundary conditions:

$$\lim_{\xi \rightarrow \pm\infty} u(\xi) = 0 \quad ; \quad (30)$$

according to (11), (30) implies ( $u = \rho$ ) that:

$$A_0 = 0 \quad , \quad \text{and} \quad V = \tilde{V} = u_0 \quad . \quad (31)$$

Consequently, the phase of solution (29) becomes linearly depending on  $x$  and  $s$ , i.e.

$$\Psi = \mathcal{H}[u] = \sqrt{u(\xi)} \exp \left\{ \frac{i}{\nu} \left[ \phi_0 - (c_0 + u_0^2) s + u_0 x \right] \right\} , \quad (32)$$

Additionally, to provide solitonlike solutions of (2),  $G[u(\xi)]$  must have a regular behaviour in such a way that

$$\lim_{\xi \rightarrow \pm\infty} |G[u(\xi)]| < \infty , \quad (33)$$

which implies

$$\lim_{\xi \rightarrow \pm\infty} \left| \frac{1}{u^2} \int G[u(\xi)] u du \right| < \infty . \quad (34)$$

Correspondingly, conditions (31) and (34) in (28) give

$$K_0 = 0 , \quad (35)$$

and the following stationary GNLSE

$$-\frac{\alpha^2}{2} \frac{d^2 u^{1/2}}{d\xi^2} + \left[ \frac{1}{u^2} \int G[u(\xi)] u du \right] u^{1/2} = E_0 u^{1/2} , \quad (36)$$

where  $E_0 = c_0 + u_0^2/2$ . In conclusion, for bright solitonlike solutions, satisfying the boundary conditions (30),  $K_0 = A_0 = 0$ , and the phase of  $\Psi$  is linear (see Eq. (32)); furthermore,  $u(\xi)$  and the constant  $E_0$  play, respectively, the role of eigenstate and eigenvalue of the GNLSE (36). In particular, the role played by the constant  $E_0$  suggests the suitability to choose it as independent parameter instead of  $c_0$ .

If non-negative solitonlike solutions (for instance, up-shifted bright solitons, gray solitons) of (2) do not satisfy the above boundary conditions (30), it is useful to introduce the following positions:

$$u(\xi) = \bar{u} + u_1(\xi) , \quad (37)$$

where the constant  $\bar{u}$  is positive <sup>2</sup> and

$$V(\xi) = V_0 + V_1(\xi) . \quad (38)$$

Thus, the following boundary are imposed:

$$\lim_{\xi \rightarrow \pm\infty} u_1(\xi) = 0 , \quad (39)$$

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<sup>2</sup>Note that in this paper we are dealing with positive solutions  $u(\xi)$ , only. This implies that  $|u_1| < \bar{u}$ , if  $u_1 < 0$ .

and, correspondingly

$$\lim_{\xi \rightarrow \pm\infty} V_1(\xi) = 0 \quad . \quad (40)$$

It follows from (11) that ( $\rho = u$ ):

$$A_0 = -(u_0 - V_0)\bar{u} \quad , \quad (41)$$

with  $V_0 \neq u_0$ , which implies that  $A_0 \neq 0$ <sup>3</sup>. Consequently, according to (29) the phase of solution  $\Psi$  is not linear, because the last term is not vanishing. Additionally, since  $A_0 \neq 0$ , it is clear that, in general (whether  $K_0$  is zero or not zero), solution  $u(\xi)$  of Eq. (28) and the constant  $E_0 = c_0 + u_0^2/2$  do not play the role of eigenstate and eigenvalue of the energy, respectively. This last circumstance can be obtained only by choosing

$$K_0 \equiv -\frac{A_0^2}{2} \quad .$$

It should be stressed that this particular choice of  $K_0$  corresponds to standard bright envelope solitonlike solutions having linear phase (i.e.  $K_0 = -A_0^2/2 = 0$ ) and satisfying the boundary conditions (30) or corresponds to envelopes having nonlinear phase (i.e.  $K_0 = -A_0^2/2 \neq 0$ ) that are eigenstates of the energy.

Let us suppose that a finite value of  $\xi$ , say  $\xi_0$  ( $|\xi_0| < \infty$ ), such that  $u(\xi_0) = 0$ , exists. Thus, in order to keep limited solution for  $V$ , Eq. (11) implies that  $A_0 = 0$  ( $u_0 = V_0$ ). Consequently, for any  $\bar{u} > 0$ , the solitonlike solutions, satisfying the relationships (37)-(40), satisfy, additionally, both the condition  $u(\xi_0) = 0$ <sup>4</sup> and the (28) with  $K_0 = 0$ . This means that they are eigenstates of the energy. Correspondingly, dark envelope solitonlike solutions of (27) have only a linear phase (in  $x$  and in  $s$ ).

The results presented above may suggest a method for finding stationary-profile solutions of an equation belonging in one family (whether GNLSE or GKdVE) if stationary-profile solutions of the associated equation, belonging in the other family (whether GKdVE or GNLSE), are known.

It should be noted that in principle Eq. (27) must be compared with the particular nonlinear Schrödinger equation that we are actually going to solve. In particular, the potential  $U[|\Psi|^2]$  must be compared with the one of Eq. (27) (i.e.  $\mathcal{U}[u = |\Psi|^2]$ ) given by (25). If  $U[|\Psi|^2]$  has a regular asymptotic behavior ( $\lim_{\xi \rightarrow \pm\infty} |U[|\Psi|^2]| < \infty$ ), the above comparison implies that  $K_0 = 0$ . Consequently, if  $u(\xi) > 0$  is a solitonlike solution satisfying the relationships (37)-(40) with  $\bar{u} > 0$ , it follows that  $A_0 \neq 0$  ( $u_0 \neq V_0$ ). Consequently,  $u(\xi)$  can be an up-shifted bright solitonlike solution (with  $u_1 > 0$ ) or a gray solitonlike solution (with  $u_1 < 0$ ).

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<sup>3</sup>The case  $V_0 = u_0$  corresponds to  $A_0 = 0$  which selects bright solitons.

<sup>4</sup> $u_1$  must be negative and they are called dark solitons.



### 3 Applications

To give some explicit examples, this method will be applied here to the family of equations represented by (3) and (4) (MNSE and MKdVE, respectively), restricting the investigation to the solitonlike solutions, only.

First of all, the simplest case concerning  $\beta = 1$  in (3), which defines the standard NLSE [9], and  $\gamma = 1$  in (4), which defines the standard KdVE [9], has been already considered in Ref.[1], where envelope solitons (bright, gray/dark) of the NLSE have been found starting from the solitonlike solution (bright, gray or dark) of the associated KdVE and using a sort of correspondence  $\mathcal{H}$ , described above. Here,  $\beta$  and  $\gamma$  are allowed to be arbitrary positive real numbers.

As first example, let us look for bright envelope solitons of (3). To this end, it is worth to observe that, for  $au_0 > 0$  and  $p_0 < 0$ , Eq. (4) admits the following positive solitonlike solution (for any positive real  $\gamma$ ):

$$u(\xi) = \left[ \frac{au_0(\gamma+2)(\gamma+1)}{2|p_0|} \right]^{1/\gamma} \operatorname{sech}^{2/\gamma} \left[ \frac{\gamma\sqrt{au_0}}{|\nu|} \xi \right] . \quad (42)$$

( $\gamma = 1$  recovers the bright KdV soliton [9]). It should be pointed out that, since  $\gamma$  is an arbitrary positive real number, solutions (42) represent an extension of the usual ones that in the literature have been given only for the case of  $\gamma$  positive integer [5]-[7]. They satisfy the boundary conditions (30). Consequently, solutions (42) are bright solitons of the MKdVE <sup>5</sup>. According to the (2), for the case under discussion we can write:

$$G[u] = p_0 u^\gamma . \quad (43)$$

Provided that the following identification

$$u = \rho = |\Psi|^2 \quad (44)$$

is made, the substitution of (43) in (27), gives the following associated MNLSE:

$$i\nu \frac{\partial \Psi}{\partial s} + \frac{\nu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \frac{p_0}{\gamma+2} |\Psi|^{2\gamma} \Psi = 0 , \quad (45)$$

where (31) and (35) have been used (note that  $K_0 = 0$ ). We must identify the potential of this equation with the one of the MNLSE (3), which is the equation

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<sup>5</sup>It should be noted that, only in the case of  $\gamma$  even integer, if  $u$  is a positive solution of (42), thus  $-u$  is a negative solution. In this case, since  $u$  is a bright soliton, solution  $-u$  could be called anti-bright soliton.

that we are going to solve. Thus, after a comparison between (3) and (45), we easily get:

$$\nu = \alpha , \quad \gamma = \beta , \quad p_0 = (\beta + 2)q_0 . \quad (46)$$

Finally, by virtue of (32) we can write the following bright envelope solitonlike solutions of (3) for any real positive  $\beta$ :

$$\begin{aligned} \Psi(x, s) = & \left[ \frac{|E_0|(\beta + 1)}{|q_0|} \right]^{1/2\beta} \operatorname{sech}^{1/\beta} \left[ \frac{\beta\sqrt{2|E_0|}}{|\alpha|} \xi \right] \times \\ & \times \exp \left\{ \frac{i}{\alpha} \left[ \phi_0 - \left( E_0 + u_0^2/2 \right) s + u_0 x \right] \right\} , \end{aligned} \quad (47)$$

where the first of (16), (42) and (46) have been used, and where  $E_0 < 0$  and  $q_0 < 0$ . TABLE I and TABLE II display the plots of  $\rho$  and  $\sqrt{\rho}$ , as function of  $\xi/\Delta$ , for some values of the parameter  $\beta$ ,  $E_0$ ,  $\alpha$ , and  $q_0$ , according to (47).

As second example, let us find envelope solitonlike solutions for the following MNLSE with a quartic potential given by (6). Since ( $\rho = |\Psi|^2$ )

$$U = U[\rho] = q_1\rho + q_2\rho^2 ,$$

according to the theory presented above, it is easy to see that we have to use the correspondence  $\mathcal{H}$  starting from the following MKdVE:

$$a \frac{\partial u}{\partial s} - \left[ p_0(u - \bar{u})^2 + \delta_0 \right] \frac{\partial u}{\partial x} + \frac{\nu^2}{4} \frac{\partial^3 u}{\partial x^3} = 0 , \quad (48)$$

where  $p_0$ ,  $\bar{u}$ , and  $\delta_0$  are constant to be determined. Thus, according to (27) the following MNLSE can be written:

$$i\nu \frac{\partial \Psi}{\partial s} + \frac{\nu^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - \left[ K_0 |\Psi|^{-4} + \frac{p_0}{4} |\Psi|^4 - \frac{2p_0\bar{u}}{3} |\Psi|^2 + \frac{1}{2} (\delta_0 + p_0\bar{u}^2) \right] \Psi = 0 , \quad (49)$$

which must be compared with the one that we have to solve. Consequently, we get:  $\nu = \alpha$ ,  $K_0 = 0$ ,  $p_0 = 4q_2$ ,  $\bar{u} = -3q_1/(8q_2)$ , and  $\delta_0 = -9q_1^2/16q_2$ .

It is easily recognized that (48) can be cast as (4) for  $\gamma = 2$ . Thus, taking into account the (42), positive solitonlike solutions of (48) are immediately found:

$$u(\xi) = \bar{u} [1 + \epsilon \operatorname{sech}(\xi/\Delta)] , \quad (50)$$

where

$$\begin{aligned} \epsilon = & \pm \sqrt{1 - 32|q_2|(u_0 - V_0)^2 / (3q_1^2)} , \\ \Delta = & |\alpha| / \left( 2\sqrt{2|E_0|} \right) , \end{aligned}$$

and

$$E'_0 = -3q_1^2 / (64|q_2|) + (u_0 - V_0)^2 / 2 \quad ,$$

provided that  $E'_0 < 0$  and  $q_2 < 0$  and

$$-\sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 < u_0 < \sqrt{\frac{3q_1^2}{32|q_2|}} + V_0 \quad .$$

Eq. (50) shows that we can distinguish the following four cases.

(a).  $0 < \epsilon < 1$  ( $u_0 - V_0 \neq 0$ ):

$$u(\xi = 0) = \bar{u}(1 + \epsilon) \quad , \quad \text{and} \quad \lim_{\xi \rightarrow \pm\infty} u(\xi) = \bar{u}$$

which corresponds to a bright soliton of maximum amplitude  $(1 + \epsilon)\bar{u}$  and up-shifted by the quantity  $\bar{u}$ . We could call it up-shifted bright soliton.

(b).  $- < \epsilon < 0$  ( $u_0 - V_0 \neq 0$ ):

$$u(\xi = 0) = \bar{u}(1 - \epsilon) \quad , \quad \text{and} \quad \lim_{\xi \rightarrow \pm\infty} u(\xi) = \bar{u}$$

which is a dark soliton with minimum amplitude  $(1 - \epsilon)\bar{u}$  and reaching asymptotically the upper limit  $\bar{u}$ . It corresponds to a standard gray soliton.

(c).  $\epsilon = 1$  ( $u_0 - V_0 = 0$ ):

$$u(\xi = 0) = 2\bar{u} \quad , \quad \text{and} \quad \lim_{\xi \rightarrow \pm\infty} u(\xi) = \bar{u}$$

which corresponds to a bright soliton of maximum amplitude  $2\bar{u}$  and up-shifted by the maximum quantity  $\bar{u}$ . We could call it upper-shifted bright soliton.

(d).  $\epsilon = -1$  ( $u_0 - V_0 = 0$ ):

$$u(\xi = 0) = 0 \quad , \quad \text{and} \quad \lim_{\xi \rightarrow \pm\infty} u(\xi) = \bar{u}$$

which is a dark soliton (zero minimum amplitude), reaching asymptotically the upper limit  $\bar{u}$ . It correspond to a standard dark soliton.

TABLE III displays the plots of  $\rho$ , as a function of  $\xi/\Delta$ , for given values of the parameters, according to (50).

Correspondingly, by means of the correspondence  $\mathcal{H}$ , we can conclude that the following MNLSE:

$$i\alpha \frac{\partial \Psi}{\partial s} + \frac{\alpha^2}{2} \frac{\partial^2 \Psi}{\partial x^2} - [q_1 |\Psi|^2 + q_2 |\Psi|^4] \Psi = 0 \quad , \quad (51)$$

has the following envelope solitonlike solutions:

$$\begin{aligned} \Psi(x, s) = & \sqrt{u} [1 + \epsilon \operatorname{sech}(\xi/\Delta)] \exp \left\{ \frac{i}{\alpha} [\phi_0 - As + u_0 x] \right\} \times \\ & \times \exp \left\{ \frac{iB}{\alpha} \left[ \frac{\xi}{\Delta} + \frac{2\epsilon}{\sqrt{1-\epsilon^2}} \arctan \left( \frac{(\epsilon-1) \tanh(\xi/2\Delta)}{\sqrt{1-\epsilon^2}} \right) \right] \right\} , \end{aligned} \quad (52)$$

where  $\phi_0$  still plays the role of arbitrary constant,

$$A = \frac{15q_1^2}{64|q_2|} + \frac{(u_0 - V_0)^2}{2} + \frac{u_0^2}{2} , \quad (53)$$

and

$$B = -\frac{|\alpha|(u_0 - V_0)}{2\sqrt{2}|E'_0|} . \quad (54)$$

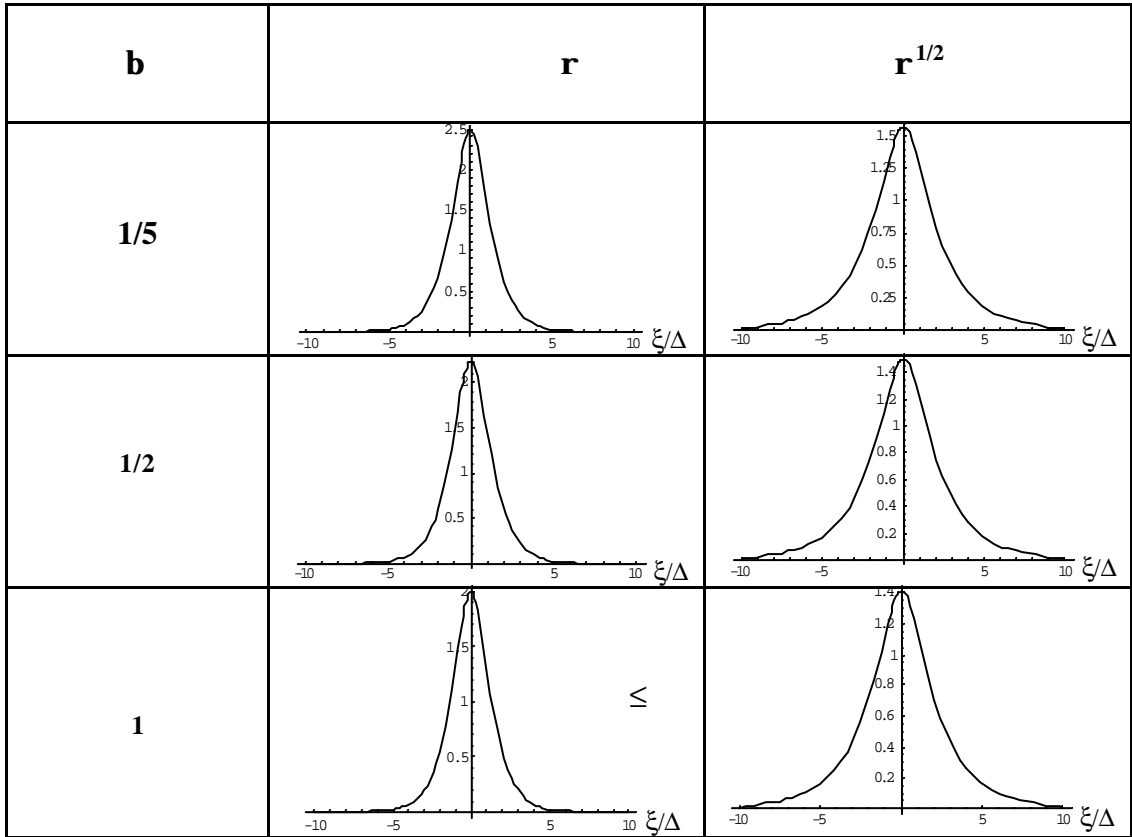
#### 4 Conclusions and remarks

In conclusion, in this paper a correspondence between solitonlike and envelope solitonlike solutions of wide families of the MKdVE and MNLSE, respectively, has been constructed within the framework of the Madelung's fluid, which play the role of the fluid description counterpart of the of the nonlinear Schrödinger equation. Under suitable constrains, this correspondence can be made invertible and, remarkably, it has been used to find bright and gray/dark envelope solitonlike solutions of a wide family of MNLSE, just starting from the knowledge of the solitonlike solutions of the associated MKdVE. In particular, on the basis of the present theory, the well known bright and gray/dark envelope solitons of the cubic NLSE have been easily recovered, starting from the soliton solutions of the corresponding standard KdVE. It is worth to point out that the above correspondence seems to be very helpful to get suitable instability criteria for the envelope solitonlike solutions of the MNLSEs presented in this paper, on the basis of the already developed know how about the instability criteria of the solitonlike solutions of the corresponding MKdVE. This field is, at present, unexplored. For instance, the theory of the instability criteria for MKdVE solitons has been developed for integer  $\gamma$  (see Eq. (4) ) [5]-[7]. The extension of this theory to any positive real  $\gamma$  should be a novelty, and the corresponding extension for the envelope solitons of the MNLSE (3) should be a novelty, as well. However, some attempt is on the way. In a future work, a pioneering contribution will be attempted.

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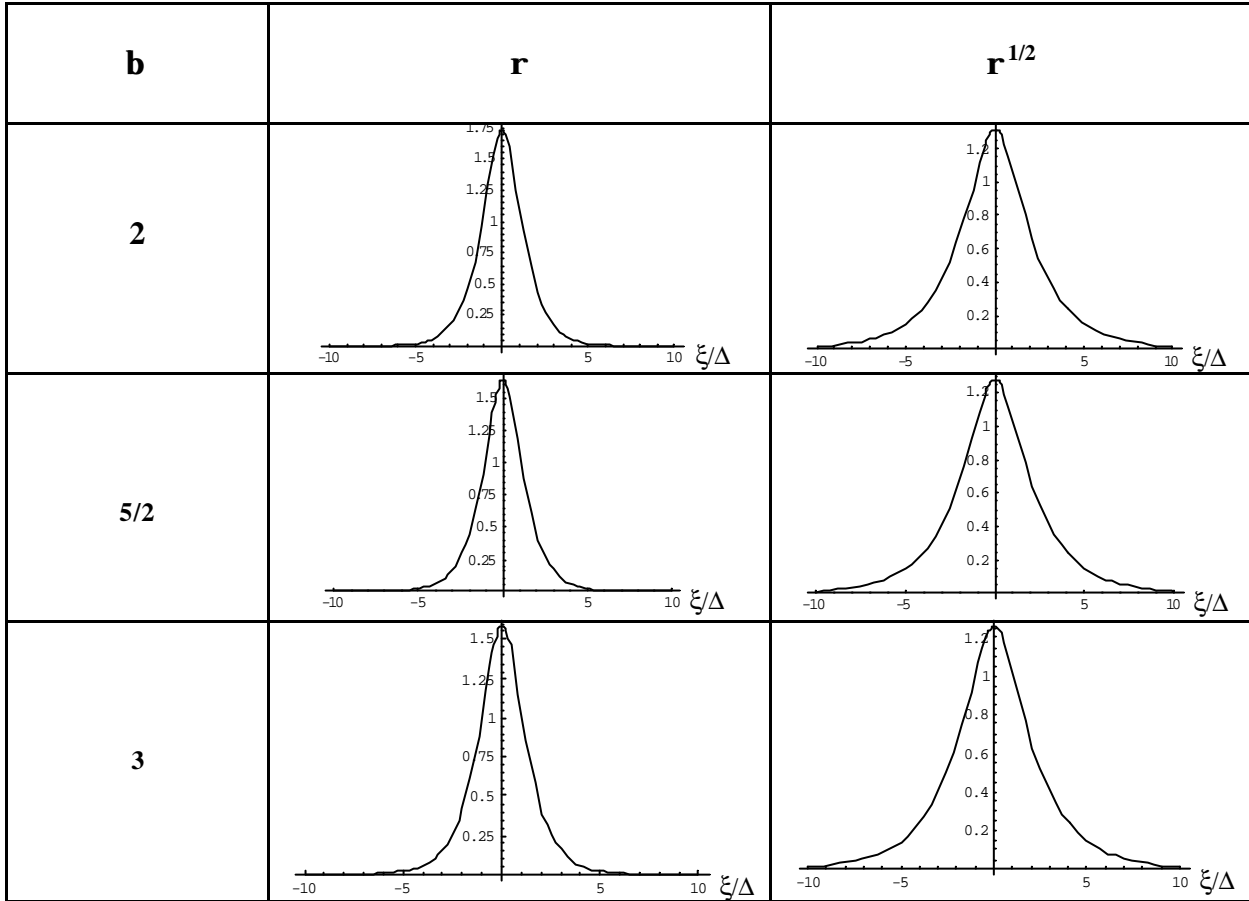
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TABLE I



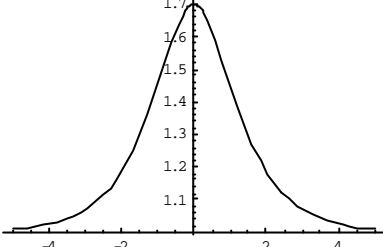
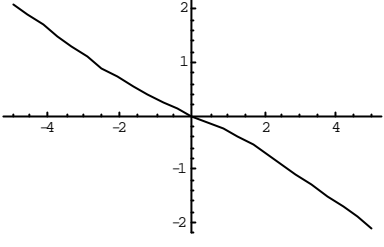
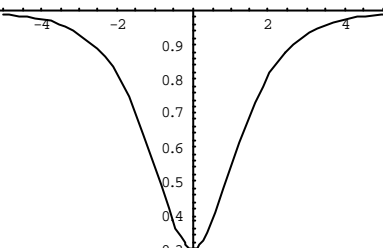
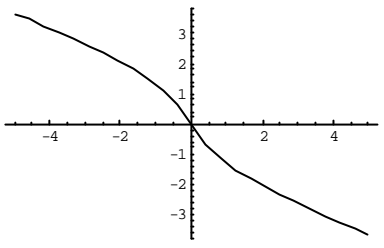
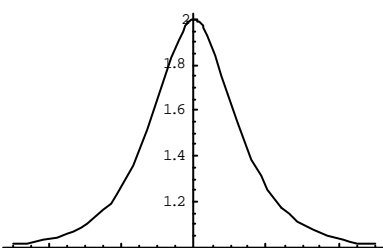
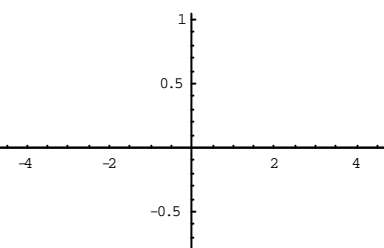
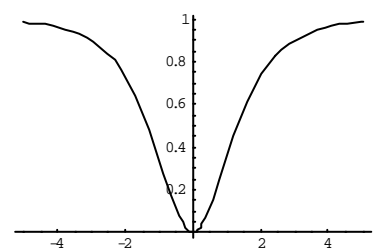
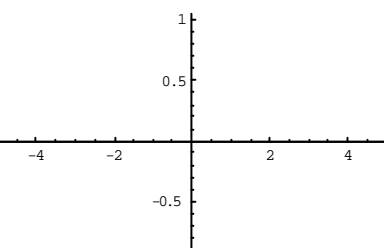
Plots of  $\rho$  and  $\rho^{1/2}$  (bright solitons) as function of  $\xi/\Delta$  for  $\beta = 1$ .  $E_0 = -1$ ,  $q_0 = -1$ ,  $\alpha = 1$ .

TABLE II



Plots of  $\rho$  and  $\rho^{1/2}$  (bright solitons) as function of  $\xi/\Delta$  for  $\beta > 1$ .  $E_0 = -1$ ,  $q_0 = -1$ ,  $\alpha = 1$ .

TABLE III

$e$	$u$	$Q_1$	Soliton's name
<b>0.7</b>			<b>up-shifted</b>
<b>-0.7</b>			<b>gray</b>
<b>1</b>		 $(\Theta_1 = 0)$	<b>upper-shifted</b>
<b>-1</b>		 $(\Theta_1 = 0)$	<b>dark</b>

Plot of solutions  $u$  and the nonlinear part of the phase  $\Theta_1 \equiv A_0 \int \frac{d\mathbf{x}}{u(\mathbf{x})}$  as function of  $\xi/\Delta$  for  $\bar{u} = 1$ ,  $e = \pm .7$  with  $u_0 - V_0 = .5$ ,  $e = \pm 1$  which corresponds to  $u_0 - V_0 = 0$ .